Problem Sheet 6

Due date: 10 October 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. The Group of Taylor Operations (**6P**): Consider the Taylor operator $T[a] := \exp[a\frac{\mathrm{d}}{\mathrm{d}x}]$ which generates one-dimensional displacements: $T[a]f(x) = f(x+a)$.

   a) (**2P**) Derive the group manifold, its dimension, and show whether it is compact.

   b) (**2P**) Find the matrix which describes how T acts on a vector made of the n functions $\{x^{n-1}, x^{n-2}, \dots, x^2, x, 1\}$, with $n \in \mathbb{N}_0$.

   c) (**1P**) What is the dimension of this representation?

   d) (**1P**) Is this representation unitary?

2. Counting Matrices (**4P**): This is some clean-up work.

   a) (**2P**) Show $\det M = \exp \operatorname{tr} \ln M$ when M is diagonalisable.

   b) (**2P**) Determine the real dimension of the Lie algebra of the group $U(N)$ by counting the number of independent entries in an anti-Hermitean matrix. You may *not* use that $\dim G = \dim L[G]$.

3. Structure Constants as Lie Algebra (**6P**): There are *four* criteria which define a Lie algebra: closure, bi-linearity, anti-symmetry, Jacobi-identity. We now interpret the structure constants f^{abc} as matrices $(t^c)^a{}_b = (\mathrm{i}f^c)^a{}_b$ which give the (ab) entry of the cth generator of the Lie algebra. That means the (ij) entry of the commutator reads a bit weird at first sight, but you will manage (remember you can call a summed index whatever you want):

$$([t^a, t^b])^i{}_j = (\mathrm{i}f^{ab}{}_c t^c)^i{}_j \implies (\mathrm{i}f^a)^i{}_k\,(\mathrm{i}f^b)^k{}_j - (\mathrm{i}f^b)^i{}_k\,(\mathrm{i}f^a)^k{}_j = \mathrm{i}f^{ab}{}_c\,(\mathrm{i}f^c)^i{}_j \;.$$

   a) (**4P**) Show that these matrices obey the definition of a Lie algebra.

   Point of Information: This representation is called the adjoint representation of the Lie algebra. It always exists as matrix representation because you just constructed it.

   b) (**2P**) Show that the generators of the adjoint representation of $L[SU(2)]$ are the angular momentum operators L_i which generate $L[SO(3)]$.

Please turn over.

4. Basis or No Basis? (**5P**) Consider the Lie algebra of strictly upper-triangular matrices (i.e. all diagonal entries are zero as well). Find a possible set of generators and discuss whether they form a complete, ortho-normalisable basis.

   Modification: If you have problems with the general case, you may specialise to the case of upper-triangular 3×3 matrices – but then you get at most **3P**.

5. Properties of $SU(2)$ and its Lie Algebra (**9P**):

   a) (**2P**) Show that every element of $SU(2)$ can be written as $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$, where a, b are complex numbers and $|a|^2 + |b|^2 = 1$. Therefore, the group-manifold of $SU(2)$ is S^3, i.e. the three-dimensional surface of a sphere in $\mathbb{R}^4$.

   b) (**2P**) Prove the identity
   $$\exp[\mathrm{i}\alpha_a \frac{\sigma^a}{2}] = 1 \cos\frac{\alpha}{2} + \mathrm{i}\,\frac{\alpha_a \sigma^a}{\alpha} \sin\frac{\alpha}{2} \quad ,$$
   where $\alpha = \sqrt{\alpha^a \alpha_a}$ is the length of the vector α^a in the space spanned by the Pauli matrices σ^a.

   Hint: The properties of the Pauli matrices come in quite handy, as does the definition of the function of a matrix by its corresponding Taylor expansion.

   c) (**3P**) Show that the mapping of its Lie algebra on the Lie group $SU(2)$ is bijective (one-to-one and onto) for $\alpha \in [0; 4\pi[$ (i.e. 4π is *not* in the bijective region). It may help to interpret $\{1, \sigma^a\}$ as the orthogonal basis vectors of 4-dimensional space, in which $SU(2) \simeq S^3$ is a sphere.

   d) (**2P**) Argue whether the manifold of $SU(2)$ is curved or not, i.e. if the only place at which the coordinate axes $\{\sigma^a/2\}$ of the Lie algebra intersect is the origin.

   Consequence: You have just proven by practical construction: (i) The tangent space of $SU(2)$ in the identity suffices to construct all of $SU(2)$; (ii) $SU(2)$ is connected; (iii) as is S^3; (iv) both are compact.

Question of the Week (bonus 3P): How heavy is DC?