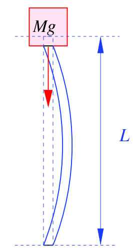
Problem Sheet 7 (last relevant for Midterm)

Due date: 17 October 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Study for the Midterm Exam (**priceless**). The exam is closed-book, on Friday 26 October at 09:15 in Corcoran 309. A sheet with formulae will be provided to you early next week.

2. Euler's Problem (**7P**): buckling of a slender column under a compressive load.
The elastic energy per unit length of a bent steel column is $YI/(2R^2)$. Here, R is the radius of curvature due to the bending, Y is Young's modulus of the steel and I is the moment of inertia of the rod's cross section about an axis through its centroid and perpendicular to the plane in which the rod is bent. A mass M is put on top of the massless column of length L, see figure. We assume that the rod is only slightly bent into the yz plane and lies close to the z axis.



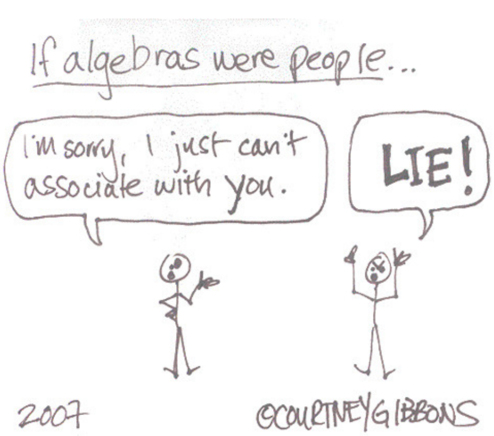
   a) (**3P**) Show that when the rod buckles slighly (i.e. deforms with both ends remaining on the z axis) the total energy, including the gravitational potential energy of the loading mass M, is approximately (the prime denotes differentiation with respect to z):

$$U[y] = \int\limits_0^L \mathrm{d}z \; \left[\frac{YI}{2}\,(y'')^2 - \frac{Mg}{2}\,(y')^2\right]$$

   b) (**4P**) Solve the resulting Euler-Lagrange equations. Carefully examine the boundary conditions: The column is nailed to the floor. Since the column is very slender, it will tripp over when the weight is not right over its foot. Is the variation of y' at the endpoints fixed?

   Hint: If you think a bit, you do not have to extend the Euler-Lagrange equation to the case that the functional depends on higher derivatives, $y''(x)$ etc. If you use the extension, prove it.

3. Casimir Operator (**4P**): The (first) Casimir operator of a Lie Algebra is the generalisation of the operator $\vec{J}^2$. It is defined as sum of all squared generators of a Lie Algebra in a given representation, $\mathrm{Cas}(L[G]) := t^a t_a$. Show: If the generators form a complete ortho-normal basis, then the Casimir operator commutes with each generator.



Please turn over.

4. LORENTZ GROUP (**9P**): Given two-dimensional, real vectors $X^\mu = \begin{pmatrix} c\mathrm{d}t \\ \mathrm{d}x \end{pmatrix}$. Interpret $\mathrm{d}t$ as the temporal and $\mathrm{d}x$ as the spatial distance between two events taking place separated by time-delay $\mathrm{d}t$ and spatial distance $\mathrm{d}x$, and $c = 1$ the speed of light. In that case, the fact that the speed of light is the same in all inertial systems can be re-formulated as $0 = (c\mathrm{d}t)^2 - (\mathrm{d}x)^2$ (just solve for c^2). Let's consider the group of two-dimensional matrices Λ which leaves the MINKOWSKI PSEUDO-SCALAR PRODUCT invariant, i.e. which leaves invariant $\langle X|X\rangle \equiv X_\mu X^\mu := (c\mathrm{d}t)^2 - (\mathrm{d}x)^2$ in component form.

   a) (**3P**) Derive that the most general, real matrix with *only positive entries on the diagonal* which leaves Minkowski's scalar product invariant can be written as

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \quad , \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad , \; -1 < \beta < 1 \; .$$

   Aside: We interpret $\beta = v/c$ for an inertial system with velocity v relative to an observer...

   b) (**2P**) Is the resulting Lie group of all two-dimensional Lorentz-transformations compact?

   Hint: Just noting that β is always finite will not give the right answer.

   c) (**4P**) Construct a complete ortho-normal basis of the Lie algebra.

5. NUCLEON-NUCLEON SCATTERING (**10P**, not relevant for Midterm Exam, but very good exercise to understand QM-I): A simpler case of decomposing a reducible representation into irreducible ones than the one discussed in the lecture is provided by the iso-spin invariance of nucleon-nucleon scattering. Recall: One defines an iso- vector as $\vec{N} := \begin{pmatrix} p \\ n \end{pmatrix}$ (p: proton, n: neutron) and assumes that the interaction H is invariant under $SU(2)$ transformations acting on it.

   a) (**8P**) Following the lecture and the mathematica notebook which accompanies it, show that the tensor product of two spin-$\frac{1}{2}$ representations is not irreducible but gives one spin-1 and one spin-0 representation. Your solution will contain:

   - the three generators $\vec{J} := \frac{\vec{\sigma}_1}{2} \otimes 1 + 1 \otimes \frac{\vec{\sigma}_2}{2}$ as matrices in the product basis $\vec{N} \otimes \vec{N}$;
   - $\vec{J}^2$ as matrix in the product basis;
   - an argument whether the resulting 4-dimensional representation is reducible;
   - the construction of a matrix which brings every generator and $\vec{J}^2$ into block-diagonal form;
   - the block diagonal forms of $\vec{J}$ and $\vec{J}^2$;
   - an argument which irreducible representations are seen in block-diagonal form and why.

   Hint: Yes, you can write this all out by hand. After all, you have to deal only with quite simple 4×4 matrices, and you only need to diagonalise a 2×2 sub-matrix. But you are also allowed to use and adjust the mathematica notebook to this case. If you do, provide a printout of your version or include it in your email submission.

   b) (**2P**) List all CLEBSCH-GORDAN COEFFICIENTS which you found in a), i.e. the coefficients $\langle JM|j_1m_1;j_2m_2\rangle$ of the transformation from the product basis to the block-diagonal basis (of irreducible representations). Because of the ambiguities in the transformation matrix, these coefficients are only fixed up to an overall sign.

Question of the Week (bonus 3P): How many paper copies are made in the US, daily?