Problem Sheet 13

Due date: 26 April 2017 **16:00**

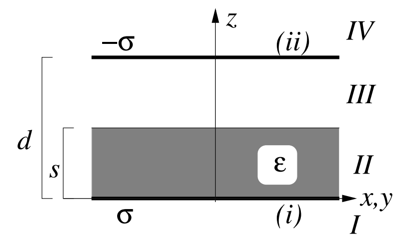
For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.

News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. PARTIALLY FILLED PLANAR CAPACITOR (**9P**): Two infinitely long, infinitesimally thin, perfectly conducting plates carry opposite charge densities and are mounted parallel to each other at distance d. The lower plate is identical with the xy-plane and carries a surface charge density $\sigma_{(i)} = \sigma > 0$. As shown in the figure, it is coated such that the space II between the plates is filled by a dielectric $\varepsilon > 1$ with thickness s. The other part of the capacitor, III, is evacuated, as are the exterior regions I and IV below and above, respectively.



Hint: You may choose to switch the order of e.g. b) and c).

a) (**3P**) Determine and sketch the electro-static potential Φ, with Φ continuous everywhere.

b) (**2P**) Determine the surface charge density of the polarisation charges at the surface II/III.

c) (**1P**) Show that the field energy per unit surface area of the plates is (determine $X > 0$!):

$$u = X\,\sigma^2\,[d + s\,(\frac{1}{\varepsilon} - 1)]\ .$$

d) (**3P**) Determine the numerical value of the electro-static pressure (force per surface unit) on the boundary II/III for a thin film of distilled water, $\varepsilon = 80$, $\sigma = 10^{-3}$ C/m^2. Assuming that the medium ε is a perfect fluid, would it flow in or out of the capacitor? Under which condition can you fill the capacitor completely with the medium when the capacitor lies on your lab table on Earth?

2. CRACKPOT-STOPPER (**4P**): As an alternative to the Lorentz-Drude model, a would-be-Physicist predicts the dielectric functions of two media with some positive constants ω_p, χ_0, Γ_1, Γ_2 as:

$$\varepsilon_1(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \mathrm{i}\omega(\Gamma_1 - \Gamma_2) + \Gamma_1\Gamma_2}\ ,\quad \varepsilon_2(\omega) = 1 + 4\pi\chi_0\ ,$$

Tell him (with reasons) for each case which fundamental principles are violated by his theory.

3. KRAMERS-KRONIG FOR METALS (**5P**): Our derivation of the Kramers-Kronig relation assumed that $\varepsilon(\omega)$ has no poles on the real axis. As we saw, this is incorrect for metals, where $\varepsilon(\omega \to 0) = \frac{4\pi\mathrm{i}\sigma_0}{\omega} +$ finite, with σ_0 the conductivity. Re-derive the Kramers-Kronig relation for metals, ending with the result already shown in the lecture ("P" denotes the principal value):

$$\mathrm{Re}[\varepsilon(\omega) - 1] = 4\pi^2\sigma_0\,\delta(\omega) + \frac{2}{\pi}P\int_0^\infty \mathrm{d}\omega'\,\frac{\omega'\,\mathrm{Im}[\varepsilon(\omega')]}{\omega'^2 - \omega^2}$$

$$\mathrm{Im}[\varepsilon(\omega)] = \frac{4\pi\sigma_0}{\omega} - \frac{2}{\pi}P\int_0^\infty \mathrm{d}\omega'\,\frac{\omega\,\mathrm{Re}[\varepsilon(\omega') - 1]}{\omega'^2 - \omega^2}$$

Hint: Pay close attention to the residue of $\frac{\varepsilon(\omega')-1}{\omega'-\omega}$ as $\omega \to 0$ and recall $\lim\limits_{x\to 0} \frac{x}{x^2-a^2} = i\pi\delta(a)$.

Please turn over.

4. Transversality in An-Isotropic Media (**5P**) Consider a homogeneous, neutral, non-conducting medium with magnetic susceptibility $\mu(\omega) = 1$, but *an-isotropic and real* dielectric function, i.e. $\varepsilon(\omega) \neq 1$ is a real tensor. Show for a plane wave with frequency ω and wave-vector $\vec{k}$: (i) $\vec{H}$ is orthogonal to $\vec{D}$ and $\vec{E}$, (ii) $\vec{D}$ and $\vec{H}$ are transverse, but (iii) $\vec{E}$ is not. Discuss under which condition(s) $\vec{E}$ is transversal.

5. Optical Tweezers (**7P**) are a standard tool in Bio-Physics to manipulate microscopic dielectric media of electric susceptibility χ_{el}, like individual cells and their components. For a demonstration, ask a graduate student in Dr. Reeves' group.

   a) (**2P**) As a warm-up, show that the energy of an induced dipole in a static electric field $\vec{E}$ is
   $$H_{\mathrm{el}} = -\frac{1}{2}\alpha\,\vec{E}^2 \ ,$$
   with α the polarisability.

   Hint: The formula $\mathcal{H}_{\mathrm{el}} = \vec{d}\cdot\vec{E}$ is valid only for the energy of a *permanent* dipole $\vec{d}$ in an external field. If the dipole is induced, $\vec{d}[\vec{E}]$, consider the infinitesimal change in energy which comes from changing the electric field.

   b) (**1P**) Derive now the total energy for a dielectric sphere $\varepsilon > 1$ with radius R:
   $$H_{\mathrm{el}} = -\frac{1}{2}\,\frac{\varepsilon-1}{\varepsilon+2}\,\vec{E}^2\,R^3 \ .$$
   Hint: You derived in the last HW that $\vec{P} = \frac{3}{4\pi}\,\frac{\varepsilon-1}{\varepsilon+2}\,\vec{E}$ for the dielectric sphere.

   c) (**2P**) Show that the medium is attracted to regions in which the magnitude of the electric field, $\left|\vec{E}\right|$, is large, irrespective of its direction.

   This is the principle used to drag dielectrics by laser beams. Lasers usually have a parabolic intensity profile $I = I_0\left(1 - r^2/R^2\right)\theta[R-r]$, where I_0 is the intensity at the centre, R the beam radius, r the (transverse) distance from the beam centre and $\theta[x]$ Heaviside's step-function.

   d) (**2P**) Show finally that a sphere of mass M describes close to the centre of the beam oscillations around the beam centre with a frequency
   $$\omega^2_{\text{tweezer}} \propto \frac{I_0 R}{Mc}$$
   and determine the constant of proportionality. For water (i.e. cells), why do we have to use $\varepsilon \approx 1.8$ instead of $\varepsilon(\omega = 0) = 80$? This gives a frequency $f_{\mathrm{res}} \approx 4$ kHz for $R = 1\,\mu$m, $I_0 = 1$ mW, which a tabletop laser can provide.