Additional Practise Sheet: Variations

Completely voluntary.

If you want, we can discuss your solutions in the **Final Question Time of the semster**.
No extra points are awarded – the values are only meant as grade of difficulty here.

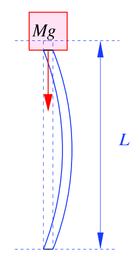
1. A DRUMMER'S PROBLEM (**7P**): The shape of a distorted drum-skin is described by the height $h(x, y)$ at point (x, y) by which the drum-skin is displaced from the flat, un-distorted skin. Thus, such a point has 3-dimensional coordinates $(x, y, h(x, y))$.

   a) (**4P**) Show: The area of the distorted drum is
   $$A[h] = \int\limits_{\text{drum boundary}} \mathrm{d}x\, \mathrm{d}y \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \ .$$

   b) (**1P**) Show: For small distortions and with $\vec{\nabla}$ the gradient w.r.t. to x and y, the area reduces to
   $$A[h \ll 1] = \text{const.} + \frac{1}{2} \int\limits_{\text{drum boundary}} \mathrm{d}x\, \mathrm{d}y \left(\vec{\nabla} h(x, y)\right)^2 \ .$$

   c) (**2P**) Use the result of problem 2 to derive the corresponding Euler-Lagrange equation and verify that A is extremal when $h(x, y)$ obeys the two-dimensional Laplace equation.

2. EULER'S PROBLEM (**8P**): Determine the stability criterion for the buckling of a slender column under a compressive load, as posed in [SG, problem 1.4.a)]:

   The elastic energy per unit length of a bent steel column is $YI/(2R^2)$. Here, R is the radius of curvature due to the bending, Y is Young's modulus of the steel and I is the moment of inertia of the rod's cross section about an axis through its centroid and perpendicular to the plane in which the rod is bent. A mass M is put on top of the massless column of length L, see figure. We assume that the rod is only slightly bent into the yz plane and lies close to the z axis.

   a) (**2P**) Show that when the rod buckles slighly (i.e. deforms with both ends remaining on the z axis) the total energy, including the gravitational potential energy of the loading mass M, is approximately (the prime denotes differentiation with respect to z):
   $$U[y] = \int\limits_0^L \mathrm{d}z \left[\frac{YI}{2}\,(y'')^2 - \frac{Mg}{2}\,(y')^2\right]$$

   b) (**3P**) Solve the resulting Euler-Lagrange equations. Carefully examine the boundary conditions: The column is nailed to the floor. Since the column is very slender, it will tripp over when the weight is not right over its foot. Is the variation of y' at the endpoints fixed?

   Hint: If you think a bit, you do not have to extend the Euler-Lagrange equation to the case that the functional depends on higher derivatives, $y''(x)$ etc. If you use the extension, prove it.

   c) (**3P**) Anrbitrary deformations of the static solution can be written as a Fourier series
   $$y(z) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{L} \ .$$
   Discuss this form and the boundary conditions it needs to fulfill. Show that the solution to the Euler-Lagrange equation is a local minimum of $U[y]$ only when $\frac{MG}{IY} < \left(\frac{n\pi}{L}\right)^2$. Interpret the implications for the stability of the column against buckling and collapse.

Please turn over.

3. A SURPRISING PERSPECTIVE TO A FAMILIAR EQUATION (**3P**) We now explore a complex scalar field $\Phi(t,\vec{r})$, i.e. $\Phi = \Phi_R + \mathrm{i}\Phi_I$ has both a real and imaginary part, and $\Phi^\dagger = \Phi_R - \mathrm{i}\Phi_I$ is its complex conjugate. The field and its complex conjugate are best treated as independent variables. Derive the Euler-Lagrange equations for Φ and $\Phi^\dagger$ from the Lagrange density

$$\mathcal{L}_\Phi = \frac{\mathrm{i}}{2}\,\Phi^\dagger\dot{\Phi} - \frac{1}{2M}\left(\vec{\nabla}\Phi^\dagger\right)\cdot\left(\vec{\nabla}\Phi\right) + V\Phi^\dagger\Phi \ .$$

Interpret and discuss the implications of your findings.

4. COMPRESSION WAVES (**1P**) in an isotropic, homogeneous and elastic medium are described by

$$\mathcal{L} = \frac{1}{2}\,\rho_0\,[\dot{\vec{\Phi}}^2 - c_s^2(\vec{\nabla}\cdot\vec{\Phi})^2] \ .$$

The field $\vec{\Phi}(t,\vec{r})$ is a three-component vector (one component for each direction in which the medium can be compressed). The two constants are: the density coefficient ρ_0; and the speed of sound c_s. Derive the Euler-Lagrange equations of this problem.

5. STANDARD TOY-MODEL OF HIGH-ENERGY PHYSICS (**1P**): We now explore the complex scalar field $\Phi(x^\mu)$ of a spin-zero particle with mass m, i.e. $\Phi = \Phi_R + \mathrm{i}\Phi_I$ has both a real and imaginary part, and $\Phi^\dagger = \Phi_R - \mathrm{i}\Phi_I$ is its complex conjugate. The Lagrangean is

$$\mathcal{L}_\Phi = \dot{\Phi}^\dagger\dot{\Phi} - \left(\vec{\nabla}\Phi^\dagger\right)\cdot\left(\vec{\nabla}\Phi\right) - m^2\Phi^\dagger\Phi \ .$$

Derive the Euler-Lagrange equations for Φ and $\Phi^\dagger$. The field and its complex conjugate are best treated as independent variables.

6. MERGING VARIATIONAL AND TENSOR CALCULUS (**6P**): Consider the functional

$$J[f_1,f_2] = \int \mathrm{d}x\,\left[f_1(x;y(x)) + f_2(x;y(x))\,\frac{\partial y}{\partial x}\right] \ ,$$

in which neither f_1 nor f_2 depend on derivatives of $y(x)$, and both functions vanish at infinity.

a) (**2P**) Show that $\dfrac{\partial f_1}{\partial y} = \dfrac{\partial f_2}{\partial x}$.

b) (**4P**) Prove that $f_1 = \dfrac{\partial}{\partial x}g(x,y)$ and $f_2 = \dfrac{\partial}{\partial y}g(x,y)$, where $g(x,y)$ is *one common function.*