Supplement on the Fourier Transform

cf. [SG, App. B.2]

Def. Fourier (Integral) Transform $\mathcal{F}[f(x)]$ of $f(x)$, Inverse Fourier (Integral) Transform $\mathcal{F}^{-1}[f(k)]$:

$$\mathcal{F}[f(x)] \equiv \tilde{f}(k) \equiv f(k) := \mathcal{N} \int_{-\infty}^{\infty} \mathrm{d}x \; \mathrm{e}^{-\mathrm{i}kx} \; f(x) \quad , \quad \mathcal{F}^{-1}[f(k)] \equiv \tilde{f}(k) \equiv f(x) := \frac{1}{2\pi\mathcal{N}} \int_{-\infty}^{\infty} \mathrm{d}k \; \mathrm{e}^{+\mathrm{i}kx} \; \tilde{f}(k)$$

$(x,\ k)$ are **conjugate variables**, $(f(x),\ \tilde{f}(k))$ are a **Fourier transform pair**.

The sign $\mathrm{e}^{-\mathrm{i}kx}$ for $\mathcal{F}[f]$ is convention, but see property (4) below.

Convention for normalisation $\mathcal{N}$ in Quantum Mechanics: $\mathcal{N}_{\mathrm{QM}} = \frac{1}{\sqrt{2\pi}}$, i.e. $\langle k|x\rangle = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{i}kx}$;

$$\text{i.e. } \mathcal{F}_{\mathrm{QM}}[f] = \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(2\pi)^{\frac{1}{2}}} \; e^{-\mathrm{i}kx} \; f(x) \quad , \quad \mathcal{F}^{-1}_{\mathrm{QM}}[f] = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{(2\pi)^{\frac{1}{2}}} \mathrm{e}^{+\mathrm{i}kx} \; \tilde{f}(k)$$

Convention for normalisation $\mathcal{N}$ in Electrodynamics/QFT: $\mathcal{N}_{\mathrm{EDyn}} = 1$,

$$\text{i.e. } \mathcal{F}_{\mathrm{EDyn}}[f] = \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-\mathrm{i}kx} \; f(x) \quad , \quad \mathcal{F}^{-1}_{\mathrm{EDyn}}[f] = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{(2\pi)} \; e^{+\mathrm{i}kx} \; \tilde{f}(k)$$

Generalisation to d dimensions:

$$\mathcal{F}[f] = \mathcal{N}^d \int_{-\infty}^{\infty} \mathrm{d}^d r \; \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}} \; f(\vec{r}) \quad , \quad \mathcal{F}^{-1}[f] = \frac{1}{(2\pi\mathcal{N})^d} \int_{-\infty}^{\infty} \mathrm{d}^d k \; \mathrm{e}^{+\mathrm{i}\vec{k}\cdot\vec{r}} \; \tilde{f}(\vec{k})$$

Properties $\forall f, g \in \mathcal{H},\ a \in \mathbb{C}$:

(1) $\mathcal{F}[f]$ exists if $\int_{-\infty}^{\infty} \mathrm{d}x \; |f(x)| < \infty$ (weaker than square-integrability)

(2) $\mathcal{F}[af + g] = a\,\mathcal{F}[f] + \mathcal{F}[g]$ linearity

(3) $\mathcal{F}[f(ax)] = \frac{1}{a}\tilde{f}\left(\frac{k}{a}\right)$ scaling

(4) $\mathcal{F}[f]^* = \mathcal{F}^{-1}[f^*]$ relation between $\mathcal{F}$ and $\mathcal{F}^{-1}$

(5) When $f(x) = f(-x)$ even in x, then $\tilde{f}(k) = \tilde{f}(-k)$ even.
When in addition $f(x) \in \mathbb{R} \Longrightarrow \tilde{f}(k) \in \mathbb{R}$
When $f(x) = -f(-x)$ odd in x, then $\tilde{f}(k) = -\tilde{f}(-k)$ odd.
When in addition $f(x) \in \mathbb{R} \Longrightarrow \tilde{f}(k) \in \mathbb{I}$

(6) $\mathcal{F}[f(x+a)] = \mathrm{e}^{\mathrm{i}ka}\;\tilde{f}(k)$ translation

(7) $\mathcal{F}[\mathrm{e}^{ax}\; f(x)] = \tilde{f}(k + \mathrm{i}a)$ exponential multiplication

(8) $\langle f|f\rangle = \int \mathrm{d}x \; |f(x)|^2 = \int \mathrm{d}k \; |\tilde{f}(k)|^2$ Parseval's relation (norm-conserving)

(9) $[f * g](x) := \mathcal{N} \int \mathrm{d}y \; f(x-y)\; g(y) = \mathcal{F}^{-1}[\tilde{f}(k)\; \tilde{g}(k)]$ Faltung/Convolution theorem

(10) $\mathcal{F}[-\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}x}] = k\,\tilde{f}(k)$ derivatives become polynomials