Problem Sheet 14

Special due date: 01 May 2017 16:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. DENSITY OF THE IONOSPHERE (**2P**): The ionosphere reflects radio-waves with a wavelength of more than 30 m. Determine its electron-density, assuming that it is a plasma of electrons and ions.

2. SPHERICAL SHELLS (**9P**): Two concentric, perfectly conducting spheres with radii a and $b > a$ are charged such that the inner sphere is at potential $\Phi_0 > 0$ relative to the outer one. The space between them is completely filled by a neutral layer with a radially dependent relative dielectric constant $\varepsilon(r)$.

   a) (**2P**) Determine the radial dependence of $\varepsilon(r)$ such that the energy density inside the layer is independent of r. Under this condition, the dielectric has no internal stresses.

      Fail-Safe Point: If you get no answer, assume $\varepsilon(r) = A\, r^{-\alpha}$ with $\alpha > 0$, A some constants.

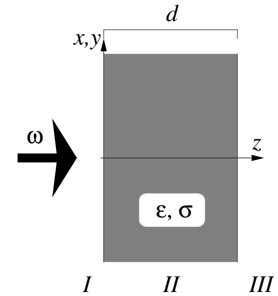
   b) (**3P**) Determine the charge densities on each of the two shells which are needed to obtain the potential difference Φ_0. Show that the resulting electric field in the layer has the form $\vec{E}(\vec{r}) = C\, r^{\alpha/2}\, \vec{e}_r$ and determine the constants C and α.

   c) (**4P**) Let's assume now that the dielectric function of the medium is microscopically described by coupling harmonic oscillators inside the medium to the microscopic electric field (Lorentz-Drude model). Argue whether and under which conditions you can find such a dielectric in which the bound surface charge densities on its two surfaces are equal in magnitude (but of course opposite).

3. SIGNAL DELAY FROM PULSARS (**4P**): A pulsar is a star which regularly emits a whole spectrum of frequencies in one short pulse. On Earth, the arrival times of these bursts are delayed for low frequencies: The signal arrives as a "whistle" whose frequency changes with time. This is attributed to dispersion in the interstellar medium which is assumed to consist of fully ionised hydrogen. Given the density of the interstellar plasma, determine the signal delay for a wave which propagates with frequency ω through the plasma, compared to a freely propagating light-ray. Show that we can determine the distance D to the pulsar if we know the electron density in the interstellar medium and measure the rate of change (frequency versus time) of the signal:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{c}{D}\,\frac{\omega^3}{\omega_p^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{3}{2}}$$

Please turn over.



4. TRANSMISSION AND REFLECTION OF ELECTROMAGNETIC WAVES ON METAL (**15P**): A plane, mono-chromatic wave with frequency ω is incident perpendicular on a homogeneous layer (II) with thickness d, dielectric constant $\varepsilon > 1$ and magnetic permeability $\mu = 1$, surrounded by vacuum (regions I and III). The layer is composed of a metal with conductivity σ, i.e. Ohm's law $\vec{j} = \sigma\,\vec{E}$ holds in (II).

   Hint: This is another problem in which algebraic manipulation programmes can save a lot of work.

   a) (**2P**) Show that the wave-equation inside the medium is

   $$\left(\nabla^2 - \frac{\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{4\pi\sigma}{c^2}\frac{\partial\vec{E}}{\partial t}$$

   and that the index of refraction is determined by $n^2 = \varepsilon + \frac{4\pi \mathrm{i}\,\sigma}{\omega}$.

   b) (**2P**) Determine those solutions of the wave-equations for $\vec{E}$ (and $\vec{B}$) in each of the regions I, II, and III which are linearly polarised.

   c) (**3P**) Find the boundary conditions for $\vec{E}$ (and $\vec{B}$): They can be written as 4 linear equations for the non-vanishing amplitudes of $\vec{E}$.

   d) (**4P**) Compute the reflectivity R (ratio of intensities between reflected and incident waves in region I), the transmission coefficient T (ratio of intensities between wave transmitted into III and incident wave in I) and the constant X:

   $$R = X\left|\frac{1-n^2}{N}\sin nkd\right|^2 \quad , \quad T = X\left|\frac{n}{N}\right|^2 \quad \text{with } N := (1+n)^2\mathrm{e}^{-\mathrm{i}nkd} - (1-n)^2\mathrm{e}^{\mathrm{i}nkd}$$

   Discuss without an explicit calculation whether $R + T = 1$ for all σ.

   e) (**2P**) Discuss and sketch R as function of the thickness d of the layer for the case $\sigma = 0$, $\varepsilon > 0$. Pay particular attention to $nkd \in \pi\mathbb{Z}$ and $nkd \in \pi(\mathbb{Z} + \frac{1}{2})$.

   f) (**2P**) Calculate the ratio T/R for $\sigma/\omega \gg \varepsilon \approx 1$. Discuss the limit in which the layer thickness is large (against what?).