Questions to Check Your Progress

(version Autumn 2010)

Note: After your Mathematical Methods course, you should be able to fill the following collection of concepts with life. Not all questions were necessarily detailed (enough) covered in the course; other important ones were certainly skipped; yet others received too much attention; and some questions are very vague. This sheet cannot do more than inspire you and be curious to think about and delve into the subject with the aid of a good textbook. The wrongest use of this list would be to memorise answers or formulae. In that case, you could just as well memorise a book. Rather, think whether you could arrive at (motivate/sketch a way to) an answer, based on the knowledge you have. Under no condition is this a survey of material for exams – neither maximal nor minimal. It might not even be of any use at all.

Qualitative Methods: See hand-outs *Mathematics: The Bare Essentials* and *Physics: The Bare Essentials*. Gauß'ian integral by Poisson's trick. Sum the geometric series. How many calories do you burn when you walk from GW to the White House? Describe the power of Dimensional Analysis to an undergraduate non-Physics major. Why can grasshoppers jump more than 30 times their size, but humans only about 5 times their size. Scaling relations, mechanical similarity and Kepler's Third Law. Obtain an approximate solution to any problem without a computer or books. Sketch and discuss functions with pen and paper.

Calculus of Variations: Linear functional, its functional derivative, Euler-Lagrange equation for several independent variables, with and without variable endpoints. Difference dependent – independent variable; total – partial derivative. Importance of cyclic variables, constants of motion, Noether's theorem, first integral/Beltrami-identity. Brachistochrone, catenoid, shortest distances, principle of least action. Twin-paradox of SRT as variational problem. Differences between Euler-Lagrange and Rayleigh-Ritz principles.

Perturbation Theory: Taylor expansion. Definition, application and "sweet-spot" of Asymptotic Series. D'Alembert's convergence criterion. Saddle-point and stationary-phase approximation, method of steepest descend. Stirling's formula. Difference between regular and singular perturbations. Solve for roots of polynomial equations perturbatively (incl. degenerate case). Solve any transcendental equation perturbatively. Provide several, justifiable error-assessments. Predict when the method you use will not work or converge poorly. Explain the difference between (numerical) precision and (mathematical) accuracy. Perturbative solution to non-linear DEq's (pendulum, an-harmonic oscillator); adjusting frequencies; a-priori assessment of range of convergence by "potential-picture". Perturbative solution to eigenvalue problem of a 3×3 matrix.

Vector Spaces: Name at least 5 different vector spaces and their scalar products. Define dual space, dual/reciprokal/adjoint basis, norm, orthogonality. Basis-dependent and basis-independent/abstract quantities. Importance of a positive definite scalar product for expanding vectors in bases. Functionals as scalar products. Basis, co-/contravariant/dual/row/column components of a vector. Philosophies behind Einstein's Summation Convention and Dirac's $\langle \mathrm{bra}|\mathrm{ket}\rangle$ notation. Compare the two: vector components, metric, basis, basis transformation. Gram-Schmidt ortho-normalisation procedure: Complete Ortho-Normal (basis) Systems. Completeness, closure, Parseval's equation. Definition function of an operator, identity, inverse, adjoint, Hermitean, self-adjoint, unitary, projection operator. Range and domain of an operator.

Group and Representation Theory: Importance of groups and Schur's Second Lemma in Physics/QM. Abelian and non-Abelian groups. Construct a group representation table. Faithful, fundamental, trivial representations. Similarity/equivalence between representations. Completely reduce a unitary representation into irreducible pieces. How does a Lie Group induce a Lie Algebra with Lie Bracket at the tangent space in the identity? Compact and connected groups. Generators of, relations between and manifolds formed by $U(1)$, $SO(2)$, $SO(3)$, $SU(2)$. Explain: exp-map, Jacobi identity, Pauli spin operators, Casimir operator. Construct angular-momentum irreps using ladder operators. Relation between $SO(4)$ and Special Relativity? Properties of groups, algebras and generators: $GL(n)$, $SL(n)$, $U(n)$, $SU(n)$, $O(n)$, $SO(n)$. Trace-Log and Campbell-Baker-Hausdorff formulae.

Please turn over.

Tensor Analysis: Metric tensor under basis changes. Importance of Dreibein, locally Orthogonal Curvilinear Coordinates, coordinate singularities, scaling factors. Div/rot/grad/nabla/Laplace/line/surface/volume elements in Cartesian/cylindrical/polar/spherical coordinates, general OCCs. ε-tensor, Levi-Civitá symbol, polar and axial/pseudo-tensor of rank n. Use the irreducible-tensor method to calculate inertia tensors. Theorems by Gauß, Stokes, Helmholtz in arbitrary numbers of dimensions. Using these theorems, derive the physical laws named after Gauß, Ampére, Faraday.

Functional Analysis: Name 3 different spaces of functions and their scalar products. Discuss a Lesbegue-integrable function which is not Riemann-integrable. Define Cauchy-completeness, Hilbert space. Properties of Dirac's δ-distribution, representation as limit of functions. Which "standard" functions can be treated only in a distributional sense. What is a test-function? What is a complete orthonormal extended basis of functions? Examples. Properties of Fourier-transformations, and transforms e.g. of Dirac's δ-distribution, Gauß' distribution function. Why does one say that Fourier transforms turn derivatives into momenta? Relation between Fourier transforms and Heisenberg's uncertainty principle? Discuss an operator which is Hermitean but not self-adjoint. Spectral decomposition. Why is the spectrum of a self-adjoint operator particularly important?

Partial Differential Equations and Green's Functions: Definition/use and utility/construction/ uniqueness of Green's functions. Give 3 different physical interpretations of Green's functions, reciprocity relation. Born approximation and propagator theory. Causality in Green's functions. Name the Green's functions of the Laplace operator "without boundaries" in 1, 2, 3 dimensions. Given the spectrum of a differential operator, construct its Green's function. Von-Neumann vs. Dirichlet boundary conditions. What is " separation of variables", Frobenius' method? Properties and importance of Legendre functions and spherical harmonics. Sketch strategies to solve the Laplace equation. Why can every function $f(\theta, \varphi)$ on the unit sphere be written as linear combination of spherical harmonics $Y_{lm}(\theta, \varphi)$? Ansatz to solve the Laplace equation in spherical coordinates. Sturm-Liouville theory of second order ordinary differential equations: relation to self-adjointness; formulate a "cooking recipe" for special functions, including Rodrigues' formulae, recursion relations, generating functions, irregular solutions.
Related to Electrodynamics: Use δ- and θ-distributions in suitable coordinate systems to describe the charge densities of point-charges, homogeneously charged spheres/cylinders (surface or volume). When rotated with constant angular velocity about a symmetry axis, what do the corresponding currents look like? Can one uniquely determine the charge distribution inside a charged sphere from its electric field at infinity? Which boundary conditions determine the solution of a particular problem uniquely? Specify the general spherical multipole decomposition of the scalar potential and its spherical multipole moments. Rôle of symmetries. Pros and cons of spherical multipoles. For which (l, m) do the spherical charge multipoles q_{lm} describe a monopole/dipole/quadrupole? How to experimentally differentiate between multipolarities of static fields? Radial dependence of lth multipole. Under which conditions is the spherical multipole expansion a good approximation?

Complex Analysis: Polar/Cartesian representation of a complex number. Cauchy-Riemann condition, analyticity, isolated/essential singularities, poles, Laurent expansion. How to calculate residues and integrals over complex functions (e.g. keyhole, "arc-at-infinity", unit-circle methods). Relation between causality, Green's functions and complex integration? Principle-value prescription, Kramers-Kronig dispersion relations. Integrate a function with different branches. Location and importance of branch cuts, Riemann surfaces. Analytically continue the Legendre polynomials.