Supplement on Dirac's δ-Distribution

cf. [SG, 2.3.2]

Def. **Function f**: a mapping $f: x \mapsto f(x)$ between sub-sets of two vector spaces.

Def. **Distribution/Generalised Function** ϕ: any object $|\phi\rangle$ for which $T[f] = \langle\phi|f\rangle$ is a continuous, linear functional for any "test function" $|f\rangle$ in a Hilbert space $\mathcal{T}$.
$\mathcal{T}$ is usually the set of infinitely often differentiable functions which are zero outside a finite region. (FAPP)

Def. **Dirac's δ-Distribution**: The distribution/generalised function which obeys

$$\int\limits_{-\infty}^{\infty} \mathrm{d}x\; f(x)\;\delta(x-x_0) = f(x_0) \qquad \text{or equivalently} \qquad \langle\delta(x-x_0)|f\rangle = f(x_0)\;\;.$$

Representation by limit of sequence of functions $\delta_\epsilon(x)$ in Hilbert space $\mathcal{H}$.

Any sequence $\delta_\epsilon(x) = \frac{1}{\epsilon}\,h\left(\frac{x}{\epsilon}\right)$ with $\lim\limits_{\epsilon\to 0}\int \mathrm{d}x\;\delta_\epsilon(x) = 1$, such that the limit $\epsilon \to 0$ is unique and independent of the particular sequence chosen. Examples in Hilbert-space $\mathcal{L}^2$ over the real line:

(1) $\frac{1}{\sqrt{\pi}}\,\frac{1}{\epsilon}\,\mathrm{e}^{-x^2/\epsilon^2} = \int\limits_{-\infty}^{\infty} \mathrm{d}k\;\mathrm{e}^{\mathrm{i}kx-\epsilon^2k^2/4}$ (Gauß'ian with its Fourier representation)

(2) $\frac{1}{\pi}\,\frac{\epsilon}{x^2+\epsilon^2} = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k}{(2\pi)}\;\mathrm{e}^{\mathrm{i}kx-\epsilon|k|}$ (Lorentz'ian with its Fourier representation)

(3) $\begin{cases} \frac{1}{\epsilon} & \text{for } |x| < \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases}$ (rectangle with height $1/\epsilon$, width ϵ, centered at 0)

(4) $\frac{1}{\pi}\,\frac{\sin\frac{x}{\epsilon}}{x} = \int\limits_{-1/\epsilon}^{1/\epsilon} \frac{\mathrm{d}k}{(2\pi)}\;\mathrm{e}^{\mathrm{i}kx}$

(5) $\frac{1}{2}\,\frac{1}{\epsilon}\,\mathrm{e}^{-|x|/\epsilon}$

Properties $\forall f \in \mathcal{H},\; a \in \mathbb{C}$:

(1) $\Longrightarrow \int\limits_{a}^{b} \mathrm{d}x\;\delta(x-x_0) = \begin{cases} 1 & \text{for } x_0 \in]a;b[\\ 0 & \text{otherwise} \end{cases}$

(2) $\Longrightarrow \theta(x) := \int\limits_{-\infty}^{x_0} \delta(x) = \begin{cases} 1 & \text{for } x_0 > 0 \\ \text{undefined} & \text{for } x = 0 \\ 0 & \text{for } x_0 < 0 \end{cases}$ "integral of $\delta(x)$": Heaviside's step distribution

(3) $\delta(ax) \text{“}=\text{”} \frac{1}{|a|}\,\delta(x)$, and in particular $\delta(x)\text{“}=\text{”}\delta(-x)$ even distribution

(4) $\int\limits_{-\infty}^{\infty} \mathrm{d}x\; f(x)\delta(x-x_0) = f(x_0)$ i.e. $f(x)\;\delta(x-x_0)\text{“}=\text{”}f(x_0)\;\delta(x-x_0)$.

(5) $\delta[f(x)]\text{“}=\text{”}\sum\limits_{x_i}\frac{1}{|f'(x_i)|}\,\delta(x-x_i)$, where x_i are all simple zeroes $f(x_i) = 0$ with $f'(x_i) \neq 0$.

(6) $\int\limits_{-\infty}^{\infty} \mathrm{d}x\;\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x_0)\right] f(x) = -\frac{\mathrm{d}}{\mathrm{d}x}f(x)\Big|_{x=x_0}$ i.e. $\left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x-x_0)\right] f(x)\text{“}=\text{”}-\left[\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right]\delta(x-x_0)$
distribution with $\langle\delta'(x-x_0)|f\rangle = -f'(x_0)$ "derivative of $\delta(x)$"

(7) $[\delta(x)]^2$, $\delta(x^2)$, $\delta(0)$, etc. are not defined.

Notes

- "$=$": statement only holds when both sides are integrated over by arbitrary "test-functions" in $\mathcal{T}$.
- Generalisation of the Kronecker-δ to a continuous, ortho-normal basis: $\langle m|n\rangle = \delta_{mn} \to \langle x|y\rangle = \delta(x-y)$
- FAPP: $\delta(x) = 0\;\;\forall\, x \neq 0$, but not quite true, see representation (4).