Problem Sheet 12

Due date: 19 April 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
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News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. Dielectric Constant of the Hydrogen Atom (**7P**): We return to the hydrogen atom already discussed in Problem Sheet 2, number 6. As you derived previously, typical field strengths are of the order of 10^9 V/m inside the atom. At first order in perturbation theory, an external electric field $\left|\vec{E}_{\text{ext}}\right| \lesssim 10^6$ V/m shifts therefore only the electron's charge density without deformation by a vector $\vec{r}_0$ against the proton.

   a) (**3P**) The electric field of only the electron in the atom with the external field switched off has at the beginning of the semester been derived as

   $$\vec{E}_{\text{electron}}(\vec{r}) = -\frac{q}{r^2}\,\vec{e}_r\left[1-\mathrm{e}^{-2r/a}\left(1+\frac{2r}{a}+\frac{2r^2}{a^2}\right)\right] \quad ,$$

   where a is the Bohr radius. Show that this shift of the electron cloud leads to a restoring force on the proton:

   $$\vec{F}_{\text{restore}} = -\frac{4q}{3a^3}\,\vec{d}_{\text{ind}} \quad \text{for } r_0 \ll a$$

   b) (**1P**) The equilibrium condition between the force exerted by the external field and $\vec{F}_{\text{restore}}$ gives $\vec{d}_{\text{ind}}$ as function of $\vec{E}_{\text{ext}}$, and hence the (classical) polarisability of the hydrogen atom. Recall that the induced dipole moment in the field $\vec{E}_{\text{ext}}$ is $\vec{d}_{\text{ind}} = Q_+\vec{R}_+ + Q_-\vec{R}_- = -q\vec{r}_0$ in our case.

   c) (**3P**) Determine finally the dielectric constant for a gas of N hydrogen *molecules*, homogeneously distributed in a volume V. Compare to the estimate in the lecture (from $\alpha_{\text{est}}(0) \approx a^3$), the QM result (from $\alpha_{\text{QM}}(0) = \frac{9}{2}\,a^3$), and explain the discrepancy to the experimental value, $\varepsilon_{H_2}(\omega = 0) = 1.00026$ at room temperature and normal pressure.

2. Microscopic Response Models (**6P**): Response functions $\chi_{\text{el}}(t)$ are better understood in time than in frequency space. For the following responses, determine the dielectric function $\varepsilon(\omega)$ in frequency space and plot its real and imaginary part. Plot its singularities in the complex ω-plane.

   Hint: $\theta(t)$ is Heaviside's step-function and χ_0, $\Gamma > 0$, ω_0, ω_p are real. Recall $\lim\limits_{\eta\searrow 0}\dfrac{\eta}{\eta^2+x^2} = \pi\delta(x)$.

   a) (**1P**) Damped medium: $\chi_{\text{el}}(t) = \chi_0\,\theta(t)\,\mathrm{e}^{-\Gamma t}$, i.e. response relaxes slowly back to original state.

   b) (**2P**) Resonant medium: $\chi_{\text{el}}(t) = \chi_0\,\theta(t)\,\sin\omega_0 t$, i.e. oscillation after applied impulse.

   c) (**3P**) Derive, sketch and interpret now the temporal response of the Lorentz-Drude model:

   $$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2-\omega^2-\mathrm{i}\omega\Gamma}$$

Please turn over.

3. Spheres in External Fields (**13P**): A neutral, homogeneous sphere with radius R and dielectric constant $\varepsilon > 1$ is set into an external, originally homogeneous electric field $\vec{E}_0$.

   a) (**1P**) Show that the boundary conditions on the surface of the sphere can be written as von-Neumann boundary conditions between the potentials Φ_i inside and Φ_o outside:

   $$-\frac{1}{R}\frac{\partial \Phi_i}{\partial \vartheta}\bigg|_{r=R} = -\frac{1}{R}\frac{\partial \Phi_o}{\partial \vartheta}\bigg|_{r=R} \quad , \quad -\varepsilon\frac{\partial \Phi_i}{\partial r}\bigg|_{r=R} = -\frac{\partial \Phi_o}{\partial r}\bigg|_{r=R}$$

   b) (**3P**) Determine the potential, electric field and dielectric displacement inside and outside.

   Hint: Symmetries! An ansatz $\Phi(\vec{r}) = \sum\limits_l [A_l r^l + B_l r^{-(l+1)}]\ P_l(\cos\vartheta)$ and a good book might be helpful. Motivation?

   c) (**1P**) Discuss the fields inside and outside. Are there induced fields/screened fields? Sketch!

   d) (**2P**) Show that the induced polarisation and total dipole moment of the sphere in vacuum is $\vec{P} = \dfrac{3}{4\pi}\dfrac{\varepsilon - 1}{\varepsilon + 2}\vec{E}$ for the dielectric sphere. In which direction is it pointing? Derive and sketch the induced surface charge density.

   e) (**2P**) By how much is the total energy changed when the dielectric is brought in?

   f) (**2P**) Determine now the fields, polarisation and surface charge density in the complementary situation of a spherical hole of vacuum inside a dielectric medium ε', with external field. In which direction does the polarisation density point now?

   g) (**2P**) What changes when you consider instead a sphere of magnetic permeability μ in an external, originally homogeneous magnetic field $\vec{B}_0$? Discuss in particular the magnetic field and induction, the magnetisation density and the direction of the magnetisation with respect to $\vec{B}_0$ for dia- and para-magnetic media. Calculate and sketch the resulting surface current density.

   Hints for f) and g): Compare field equations and boundary conditions with the ones in a).
   Note: Yes, "the answer" is in Jackson. But can you (re-)produce and understand it such that you can present it as your own?

4. Scaling of Electrostatic Forces (**4P**): Two solid, homogeneous, dielectric spheres of radius R are separated by a distance $a \gg R$. The first carries a charge Q, while the second one is neutral.

   Increase now the distance between the spheres by a factor of 2. By how much do you have to change the charge Q on the first sphere such that the force between the spheres remains the same?

   Hint: If your answer is 2 or 4, you did not properly take into account the influence of the charged sphere on the uncharged one.