Supplement on Spherical Harmonics

Conventions from Arfken/Weber: Mathematical Methods for Physicists. $\Omega = (\theta, \phi)$: solid angle.

Eigenfunctions of the Laplace Operator: $$\Delta\, Y_{lm}(\Omega) = -\frac{l(l+1)}{r^2}\, Y_{lm}(\Omega) \tag{1}$$

Definition: $$Y_{lm}(\Omega) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, \mathrm{e}^{\mathrm{i}m\phi} \tag{2}$$

Orthonormality: $$\int \mathrm{d}\Omega\, Y^*_{lm}(\Omega)\, Y_{l'm'}(\Omega) = \delta_{ll'}\, \delta_{mm'} \tag{3}$$

Completeness/Closure: $$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega') = \delta(\Omega - \Omega') = \delta(\cos\theta - \cos\theta')\, \delta(\phi - \phi') \tag{4}$$

$$Y^*_{lm}(\Omega) = (-1)^m\, Y_{l,-m}(\Omega) \tag{5}$$

Rodrigues' formula for Associated Legendre-Polynomials:

$$P_l^m(u) = \frac{1}{2^l l!}\, (1-u^2)^{m/2}\, \frac{\mathrm{d}^{l+m}}{\mathrm{d}u^{l+m}} (u^2-1)^l \quad \forall m \in [-l; l] \tag{6}$$

Legendre-function: eigenfunction to Legendre ODE: $$\frac{\mathrm{d}}{\mathrm{d}u}\left[\left(1-u^2\right)\frac{\mathrm{d}}{\mathrm{d}u}\right] P_l(u) = -l(l+1)\, P_l(u) \tag{7}$$

Rodrigues' formula for Legendre-Polynomials: $$P_l(u) = \frac{1}{2^l l!}\, \frac{\mathrm{d}^l}{\mathrm{d}u^l}(u^2-1)^l \tag{8}$$

Generating Function: $$\frac{1}{\sqrt{1+\rho^2-2\rho u}} = \sum_{l=0}^{\infty} \rho^l\, P_l(u) \quad \text{for } |\rho| < 1 \tag{9}$$

Normalisation: $$\int_{-1}^{1} \mathrm{d}u\, P_l(u)\, P_{l'}(u) = \frac{2}{2l+1}\, \delta_{ll'} \tag{10}$$

Addition Theorem [with α the angle between (θ, ϕ) and (θ', ϕ')]:

$$P_l(\cos\alpha) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega') = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\alpha, 0) \tag{11}$$

Some Examples:

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}\,(3x^2-1) \tag{12}$$

$$Y_{00}(\Omega) = \frac{1}{\sqrt{4\pi}} \tag{13}$$

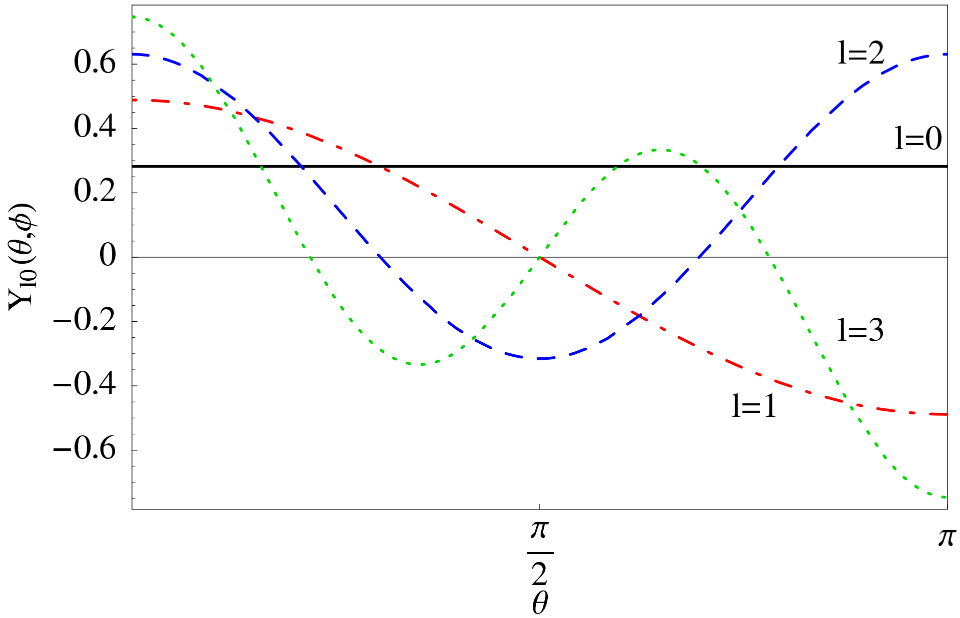
$$Y_{10}(\Omega) = \sqrt{\frac{3}{4\pi}}\, \cos\theta = \sqrt{\frac{3}{4\pi}}\, \frac{z}{r} \qquad Y_{1\pm1}(\Omega) = \mp\sqrt{\frac{3}{8\pi}}\, \sin\theta\, \mathrm{e}^{\pm\mathrm{i}\varphi} = \mp\sqrt{\frac{3}{8\pi}}\, \frac{x \pm \mathrm{i}y}{r} \tag{14}$$

$$Y_{20}(\Omega) = \sqrt{\frac{5}{16\pi}}\, (3\cos^2\theta - 1) = \sqrt{\frac{5}{16\pi}}\, \frac{3z^2 - r^2}{r^2} \tag{15}$$

$$Y_{2\pm1}(\Omega) = \mp\sqrt{\frac{15}{8\pi}}\, \cos\theta\, \sin\theta\, \mathrm{e}^{\pm\mathrm{i}\varphi} = \mp\sqrt{\frac{15}{8\pi}}\, \frac{z(x \pm \mathrm{i}y)}{r^2} \tag{16}$$

$$Y_{2\pm2}(\Omega) = \sqrt{\frac{15}{32\pi}}\, \sin^2\theta\, \mathrm{e}^{\pm2\mathrm{i}\varphi} = \sqrt{\frac{15}{32\pi}}\, \frac{(x \pm \mathrm{i}y)^2}{r^2} \tag{17}$$

Graph of the first Spherical Harmonics and Legendre-Polynomials $Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\; P_l(\cos\theta)$



Visualising as Deformation of a Sphere: $R(\vartheta,\varphi) = 1 + \alpha_{l0}\; Y_{l0}(\vartheta,\varphi)\;\;,\; \alpha \ll 1$