Problem Sheet 2

Due date: 12 September 2018 **16:00**

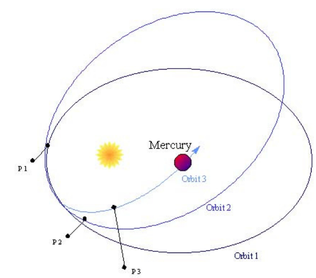
For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/math-methods18/math-methods18.html.

1. Dimensional Analysis II (**2P**): By matching dimensions, we can discover some very fundamental relations of modern physics – without understanding the underlying theory.

   Let's pretend that the mass of the electron is somehow generated by its charge via electromagnetic interactions (so electromagnetic fundamental constants can enter). Estimate the classical charge radius of the electron.

2. Perihelion Shift of Mercury (**7P**): An important success of General Relativity is the correct prediction of the angle $\Delta\phi = 43''$ per century by which the perihelion (minimum distance to the Sun) of Mercury rotates. This effect, first discovered by Leverrier in 1859, could not be explained by Newtonian gravity. In 1915, Einstein found that exact value from the first post-Newtonian correction to the gravitational potential between two bodies. We now make a back-of-the-envelope estimate of the size of the effect – something which Einstein presumably did before he started on the hard part. The problem is also known as Feynman's (Oral) Exam Question.

   In the lecture, we already talked about some aspects: We expect that the post-relativistic potential can be written as a Taylor expansion in squares of negative powers of the speed of light, c; with r: distance Sun-Mercury, M: solar mass, m: Mercury mass, G_N: Newton's gravitational constant:

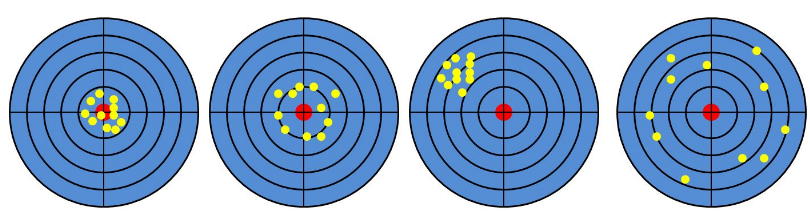
   $$U(r) = -\frac{G_N M m}{r}\left[1 + \frac{\beta}{c^2} + \mathcal{O}(c^{-3})\right]$$

   a) (**1P**) Argue how the result depends on M, m, r, period of orbit, other parameters and constants.
   b) (**2P**) Determine how the parameter β scales in terms of these parameters.
   c) (**4P**) If the potential were $1/r$, orbits would be closed, i.e. after one orbit $\phi \to \phi + 2\pi$, the planet would return exactly to its previous position. The correction results in the orbit being shifted by an angle $\delta\phi$ after each rotation. Estimate, using the values below, the precession $\delta\phi$ after a full orbit from the post-Newtonian correction. From that, estimate the perihelion shift $\Delta\phi$ per century. Compare with the experimental value.

   Numbers: Sun's Schwarzschild radius: $\frac{2G_N M}{c^2} = 3$ km; Mercury's revolution period: 0.249 years.

3. Understand the Difference between "Accuracy" and "Precision" **(0P)**

   Notice that mathematica stubbornly interchanges the two.

   Accurate & Precise | Accurate & Imprecise | Inaccurate & Precise | Inaccurate & Imprecise

Please turn over.

4. Asymptotic Series (**8P**): Consider the integral $\displaystyle\int\limits_0^\infty \mathrm{d}t\; \frac{\mathrm{e}^{-xt}}{1+t^2}$ for $x \to \infty$.

   a) (**4P**) Using integration by parts or any other method, derive a series representation with remainder $R_m(x)$.

   Hint: Depending on which approach you choose, it might be smart to *not* perform derivatives of $1/(1+t^2)$ explicitly but leave them in implicit form till very late. Then, look at a few derivatives and find a pattern.

   b) (**1P**) Prove that the series is an asymptotic series.

   Hint: If your remainder in a) is too complicated, start from $R_m(x) = \displaystyle\int\limits_0^\infty \mathrm{d}t\; \frac{(-t^2)^{m+1}\mathrm{e}^{-xt}}{1+t^2}$.

   c) (**1P**) Derive a criterion at which point the error in the series should start increasing again. What's the "sweet spot", as function of x?

   d) (**2P**) Numerically investigate for $x = 10$ the dependence on the number of terms m in the sum retained, up to suitable m. Plot! Provide a prognosis for the "exact" integral at $x = 10$, including an error-bar.

5. Root-Finding (**4P**): Find the perturbative roots of the following polynomial to next-to-next-to-leading order and provide an error-assessment, dependent on ϵ with $|\epsilon|$ small:

$$\epsilon^2 x^7 - \epsilon x^5 + x - a = 0 \quad a \;\text{ real}$$

For which a is this a good approximation, for which a bad one?

6. An-Harmonic Oscillator (**9P**): A pendulum is released at rest from an amplitude A. Its motion is described by the non-linear equation of motion

$$\ddot{x} + \Omega^2 x - \eta x^2 = 0 \;\; .$$

   a) (**3P**) You see that η is not a dimension-less parameter, so we cannot pretend that $\eta \ll 1$. By considering the corresponding potential, derive a criterion for η such that perturbation theory is useful. What is the dimension-less parameter which can be assumed to be small?

   b) (**6P**) Assuming your parameter in a) is indeed small, solve for the oscillation pattern $x(t)$ to first order perturbation theory. Is it necessary to adjust the frequency of the leading-order solution? Give a reason!

Question of the Week (bonus 3P): *From time to time, I offer "Guesstimate" questions for extra credit. Your answer should be a short "essay" that explains how you come up with the result.*

How many M&M's fit into a Smart car?