Problem Sheet 8

Due date: 24 October 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Study for the Midterm Exam (**priceless**). The exam is closed-book, on Friday 26 October at 09:15 in Corcoran 309. A sheet with formulae has been provided to you.

2. ϵ-Tensor and Kronecker-δ, Part II (**4P**): While the ϵ-tensor is coordinate independent and hence commutes with derivatives, the arbitrary, three-dimensional vector functions $\vec{A}(\vec{r}), \vec{B}(\vec{r})$ are not. Calculate or prove using the ϵ-tensor and Einstein's summation convention, and *not* using some other technique:

   a) (**2P**) If M_S is a symmetric matrix, then $\epsilon_a{}^{bc}\,(M_\mathrm{S})^a{}_b = 0$.

      Interpretation: The ϵ-tensor projects out the anti-symmetric component of an arbitrary matrix. Colloquially, this means "symmetric times antisymmetric is zero".

   b) (**1P**) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{r})) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}(\vec{r})) - \Delta\vec{A}(\vec{r})$;

   c) (**1P**) If $\vec{A}(\vec{r})$ and $\vec{B}(\vec{r})$ are source-free, what are the sources of $\vec{A}(\vec{r})\times\vec{B}(\vec{r})$?

3. A Skewed Basis (**4P**): Given the following basis of two-dimensional Euclidean space:

$$e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \quad e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   a) (**2P**) Construct the metric tensor g_{ij} and dual basis ϵ^i.

   b) (**1P**) Determine the *covariant* components of the vector $r = 2e_x - e_y$ in this basis, with e_x, e_y the ordinary, orthonormal basis vectors of two-dimensional Euclidean space.

   c) (**1P**) Determine the length of a vector $r = 3e_1 + 2e_2$ *without* using the Cartesian basis.

Please turn over.

4. HYPERBOLIC COORDINATES (**8P**): The relation between hyperbolic coordinates $(u,\ v,\ \phi)$ and Cartesian ones in three-dimensional Euclidean space is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh u\ \cos v\ \cos\phi \\ \cosh u\ \cos v\ \sin\phi \\ \sinh u\ \sin v \end{pmatrix} .$$

I know you know you can find the solution in wikipedia, but if I would be interested in the solution, I would do the problem myself. I am interested in how you *get* the solution.

a) (**2P**) Derive the transformation matrix from the change $\mathrm{d}s^i_{\text{Cartes}} = d^i{}_j\ \mathrm{d}s^j_{\text{hyp}}$ of the line element.
b) (**2P**) When you calculate the metric tensor, you find that these are orthogonal curvilinear coordinates. Show that the scale factors are $h_{(u)} = h_{(v)} = \sqrt{\cosh^2 u - \cos^2 v}$, $h_{(\phi)} = \cosh u\ \cos v$.
c) (**4P**) Sketch the "coordinate grid", i.e. the lines mapped out when all but one of the coordinates u, v and ϕ are kept fixed. Pay particular attention to their intersections. Identify possible coordinate singularities, i.e. points at which the coordinate transformation is not well-defined.

5. IRREDUCIBLE TENSORS AND INTEGRATIONS (**6P**): In this exercise, you will see the advantage of using the irreducible-tensor decomposition for rank-2 tensors. $\mathrm{d}\Omega$ is the surface element of the unit sphere; $\vec{e}_a$ and $\vec{e}_b$ unit vectors pointing in some un-specified, fixed directions.

a) (**1P**) Calculate $\int \mathrm{d}\Omega\ e^i_r\, e^j_r$ in *two* dimensions, where $\mathrm{d}\Omega$ is the surface element of the unit sphere in two dimensions, i.e. of the circle. This one is independent of the two next problems; and you can rehash what we did in the lecture for the same object in three spatial dimensions.
b) (**3P**) Using the irreducible-tensor decomposition for rank-2 tensors, calculate in three spatial dimensions $\int \mathrm{d}\Omega\ \dfrac{\vec{e}^{\,i}_r \vec{e}^{\,j}_r}{\vec{e}_r\cdot\vec{e}_a + b}$, with $b > 1$.
c) (**2P**) Consider $\int \mathrm{d}\Omega\ \dfrac{\vec{e}_r \times \vec{e}_b}{\vec{e}_r\cdot\vec{e}_a + c}$ for $c > 1$ in three spatial dimensions. To see how painful this can be, set in Cartesian coordinates $\vec{e}_a = (0,0,1)$ and $\vec{e}_b = (\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')$, with θ', ϕ' fixed angles (not integrated over). Attempt (but do not proceed too far) to calculate the cross and scalar products as well as the integrals explicitly. After you think you have suffered enough, use the methods developed in the lecture to write down the result for arbitrary $\vec{e}_a$ and $\vec{e}_b$ in a few lines.

Question of the Week (bonus 3P): Why is muscular power directly proportional to the section of the muscle, not to its volume?