Problem Sheet 3

Due date: 19 September 2018 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

Question of the Week (bonus 3P): Why does a horse run uphill slower than a dog, although both have the same running speed on level ground?

1. ASYMPTOTICS OF THE ERROR FUNCTION (**6P**): As you know, the Gauß'ian distribution plays a prominent rôle in Statistics. The "complementary error function" is the probability that an event which follows a Gaußian distribution has a value between x and ∞.

   a) (**5P**) Derive that for large $x \to \infty$, the integral can be asymptotically approximated as
   $$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int\limits_x^\infty \mathrm{d}t\; \mathrm{e}^{-t^2} \to \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi} x} \sum_{n=0}^{m} (-1)^n \frac{(2n-1)!!}{2^n x^{2n}} + R_m(x) \;\;,$$
   where $(2n-1)!! = 1 \times 3 \times 5 \cdots \times (2n-1)$. Determine the remainder as an integral and estimate its size.

   b) (**1P**) Where is its "sweet-spot"?

2. SADDLE-POINT APPROXIMATION (**4P**): We approximately solve $I(n) := \int\limits_{-\pi/2}^{\pi/2} \mathrm{d}t\; \cos^n t$ for $n \to \infty$.

   a) (**2P**) Apply the saddle-point approximation to the integrand. You see that its integral cannot be done in closed form. By plotting the integrand of the original and approximated function, shown that one can without significant error increase the integration of the approximated integral to have the limits $\pm\infty$.

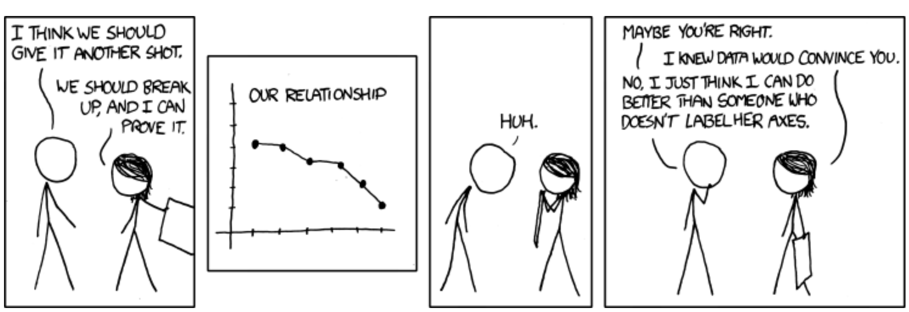
   b) (**2P**) Perform the approximation. Compare with the "exact" result, $I(n) = \dfrac{\pi}{2^n} \dfrac{n!}{[(n/2)!]^2}$, n even.

3. PERTURBATION THEORY FOR MATRICES (**6P**): Consider the matrix of example (ii) in the lecture, split into un-perturbed and perturbed parts as:
   $$\begin{pmatrix} 7.5 & 0.4 & -0.7 \\ 0.4 & -3.1 & 0.1 \\ -0.7 & 0.1 & 2.6 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 0.5 & 0.4 & -0.7 \\ 0.4 & -0.1 & 0.1 \\ -0.7 & 0.1 & 0.6 \end{pmatrix}$$

   a) (**2P**) Perturbatively find one of the eigenvalues and eigenvectors to next-to-leading order (NLO). Of course, you should do this by hand, as practise in matrix multiplications.

   b) (**2P**) Perturbatively find one of the eigenvalues at next-to-next-to-leading order (N^2LO).

   c) (**2P**) Provide a defendable estimate for the eigenvalue and eigenvector, including an error-bar.



Please turn over.

4. PERIHELION SHIFT OF MERCURY, AGAIN (**6P**): Including the first post-Newtonian correction, the equation of motion of the distance r of a mass m as function of time t in the gravitational field of a much heavier body $M \gg m$ is:

$$\ddot{r}(t) = \frac{l^2}{m^2 r^3} - \frac{G_N M}{r^2} + \frac{6 G_N^2 M^2}{c^2 r^3} \quad ,$$

where l is the system's angular momentum (cf. entrifugal barrier). We surmised in HW II.3 $\beta \sim \dfrac{G_N M}{r}$, now we know $\beta = \dfrac{3 G_N M}{r}$ – so a tideous non-relativistic reduction only provides the factor "3".
You will learn in the Mechanics lectures that one performs a change of variables from $r(t)$ to $u(\theta)$, where θ parametrises the planet's orbit. The result is an equation which we need to solve by approximation:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \omega_0^2 \, u(\theta) = \frac{G_N M m^2}{l^2} - \eta \, u(\theta)$$

with $\eta := \dfrac{6 G_N^2 M^2 m^2}{l^2 c^2} \ll 1$ the small parameter. The problem starts now.

a) (**1P**) Show: For $\eta = 0$ and with ε the eccentricity of the orbit, the equation is solved by

$$u_0(\theta) = \frac{G_N M m^2}{l^2} \left[1 + \varepsilon \, \cos \omega_0 \theta\right] \quad \text{when} \quad \omega_0^2 = 1 \ .$$

Notice that ω_0 is not really a frequency because θ does not have dimensions of time. We are however free to choose $\omega_0^2 = 1$, so that θ plays the rôle of an angle: $\cos \omega_0 \theta = 1$ for $\theta = 0, 2\pi, 4\pi \ldots$.

b) (**3P**) Now expand the full equation to first order in η (re-adjust "frequency"!):

$$u = u_0 + \eta \, u_1 + \mathcal{O}(\eta^2) \quad \text{and} \quad \omega^2 = \omega_0^2 + \eta \, \omega_1^2 + \mathcal{O}(\eta^2)$$

Fix ω_1^2 to cancel any problematic terms. Do not solve the equation of motion completely.

c) (**2P**) Consider finally one complete period *with* the perturbation. You see that ω is changed, so maxima of $\cos \omega\theta$ are not 2π apart. By how much does the distance $\Delta\theta$ of two maxima differ from the unperturbed case? That difference is called the *perihelion shift*. By recycling last week's result, compare the perihelion shift of Mercury with LEVERRIER's observation, $43''$ per century. Alternatively, you can use: solar mass $M = 2.0 \times 10^{30}$ kg, Mercury mass $m = 3.3 \times 10^{23}$ kg, $l = 9.15 \times 10^{38}\,\mathrm{kg\,m^2\,s^{-1}}$ for the Mercury-Sun system, Mercury's rotation period 0.24885 years.

5. A FATA MORGANA (MIRAGE) (**8P**): According to FERMAT'S PRINCIPLE, in a medium with varying speed of light $v(\vec{r})$, light takes the path between two given points P and Q with the least travel time,

$$\text{i.e. the path of light minimises} \quad J[\vec{r}(s)] = \int\limits_P^Q \frac{\mathrm{d}s}{v(\vec{r})} \quad , \quad \text{where } s \text{ parameterises the path.}$$

a) (**3P**) To simplify, we concentrate on a two-dimensional problem, i.e. $\vec{r} = (x, y)$. In addition, we pick a medium in which v varies only with y, e.g. air whose light speed only changes with height (different layers/densities). Reduce the solution to the light path to a QUADRATURE, i.e. find the parameterisation of the path $x(y)$ or $y(x)$ in a form which contains only one integral (which you cannot do in general). Start with a functional with independent variable x, not s.

b) (**4P**) Turn now to the special case $v(x, y) = v_0 + \gamma \, y$ with v_0, γ constants. This parameterisation only makes sense for the range of y where $v > 0$. Construct the light path between two points on the surface of the (flat) Earth $P = (x = -L, y = 0)$ and $Q = (x = L, y = 0)$. Sketch, describe and discuss the form of the path. Is it unique? Differentiate between $\gamma > 0$ and $\gamma < 0$.

c) (**1P**) Is v larger on the top or on the bottom of the dune? Give a physics reason!