Problem Sheet 10

Due date: 05 April 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. THE CRAB SYNCHROTRON (**8P**): The Crab nebula was created by a supernova explosion seen on Earth in 1054. Observations reveal that in the nebula, electrons with energies of up to 10^{13} eV are bent in magnetic fields of up to 3×10^{-4} Gauß.

   a) (**3P**) Determine the maximal curvature (minimal bending radius), the natural (typical) synchrotron frequency and energy associated with these orbits, and the critical frequency and photon energy.

   b) (**3P**) Picking up last week's problem on synchrotrons and their radii, show that (in the ultra-relativistic limit) the particle radiates half of its energy E_0 within a time

   $$\tau(E_0) = \frac{3m^3c^5}{2q^4B^2}\,\frac{mc^2}{E_0}$$

   c) (**2P**) Determine the "half-life" of the synchrotron radiation in the Crab nebula. Does the energy have to be constantly replenished? From where?

2. A PARTICULAR CASE OF BREMSSTRAHLUNG (**8P**): A non-relativistic particle of mass m, charge q and incident kinetic energy E_0 makes a head-on collision with a fixed central force field $V(r)$. The potential is repulsive, and there is a point at which $V(r_0) = E_0$. This is a nice "inter-departmental" problem between Mechanics and Electrodynamics.

   a) (**4P**) Show from Larmor's formula that the total energy radiated is

   $$\Delta W = \frac{4q^2}{3m^2c^3}\sqrt{\frac{m}{2}}\int\limits_{r_0}^{\infty}\frac{\mathrm{d}r}{\sqrt{V(r_0)-V(r)}}\left|\frac{\mathrm{d}V}{\mathrm{d}r}\right|^2$$

   b) (**3P**) Show that for (repulsive) Coulomb-interaction between q and another, heavy charge $Q \gg q$, the total energy radiated is in terms of the velocity v_0 of q at infinity (and determine the pure number Γ):

   $$\Delta W = \Gamma\,\frac{qmv_0^5}{Qc^3}\quad .$$

   c) (**1P**) It's remarkable that the radiation loss disappears as $q/Q \to 0$. Give a simple argument why this should be right.

3. SHADOW BEHIND A PARTICLE (**7P**): A non-relativistic, charged particle is initially at rest and illuminated by a monochromatic plane wave.

   Consider the Liénard-Wiechert form of the radiation field emitted by the particle as the Lorentz force acts on it. Show that the incident and radiated fields interfere destructively on the axis directly behind the particle, i.e. that a "shadow" is created. It is enough to consider the far-field in the non-relativistic approximation. At which typical distance does your approximation break down? Relate to known length-scales. Is the field in front of the charge also depleted?

Please turn over.

4. Why are Sunsets Red? (**3P**)

5. Compton Scattering Off Induced Electric and Magnetic Dipoles, Part I (**4P**): In the lecture, we considered the case that an incoming electro-magnetic wave induces an electric dipole, whose strength is proportional to the strength of the incoming elecric field, and which oscillates in the same direction as the incoming field. Now, we extend the same set-up to magnetic responses: A monochromatic, plane electro-magnetic wave with electric and magnetic fields $\vec{E}_{\mathrm{in}}(\omega) = \vec{E}_0\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}}$ and $\vec{B}_{\mathrm{in}}(\omega) = \vec{e}_k \times \vec{E}_0\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{r}}$ induces a frequency-dependent electric and magnetic dipole moment in a charge distribution, i.e.

$$\vec{d}_{\mathrm{ind}}(\omega) = \alpha(\omega)\;\vec{E}_{\mathrm{in}}(\omega)\;\;,\;\;\vec{m}_{\mathrm{ind}}(\omega) = \beta(\omega)\;\vec{B}_{\mathrm{in}}(\omega)\;\;.$$

Electric and magnetic polarisabilities α and β are for us at present just real constants of proportionality.

a) (**1P**) Determine the electric field $\vec{E}_{\mathrm{out}}$ and magnetic field $\vec{B}_{\mathrm{out}}$ of the scattered wave which comes from the induced *electric* dipole $\vec{d}_{\mathrm{ind}}$.

b) (**2P**) Determine the electric and magnetic fields of the scattered wave which come from the induced *magnetic* dipole $\vec{m}_{\mathrm{ind}}$.

c) (**1P**) Find now the total electric and magnetic fields of the scattered wave: Super-impose the electric and magnetic fields coming from the electric dipole with those from the magnetic dipole. Write your result such that the outgoing electric field depends only on the polarisation of the incoming wave (and α and β), but *not* on the incoming magnetic field.