Problem Sheet 11

Due date: 12 April 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. COMPTON SCATTERING OFF INDUCED ELECTRIC AND MAGNETIC DIPOLES, PART II (**10P**): In the lecture, we considered the case that an incoming electro-magnetic wave induces an electric dipole, whose strength is proportional to the strength of the incoming elecric field, and which oscillates in the same direction as the incoming field. Now, we extend the same set-up to magnetic responses: A monochromatic, plane electro-magnetic wave with electric and magnetic fields $\vec{E}_{\text{in}}(\omega) = \vec{E}_0\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}}$ and $\vec{B}_{\text{in}}(\omega) = \vec{\mathrm{e}}_k \times \vec{E}_0\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}}$ induces a frequency-dependent electric and magnetic dipole moment in a charge distribution, i.e.

$$\vec{d}_{\text{ind}}(\omega) = \alpha(\omega)\ \vec{E}_{\text{in}}(\omega)\ \ ,\ \ \vec{m}_{\text{ind}}(\omega) = \beta(\omega)\ \vec{B}_{\text{in}}(\omega)\ \ .$$

   Electric and magnetic polarisabilities α and β are for us here just real constants of proportionality.

   a) (**4P**) Parallel to the steps in the lecture, derive now the differential cross-section of linearly polarised light off this system. Your answer depends on the angle θ_{scatt} between incoming and outgoing wave, and the angle ϕ_d between the scattering plane and the polarisation of the incoming wave, see lecture, via:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\omega^4}{c^4}\left[\alpha^2\left(\cos^2\theta_{\text{scatt}}\cos^2\phi_d + \sin^2\phi_d\right) + \beta^2\left(\cos^2\theta_{\text{scatt}}\sin^2\phi_d + \cos^2\phi_d\right) + 2\alpha\beta\cos\theta_{\text{scatt}}\right]$$

   Hint: Follow the steps in the lecture!

   b) (**4P**) Find the differential and total cross-section for an un-polarised incoming wave and determine the polarisation asymmetry.

   c) (**2P**) With the aid of simple simple physical arguments, predict whether the polarisation asymmetry for $\theta_{\text{scatt}} \in \{0; \frac{\pi}{2}; \pi\}$ is zero or not.

2. RADIATION FROM A DAMPED OSCILLATOR (**7P**): A harmonically bound charge q oscillates at a frequency ω_0, weakly damped at a rate α ($\alpha \ll \omega_0$, e.g. because of energy loss by radiation). The charge is initially at rest, and then is excited at $t = 0$ such that its velocity is

$$\vec{v}(t) = \vec{v}_0\ \cos\omega_0 t\ \mathrm{e}^{-\alpha t}\quad \text{for } t > 0\ \ ,$$

   with $\vec{v}_0$ constant, $|\vec{v}_0| \ll c$. Recall that we motivated in the lecture for observers far from the source $\vec{s}(t)$ of radiation (e.g. Jackson, Sect. 14.5):

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\omega\ \mathrm{d}\Omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int \mathrm{d}t\ \left[\vec{\mathrm{e}}_R \times \left(\vec{\mathrm{e}}_R \times \vec{\beta}\right)\right]\ \mathrm{e}^{\mathrm{i}\omega\left[t - \frac{\vec{\mathrm{e}}_R \circ \vec{s}}{c}\right]}\right|^2$$

   a) (**5P**) Show that in these approximations, the spectral and angular distribution of the radiation produced as the oscillations die out is near the resonance, i.e. for $(\omega - \omega_0) \ll \omega_0$:

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\omega\ \mathrm{d}\Omega} = \Gamma\ \frac{\omega^2\sin^2\theta}{(\omega-\omega_0)^2 + \alpha^2}\ \ ,$$

   with θ the angle between the directions of observation and of oscillation. Determine $\Gamma(q, v_0)$.

   b) (**2P**) Discuss and sketch the spectrum of the total radiation.

Please turn over.

3. WAVEGUIDES (**13P**): The electric and magnetic fields of a "free wave" which propagates only in a finite volume are *not always* perpendicular to its propagation directions. As an example, consider an infinitely long waveguide, i.e. the evacuated interior of an infinitely long tube whose walls are made of a perfectly conducting material. It has a quadratic profile with dimensions $z \in]-\infty;\infty[$, $x, y \in [0; L]$. We will show that waves which travel along the positive z-direction also have a *longitudinal* component.

   a) (**2P**) Show that the components of the electric field $\vec{E}$ tangential to the surface must vanish at the surface. Show the same for the components of the magnetic field $\vec{B}$ normal to the surface.

   b) (**5P**) *Confirm* that the following electric and magnetic fields solve all of Maxwell's equations in vacuum and obey *all* boundary conditions:

$$\vec{E}(t,\vec{r}) = \mathrm{e}^{-\mathrm{i}(\omega t - kz)} \begin{pmatrix} C_x \cos\dfrac{n_x\pi x}{L} \sin\dfrac{n_y\pi y}{L} \\ C_y \sin\dfrac{n_x\pi x}{L} \cos\dfrac{n_y\pi y}{L} \\ C_z \sin\dfrac{n_x\pi x}{L} \sin\dfrac{n_y\pi y}{L} \end{pmatrix}$$

$$\vec{B}(t,\vec{r}) = \frac{c}{\omega}\,\mathrm{e}^{-\mathrm{i}(\omega t - kz)} \begin{pmatrix} \left(-\mathrm{i}C_z\dfrac{\pi n_y}{L} - C_y k\right) \sin\dfrac{n_x\pi x}{L} \cos\dfrac{n_y\pi y}{L} \\ \left(\mathrm{i}C_z\dfrac{\pi n_x}{L} + C_x k\right) \cos\dfrac{n_x\pi x}{L} \sin\dfrac{n_y\pi y}{L} \\ \dfrac{\mathrm{i}\pi}{L}\left(C_x n_y - C_y n_x\right) \cos\dfrac{n_x\pi x}{L} \cos\dfrac{n_y\pi y}{L} \end{pmatrix}$$

   Here, C_x, C_y, C_z are constants, n_x, n_y non-negative integers, k real and positive if the wave travels in the positive z-direction, and

$$\omega = c\sqrt{k^2 + \left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2}\ ,\quad C_x\frac{\pi n_x}{L} + C_y\frac{\pi n_y}{L} - \mathrm{i}C_z k = 0$$

   These are a lot of conditions, so be sure you do not miss to check any! This is indeed the *most general* solution, but you do not have to show that (you may, of course, for extra credit).

   Hint: This may be a bit tedious, so I suggest you at first take my word for it and solve the following sub-problems. Come back to this one when you have the time.

   c) (**4P**) Now, deduce from all this information: If $E_z = B_z = 0$, then $\vec{E} = \vec{B} = 0$, i.e. if the wave would be purely transversal, it would disappear altogether.

   Hint: There is a very clever way which shows that the electric and magnetic fields have in that case neither curl nor divergence if $E_z = B_z = 0$. If you start from the expressions above and use another way, you need to differentiate between three cases: (i) $n_x = 0$; (ii) $n_y = 0$; or (iii) $C_z = 0$.

   This means we can classify all waves in the guide as either of two:

   (i) transverse electric (TE) modes: $E_z = 0$ and $\left.\dfrac{\partial B_z}{\partial n}\right|_S = 0$;

   (ii) transverse magnetic (TM) modes: $B_z = 0$ and $E_z|_S = 0$.

   $\dfrac{\partial}{\partial n}$ denotes the derivative normal to the surface S of the waveguide.

   d) (**2P**) Finally, show that minimum frequencies exist for both waves: $\omega_{\text{min}}^{\text{TE}} = \dfrac{c\pi}{L}$, $\omega_{\text{min}}^{\text{TM}} = \sqrt{2}\,\dfrac{c\pi}{L}$.

   Note: So, a rectangular waveguide can serve as frequency filter for electro-magnetic waves. This result depends however on the geometry: For example, no critical frequency exists for a round profile, i.e. for a coaxial cable.

$\boxed{\text{The theory of relativity is relative only to how Chuck Norris interprets it.}}$