Supplement on Vector Spaces

Def. Vector Space ($\mathbb{M}$,+,×): a set $\mathbb{M}$ with operations

$$\begin{array}{llrcl} \text{addition “+”} & : & \mathbb{M}\times\mathbb{M} & \rightarrow & \mathbb{M} \\ & & (x,y) & \mapsto & x+y \\ \text{multiplication “}\times\text{”} & : & (\mathbb{C}\text{ or }\mathbb{R})\times\mathbb{M} & \rightarrow & \mathbb{M} \\ & & (\lambda,x) & \mapsto & \lambda\times x \end{array}$$

such that

(A1) $(\mathbb{M},+)$ is an Abelian group with id $=0$ the “zero vector”, i.e.

- (A1a) $\forall\, x,y,z\in\mathbb{M}:\ (x+y)+z=x+(y+z)$ — associativity in +
- (A1b) $\forall\, x,y\in\mathbb{M}:\ x+y=y+x$ — commutativity (Abelian)
- (A1c) $\exists$ “zero vector” id $\equiv 0:\ 0+x=x\ \forall x\in\mathbb{M}$ — zero vector as identity/unit element
- (A1d) $\forall\, x\in\mathbb{M}\ \exists x^{-1}:\ x+x^{-1}=\text{id}\equiv 0$ — inverse, usually denoted by $x^{-1}=-x$

(A2) $\forall\, x,y\in\mathbb{M},\ \lambda,\mu\in\mathbb{C}$ or $\mathbb{R}$:

- (A2a) $(\lambda+\mu)\times(x+y)=\lambda\times x+\lambda\times y+\mu\times x+\mu\times y$ — distributive multiplication
- (A2b) $(\lambda\mu)\times x=\lambda\times(\mu\times x)$ — associativity in ×
- (A2c) $1\times x=x$ — unit element of multiplication

This is not the most general definition possible, but it suffices for us: “FAPP”

Def. Scalar/Inner Product $\langle .,.\rangle$:

$$\begin{array}{rcl} \overline{\mathbb{M}}\times\mathbb{M} & \rightarrow & (\mathbb{C}\text{ or }\mathbb{R}) \\ (y,x) & \mapsto & \langle y,x\rangle \end{array}$$

maps vectors in $\mathbb{M}$ with vectors in $\overline{\mathbb{M}}$ into numbers such that $\forall\, x,x'\in\mathbb{M},\ y,y'\in\overline{\mathbb{M}},\ \mu\in\mathbb{C}$:

(A1) $\langle y,\lambda x+x'\rangle=\lambda\langle y,x\rangle+\langle y,x'\rangle$, i.e. linear in $\mathbb{M}$ — sesqui-linear

$\langle\lambda y+y',x\rangle=\lambda^*\langle y,x\rangle+\langle y',x\rangle$, i.e. anti-linear in $\overline{\mathbb{M}}$

(A2) $\langle y,x\rangle=\langle x,y\rangle^*$ — Hermitean/conjugate symmetric

A vector space $\overline{\mathbb{M}}$ which does achieve this is the **dual space to** $\mathbb{M}$.

Def. Posititve definite scalar product: $\forall\, x\in\mathbb{M}:\ \langle x,x\rangle\geq 0$ and $\langle x,x\rangle=0\iff x=0$

Def. Length/Norm of x: $||x||:=\sqrt{\langle x,x\rangle}$

Def. x and y are orthogonal: $\langle y,x\rangle=0$

Triangle Inequality: $||x+y||\leq||x||+||y||$, and “=” only iff $x=\lambda y$, i.e. x and y are **parallel**.

Cauchy-Schwarz Inequality: $\langle y,x\rangle^2\leq\langle x,x\rangle\ \langle y,y\rangle$, and “=” only iff $x=\lambda y$, i.e. x and y are **parallel**.