Additional Practise Sheet: Complex Analysis

Completely voluntary.

If you want, we can discuss your solutions in the **Final Question Time of the semster**.
No extra points are awarded – the values are only meant as grade of difficulty here.

1. Complex Functions (**7P**): Let $z = x + \mathrm{i}y$ be complex, with x (y) its real (imaginary) part. Given the complex functions:

$$\frac{1+z^2}{1-z^2} \quad , \quad \cos z^2 \quad , \quad \sqrt{z}$$

Decompose each function into its real and imaginary part, $u + \mathrm{i}v$. Determine whether it obeys the Cauchy-Riemann condition $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$ everywhere, or at nearly all points (and if so, state at which it does not). Is the function well-defined/analytic everywhere in the complex plane? Determine the nature of each of its singularities and their residues.

2. Complex Mapping (**2P**): This problem is really beyond the mainstream of the lecture. A complex function maps complex numbers into complex numbers. Since complex numbers can be interpreted as coordinates in a 2-dimensional plane, it is natural to study how a geometric figure in 2 dimensions is mapped from the complex z-plane into the complex w-plane by a complex function w.

Into which figure on the w-plane is the rectangle $\{z = x + \mathrm{i}y : 0 \leq x \leq 1,\ 0 \leq y \leq \pi\}$ mapped by the complex exponential $w = \mathrm{e}^z$?

Point of information: This technique is again quite useful in two-dimensional Electrostatics. Say you have solved a problem without charges on that rectangle with given boundary conditions, getting $\Phi(x, y)$ as the real part of a complex function. Then you map this region via $z \to w(z)$ into a new region, and Φ to $\Phi(u(x, y), v(x, y))$. If the mapping w is analytic, the new Φ will again obey the Laplace equation, because the analytic function with real part $\Phi(x, y)$ is again analytic, i.e. its real and imaginary parts have to be harmonic. The "only" problem is to find that function w which maps your simple problem to the complicated problem you actually want to solve.

3. Complex Integration (**4P**): You can check your final results with an algebraic manipulation programme. If you use contour integration with neglecting an arc at infinity, ***discuss in detail that your function vanishes indeed on that arc.***

   a) (**2P**) Turn the following integral into a contour integral around the unit circle and evaluate:

$$\int_0^{2\pi} \frac{\mathrm{d}\vartheta}{a + \cos\vartheta} \quad , \quad a > 1$$

   b) (**2P**) Evaluate the integral $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{6x^4 + 5x^2 + 1}$

   c) (**2P**) Calculate $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{\mathrm{e}^{-\mathrm{i}ax}}{x^4 + 5x^2 + 4}$ for $a > 0$ and for $a < 0$.

Please turn over.

4. Heaviside's Step-Function (**4P**): Show that the step-function has the integral representation

$$\theta(x) := \left\{ \begin{array}{l} 1 \text{ for } x > 0 \\ 0 \text{ for } x < 0 \end{array} \right\} = -\frac{1}{2\pi\mathrm{i}} \lim_{\epsilon \searrow 0} \int_{-\infty}^{\infty} \mathrm{d}\omega\ \frac{\mathrm{e}^{-\mathrm{i}\omega x}}{\omega + \mathrm{i}\epsilon}, \quad \epsilon > 0$$

and that its derivative is therefore Dirac's δ-distribution with the integral representation

$$\delta(x) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{-\mathrm{i}\omega x} \quad .$$

Show also the following useful generalisation (a arbitrary):

$$\mp \mathrm{i}\, \mathrm{e}^{\mathrm{i}ax}\, \theta(\pm x) = \lim_{\epsilon \searrow 0} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{(2\pi)} \frac{\mathrm{e}^{-\mathrm{i}\omega x}}{\omega - (a \mp \mathrm{i}\epsilon)}, \quad \epsilon > 0$$

5. Analytic Continuation (**2P**):

   a) (**1P**) Show $\sin^2 z + \cos^2 z = 1$ for $z \in \mathbb{C}$ by analytic continuation from $\sin^2 x + \cos^2 x = 1\ \forall x \in \mathbb{R}$.

   b) (**3P**) Let $f(z)$ be analytic at $z = 0$ and $f(\frac{1}{n}) = \frac{1}{n^2}$ for $n = 1, 2, \ldots$. What is $f(z)$?

6. Disperson Relation (**3P**): Apply the Kramers-Kronig relation to a medium which shows no absorption except at a frequency ω_0. That means the imaginary part of the frequency-dependent dielectric susceptibility is found to be strongly peaked around ω_0: $\mathrm{Im}[\chi(\omega)] = \alpha\delta(\omega - \omega_0)$. Determine the real part of $\chi(\omega)$ from your observation. You may assume that $\chi(\omega)$ has no poles on the real axis and is causal.