Additional Practise Sheet: Tensor Analysis

Completely voluntary.

If you want, we can discuss your solutions in the **Final Question Time of the semster**.
No extra points are awarded – the values are only meant as grade of difficulty here.

1. (**2P**) Let $\vec{U}$, $\vec{V}$ arbitrary vector fields without source and curl. Which sources and curls has $\vec{U} \times \vec{V}$?

2. Integration Theorems (**6P**): The following problems are independent of each other.

   a) (**2P**) Calculate the surface integral over a cube, sphere and torus, each with volume V, in the field $\vec{A}(\vec{r}) = \vec{r}$. Sketch $\vec{A}$.

   Hint: Gauss' theorem allows you to avoid to actually calculate the surface integral.

   b) (**2P**) Determine all exponents α for which the spherically symmetric vector field $\vec{A}(\vec{r}) = |\vec{r}|^\alpha \, \vec{e}_r$ is source-free everywhere.

   c) Integral Theorems: Verify Stokes' theorem by explicit calculation in the field $\vec{A} = (xy, 4yz, 3x^4 z)$. Pick as surface of integration a square with sides L, centred at the origin in the xy-plane.

   d) (**2P**) Prove Green's theorems for arbitrary scalar fields Φ, Ψ (corollaries of Gauss' theorem):

$$\int \mathrm{d}^3 r \, \Big((\vec{\nabla}\Phi) \cdot (\vec{\nabla}\Psi) + \Phi \Delta \Psi \Big) = \oint \mathrm{d}^2 \vec{s} \cdot (\vec{\nabla}\Psi) \, \Phi$$

$$\int \mathrm{d}^3 r \, (\Phi \Delta \Psi - \Psi \Delta \Phi) = \oint \mathrm{d}^2 \vec{s} \cdot (\Phi \vec{\nabla} \Psi - \Psi \vec{\nabla} \Phi)$$

3. Vector Potential (**4P**): Given the vector potential in spherical coordinates:

$$\vec{A}(r, \theta, \phi) = g_{\mathrm{M}} \, \frac{1 - \cos\theta}{r \sin\theta} \, \vec{e}_\phi$$

   Determine its magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$, and show that $\vec{A}$ has no sources anywhere.

4. Vector Potential (**4P**): Given the vector potential in cylindrical coordinates:

$$\vec{A}(\rho, \phi, z) = -I \, \vec{e}_z \, \ln \frac{\rho}{\rho_0} \quad \text{with } I, \, \rho_0 \text{ constants.}$$

   Find its magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$, and carefully show that $\vec{A}$ has no sources anywhere.

5. Covariant ϵ-Tensor (**4P**):

   a) (**2P**) Show $\tilde{\epsilon}^{ijk} \sqrt{\det \tilde{g}} = \epsilon^{ijk} \sqrt{\det g}$ under a change of basis for the *contravariant* components in 3 dimensions.

   b) (**2P**) Determine now the transformation of the *covariant components* ϵ_{ijk}. You may use that $\epsilon^{ijk} \epsilon_{ijk} = 6$, after you have proven it.

   Hint: Sub-section a) was not in vain.

Please turn over.

6. Proper and Pseudo-Tensors (**1P**) Convince yourself that the following matrix describes a rotation about the z-axis by an angle ϕ and a reflection at the xy-plane:

$$\vec{a}\,' = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & -1 \end{pmatrix} \vec{a}$$

Compare the transformation property of the two vectors $\vec{a}$, $\vec{b}$ with the one of $\vec{a}\times\vec{b}$. Why is $\vec{a}\times\vec{b}$ also called a "pseudo- or axial-vector" and ϵ^{ijk} a "pseudo-tensor"?

7. Volume Element as Tensor (**5P**): Consider two sets of coordinates $(\tilde{v}^1,\dots,\tilde{v}^n)$ and $(v^1,\dots,v^n)$ of an n-dimensional vector space.

   a) (**P**) Show that the volume element $\mathrm{d}V = \mathrm{d}\tilde{v}^1\,\mathrm{d}\tilde{v}^2\cdots\mathrm{d}\tilde{v}^n$ transforms under transformations from one set of coordinates $(\tilde{v}^1,\dots,\tilde{v}^n)$ of an n-dimensional vector space to another set $(v^1,\dots,v^n)$ as

8. Parabolic-Cylindrical Coordinates (**9P**) $(u,\ v,\ z)$ are defined in three-dimensional Euclidean space as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u\,v \\ \frac{1}{2}(u^2-v^2) \\ z \end{pmatrix} \quad .$$

   a) (**2P**) Derive the transformation matrix for the change of basis.
   b) (**3P**) Calculate the metric tensor to verify that these are orthogonal curvilinear coordinates. Show that the scale factors are $h_{(u)} = h_{(v)} = \sqrt{u^2+v^2}$, $h_{(z)} = 1$.
   c) (**4P**) Sketch the "coordinate grid". If they exist, identify coordinate singularities.
   d) (**2P**) Calculate the volume of the parameter region $u\in[0;1]$, $v\in[0;1]$, $z\in[0;1]$.