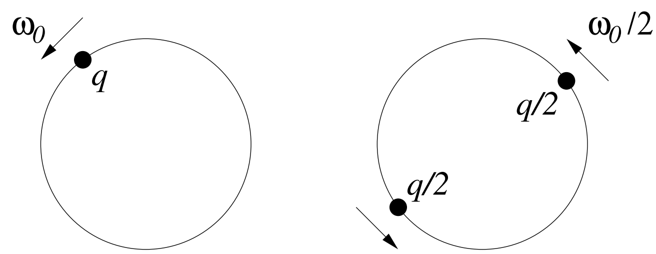
Problem Sheet 7

Due date: 8 March 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. Atomic Dipole- and Quadrupole-Radiation, Part II (**10P**): Electric quadrupole radiation in atoms is considerably weaker than electric dipole radiation. It can thus be neglected – except if electric dipole radiation is forbidden e.g. due to selection rules. We continue to discuss a classical analogon.

   Point-charges move in the xy-plane on a circle with radius l in the mathematically positive sense with constant, non-relativistic angular velocity. The atomic nucleus rests at the centre, making the atom as a whole electrically neutral. We dealt with the electric dipole last week. The quadrupole can now be thought of as two point-charges $q/2$ which are opposite to each other and rotate with angular velocity $\omega_0/2$; see figures. Consider only the far-zone, in the lowest non-vanishing multipole approximation.



   After last week's dipole, we now concentrate on the the quadrupole, *right* figure.

   Hint: You can again copy equations from textbooks. Or you use the opportunity to start from the wave-equation and derive step-by-step the solution by yourself, understanding radiation theory in this simple example. You may also choose to solve the problem with the alternative radiation formula in coordinate space presented in problem 5 below.

   a) (**2P**) Find the time-dependent, Cartesian dipole and quadrupole moments in the right figure.
   b) (**2P**) Determine the retarded vector-potential $\vec{A}_{\text{ret}}$ in the far-zone and the frequency (or wavelength) of the emitted radiation.
   c) (**3P**) Determine the *time-dependent* electric and magnetic field in the far-zone.

      Hint: You might have to derive $\vec{e}_r \times (\vec{e}_x + \mathrm{i}\,\vec{e}_y) = \mathrm{e}^{\mathrm{i}\phi}\ (\vec{e}_\phi\ \cos\theta - \mathrm{i}\vec{e}_\theta)$
   d) (**3P**) Determine the time-averaged power radiated into an arbitrary solid angle, and the total radiated power. Discuss and sketch the radiation characteristics (angular distribution) of this quadrupole. Compare in particular to last week's dipole.

      Some possible answers in Cartesian and spherical coordinates (r, ϑ, φ), respectively:

$$\vec{A}^{(E2)}(t,\vec{r}) = -\frac{ql^2\omega_0^2}{4rc^2}\ (\vec{e}_x + \mathrm{i}\vec{e}_y)\,[\vec{e}_r \cdot (\vec{e}_x + \mathrm{i}\vec{e}_y)]\exp[\mathrm{i}(kr - \omega_0 t)]$$

$$\frac{\mathrm{d}P^{(E2)}}{\mathrm{d}\Omega} = \frac{q^2 l^4 \omega_0^6}{128\pi c^5}\ \sin^2\vartheta\left(\cos^2\vartheta + 1\right)$$

Please turn over.

2. Radiating Hydrogen Atom (**3P**): Using the results you derived last week, estimate the life-time τ of the $2p$-state in hydrogen, assuming that the system is classical and the frequency is unchanged despite of radiative losses, cf. the Bohr-Sommerfeld model of the atom. Compare to the experimentally measured value $\tau \approx 1.6 \times 10^{-9}$ s. About how many rotations does the electron make in that time-span?

3. Now we combine what we learned about atomic systems (**2P**): Compare the total, time-averaged power of dipole and quadrupole radiation for typical atomic extensions and wave-lengths: $l \sim 1$ Å, $\lambda \sim 1000$ Å. Compare to the case that atomic radius and wavelength are comparable.

4. No Radiation (**3P**): Consider a spherically symmetric charge distribution which oscillates only radially with time. Show: This system does not radiate.

5. Another Radiation Formula (**3P**): We derived in the lecture the general solution of a radiative problem as

$$\vec{A}(t,\vec{r}) = \frac{1}{c} \int \mathrm{d}^3r' \; \frac{1}{|\vec{r}-\vec{r}'|} \; \vec{j}(t - \frac{|\vec{r}-\vec{r}'|}{c}, \vec{r}')$$

Show now that at large distances from the source:

$$\vec{A}(t,\vec{r}) \approx \frac{1}{cr} \int \mathrm{d}^3r' \; \left[\vec{j}(\vec{r}', t - \frac{r}{c}) + \frac{\vec{r}\cdot\vec{r}'}{cr} \, \frac{\partial}{\partial t} \, \vec{j}(\vec{r}', t - \frac{r}{c}) \right]$$

Convince yourself that the first term describes electric dipole radiation, while the second one encodes electric quadrupole and magnetic dipole radiation – no intricate details necessary. Discuss however in detail how the long-wavelength approximation must be invoked to make sense of the expansion.

6. Einstein and the Light-Ray (**4P**): A mono-chromatic light-wave of frequency $\omega_{\mathbf{I}}$ and energy density $\mathcal{H}_{\mathbf{I}}$ travels along the $\vec{e}_x$-axis in an inertial frame **I**. Determine the frequency and energy density in another inertial frame **II** which moves at velocity $\vec{v}$ relative to **I** (usually $\vec{v} \not\propto \vec{e}_x$!).

Hint: One could e.g. show that $\mathcal{H} = \dfrac{\omega^2}{4\pi c^2} a^\mu a_\mu \; \cos^2 k\cdot x$ for a light-wave $A^\mu(x) = a^\mu \; \cos k\cdot x$.