Mathematics: The Bare Essentials

Know these tricks by heart! This is an incomplete list of the bare minimum of "Mental Math", i.e. skills and numbers most commonly used for estimates, back-of-the-envelope calculations, etc. Physicists spend a lot of time solving complicated problems, so we want to always start with an idea of the result. We have ideas when we are hiking, cycling, in the shower, etc., and usually not on our desk. We discuss them with colleagues on the blackboard, and we cannot waste their and our time with looking stuff up. Therefore, we need to be able to do calculations without computers, books or calculators, i.e. in our head or with a piece of scrap paper and a dull pencil.
Here are some items of the *Mathematics tool-chest* which you need for that purpose but which are often overlooked. You should know the following by heart when woken up at night. The list is not exhaustive, not meant to be relevant and may not be useful. Your mileage may vary. It's better to know too much than not enough. You should check these values and understand how they come about, and their limitations. Utter trivialities are not listed, e.g. the solution of $ax^2 + bx + c = 0$.
If you know more things worth remembering, let me know!

- **Multiplication table** up to and including 12×12 (both ways, of course).
- **Squares** up to and including $30 \times 30 = 900$ (both ways, of course).
- **Integer powers** of 2 up to $2^{10} = 1024$, of 3 up to $3^4 = 81$, of 5 up to $5^4 = 625$ (both ways, of course).
- **Recognising Fractions** $\frac{1}{n}$ as decimals, with n from 1 to 11, including $0.167 \approx \frac{1}{6}$, $0.142 \approx \frac{1}{7}$, $0.111 \approx \frac{1}{9}$, $0.091 \approx \frac{1}{11}$.
- **Graphs, differentiation and integration** of elementary functions: x^α (α arbitrary!), e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$ (plus their inverse – or at least a way to get these quickly).
- **Euler's formula** $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$, voted the *most beautiful equation in Mathematics* in 2007.
- **Trigonometric and hyperbolic functions** in terms of the exp function, and of each other, e.g. $\sin x = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x})$, $\tan x = \frac{\sin x}{\cos x}$.
- **De Moivre's Theorem** to relate powers of trigonometric functions with those of multiple angle: $(\cos x + \mathrm{i}\sin x)^n = \cos nx + \mathrm{i}\sin nx$.
- **Gauß' Integral** $\int\limits_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$ for $\mathrm{Re}[\alpha] > 0$.
- **Taylor expansion** of elementary functions up to the *second non-trivial order*, for $|x| \ll 1$:

  NB: *"nth non-trivial order"* is the nth term in an expansion which is not zero or any other pure number, and depends on the expansion parameter.

  $$\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \mathcal{O}(x^3) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

  $$\sin x = x - \frac{x^3}{3! = 6} + \mathcal{O}(x^5)$$

  $$\cos x = 1 - \frac{x^2}{2! = 2} + \frac{x^4}{4! = 24} + \mathcal{O}(x^6)$$

  $$\ln(1+x) = x - \frac{x^2}{2} + \mathcal{O}(x^3) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$$

$(1+x)^\alpha = 1 + \alpha\, x + \dfrac{\alpha(\alpha-1)}{2! = 2}\, x^2 + \mathcal{O}(x^3)$

with the special cases

geometric series ($\alpha = -1$): $1 + x + x^2 + x^3 + \dots = \displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

binomial ($\alpha = 2$): $1 + 2x + x^2 = (1+x)^2$

polynomial ($\alpha \in \mathbb{N}_0^+$): $(1+x)^n = \displaystyle\sum_{m=0}^{n} \binom{n}{m} x^m$, with $\displaystyle\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$

square root ($\alpha = \frac{1}{2}$): $\sqrt{1+x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} + \mathcal{O}(x^3)$

- **Some Essential Numbers** $\approx$ rounded for easier memorising – it often suffices to know the second significant figure, but I put down more if I remembered more. $a \approx b$ (-1%) means that a and b are nearly equal, but that b is $\approx 1\%$ bigger than a. Example: $\pi^2 \approx 10$ (-1%), because $\pi^2 = 9.869\dots$.

  $\pi = 3.14159265\dots \approx \dfrac{355}{113}$ $(-0.8\times 10^{-7}$!!!)(mnemonic: $11, 33, 55$ with fraction in middle)

  $\approx 3\dfrac{1}{7}$ $(+0.05\%) \approx 3$ (-5%)

  $\mathrm{e} = 2.718281\dots \approx 2.7$

  $\ln 2 \approx 0.7$ (-1%)

  $\ln 3 \approx 1.1$ (-0.1%)

  $\ln 10 \approx 2.3$ (-0.1%)

  $\sqrt{2} \approx 1.41 \approx \frac{10}{7}$ (-1%)

  $\sqrt{3} \approx 1.73$,

  Larger numbers:

  $\pi^2 \approx 10$ (-1%)

  $\mathrm{e}^3 \approx 20$ $(+0.4\%)$

  $2^{10} \approx 1000$ $(+2\%)$

  Trigonometric numbers:

  $\sin\dfrac{\pi}{6} = \sin 30^\circ = \cos\dfrac{\pi}{3} = \cos 60^\circ = \dfrac{\sqrt{1}}{2} = \dfrac{1}{2}$

  $\sin\dfrac{\pi}{4} = \sin 45^\circ = \cos\dfrac{\pi}{4} = \cos 45^\circ = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}} \approx 0.7$

  $\sin\dfrac{\pi}{3} = \sin 60^\circ = \cos\dfrac{\pi}{6} = \cos 30^\circ = \dfrac{\sqrt{3}}{2} = \dfrac{1}{2}\sqrt{3} \approx 0.87$

  Curiosities: $\ln 2 \approx 1/\sqrt{2} \approx \mathrm{e} - 2$

Conventions

I define and normalise special functions (like Legendre Polynomials $P_l(z)$, spherical harmonics $Y_{lm}(\vartheta,\varphi)$ etc.) as in Gradshteyn/Ryzhik [GR] = Arfken/Weber [AW].

Einstein's summation convention: sum over all indices which appear exactly once as subscript and exactly once as superscript, except if stated otherwise.

Physics: The Bare Essentials

Formulae are nice, but we also want to explain natural phenomena quantitatively. We Physicists can not only do math but use it to answer real-life problems. As generalists, we are valued in academia and industry for our ability to apply our skills to a broad set of problems in which we are not experts. Getting order-or-magnitude estimates quickly and with little detailed knowledge is an art as much as a skill, known as **Back-of-the-Envelope or Order-of-Magnitude Calculations, Fermi Problems, Guesstimates,. . . .**
This is an incomplete list of the bare minimum of "Mental Physics", i.e. numbers most commonly used. Physicists spend a lot of time solving complicated problems, so we want to always start with an idea of the result. We have ideas when we are hiking, cycling, in the shower, etc., and usually not on our desk. We discuss them with colleagues on the blackboard, and we cannot waste their and our time with looking stuff up. Therefore, we need to be able to do calculations without computers, books or calculators, i.e. in our head or with a piece of scrap paper and a dull pencil.
Here are some items of the *Physics tool-chest* you need for that but which are often overlooked. You should know them by heart when woken up at night. The list is biased towards "Modern Physics", i.e. Quantum Mechanical systems, is not exhaustive, not meant to be relevant and may not be useful. Your mileage may vary. It's better to know too much than not enough. You should check these values and understand how they come about, and their limitations. Trivialities are not listed, e.g. "1 litre of water weights 1 kg".
If you have more things worth remembering, let me know!

Foundations of Guesstimating

One always uses too many significant figures. Example: $147.2/13 = 11.3231\ldots$ for Mathematicians, but really only "≈ 11" for Physicists, since one number, namely 13, is only know to 2 significant figures.
The Geometric Mean as Just-Right (or "Goldilocks") Prescription: First aim a bit too high, then aim a bit too low, and finally take the geometric mean. Example: Estimate the number of graduates in GW Physics. 100 seems too many but 10 seems not enough. Take the geometric mean $\sqrt{10 \times 100} \approx 30$ and you are about right. (Notice the significant figure!)

Everyday Numbers

World population in 2022: 8×10^9

Conversions
1 m s^{-1} = 3.6 km h^{-1}
1 calory cal = energy necessary to warm 1 g of water from 20° to 21° Celsius (spec. heat) $\approx$ 4 Joule J

General Physics
Galilei's gravitational constant on Earth $g \approx 9.81$ m s^{-2}
Speed of sound in Earth's atmosphere, STP 343 m s^{-1}

Astronomy
Earth radius 6 400 km, circumference 40 000 km, mass 6×10^{24} kg
Number of stars visible to the naked eye from Earth, at at clear night, both hemispheres together: 6 000
Sun radius 7×10^8 m$\approx$ 100 earth radii, mass 2×10^{30} kg
1 year $\pi \times 10^7$ s (The π comes about because Earth travels on a circle around the sun. . .)
Distance Earth – moon 400 000 km $\approx$ 1.3 light-seconds
Distance Earth – sun 1 Astronomical Unit AU $\approx$ 150 million km $\approx$ 8 light-minutes
Flux of Solar Radiation at Earth orbit 1 400 W m^{-2}
1 light-year $\approx 9.5 \times 10^{12}$ km (9 trillion km)
Distance sun – next star 3 light-years
Distance sun – centre of the Milky Way $\approx 10^{20}$ m $\approx$ 25 000 light-years
Distance between galaxies $\approx 10^{23}$ m $\approx$ 25 million light-years
Age of the Universe 14 000 000 000 years (14 billion years)

Systems of Units are quite arbitrary: Some calculate in metres, grams and Celsius, while others prefer feet, pounds and Fahrenheit. Still, some systems facilitate conversions and calculations more than others, and some systems have simplicity and natural numbers built in, like “1 litre of water weights 1 kg” in SI. A very efficient system is the **Natural System of Units** in which as many fundamental constants as possible have as simple a value as possible. It is particularly popular in Nuclear and High-Energy Physics.

Set the speed of light and Planck's quantum (more accurately: $\hbar = h/(2\pi)$) to $c = \hbar = 1$. This expresses velocities in units of c, and actions and angular momenta in units of $\hbar$. Then, only one fundamental unit remains, namely either an energy- or a length-scale. Time-scales have the same units as length-scales. When one sets on top of that Boltzmann's constant $k_\mathrm{B} = 1$, energy and temperature have the same units. This is particularly nice because one only has to memorise a handful of really relevant numbers for quick estimates. In this system, “matching units” between different parts of an equation becomes very easy: $E = m$ instead of $E = mc^2$, etc. This example also shows how to “restore” the SI answer from an answer in “natural units”: Throw in enough powers of $c\,[\mathrm{m/s}]$ and $\hbar\,[\mathrm{kg\,m^2/s}]$ till you match the proper SI units. In the example, you have to convert $\mathrm{kg\,m^2/s^2}$ into kg, i.e. add two powers of m/s, which shows you have to multiply with c^2. [As a last step, one could set Newton's gravitational constant $G_N = 1$, which eliminates *any* dimension-ful unit – but that is done only by String Theorists.]
$\approx$: number rounded for easier memorising – it suffices to know the first significant figure.
$\widehat{=}$, $\widehat{\approx}$: correspondences are correct only in the natural system of units.

Quantity	**Value**
speed of light in vacuum	$c := 2.997\,924\,58 \times 10^8$ m s^{-1}(definition!)$\approx 3 \times 10^8$ m s^{-1}
Newton's gravitational constant	$G_N = 6.67 \times 10^{-11}$ N m^2 kg^{-2}
energy electron gains when accelerated by 1 V	1 electron Volt (eV) $\approx 1.6 \times 10^{-19}$ Joule $= 1\dfrac{q}{[\mathrm{C}]}$ J
conversion factor	1 J $\approx 6 \times 10^{18}$ eV
typ. size of a cell	$\approx 5 \times 10^{-6}$ m $= 5\ \mu$m
typ. length-scale in Atomic Physics (size of H-atom, atom-spacing in solid)	1 Ångstrom (Å) $= 10^{-10}$ m $= 0.1$ nm
typ. length-scale in Sub-Atomic Physics (size of proton, neutron)	1 fermi (femtometre, fm) $= 10^{-15}$ m
conversion factor energy – length	$\hbar c \,\widehat{=}\, 1 \approx 200$ MeV fm $\approx 2 \times 10^{-7}$ eV m ≈ 2000 eV Å
$\Longrightarrow$ conversion factor distance – time (nat. units) (time for light to travel a typ. distance-scale)	1 fm $\widehat{\approx} \frac{1}{3} \times 10^{-23}$ s; 1 Å $\widehat{\approx} \frac{1}{3} \times 10^{-18}$ s
conversion factor energy – temperature: $E = k_\mathrm{B} T$	1 eV $\widehat{\approx}$ 11 600 Kelvin
fine-structure constant/measure of el. strength	$\alpha = \left.\dfrac{q^2}{\hbar c}\right\vert_{\text{Gauß}} = \left.\dfrac{q^2}{4\pi\varepsilon_0 \hbar c}\right\vert_{\text{SI}} \approx \dfrac{1}{137}$ (no units!)
electron mass	$m_\mathrm{e} \approx 511$ keV
“classical” electron radius	$r_\mathrm{e} = \left.\dfrac{q^2}{m_\mathrm{e} c^2}\right\vert_{\text{Gauß}} \approx 3$ fm
nucleon (proton, neutron) mass	$m_\mathrm{N} \approx 940$ MeV $\approx 1800\, m_\mathrm{e}$
energy levels of the Hydrogen-atom	$E_n = -\dfrac{\alpha^2 m_e c^2}{2n^2} \approx -\dfrac{13.6\ \mathrm{eV}}{n^2}$, $n = 1, 2, 3, \ldots$
Bohr-radius of the Hydrogen-atom	$a_\mathrm{B} = \dfrac{\hbar}{\alpha m_e c} \approx 0.5$ Å
typ. energy & wave-length of visible light	$E_\text{vis} \approx [2;3]\,\mathrm{eV}$, $\lambda_\text{vis} \approx [400;700]\,\mathrm{nm} \approx [4000;7000]$ Å
typ. chemical binding energy	1.5 eV
Avogadro's number of particles per mol	$N_\mathrm{A} := 6.02214076 \times 10^{23}$(definition!)$\approx 6 \times 10^{23}$ mol^{-1}
volume of 1 mol ideal gas at STP (1 bar, 273 K)	22.4 litres (l) mol^{-1}
nuclear cross-section unit	1 barn b$= 100$ fm$^2 = 10^{-28}$ m^2

A Bibliography for learning guesstimates by doing. Numbers taken from

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