## Problem Sheet 6

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on $5 \times 5$ quadrille paper; electronic solutions must be in .pdf format. I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework. News and .pdf-files of Problems also at http://home.gwu.edu/ ${ }^{\text {hgrie/lectures/edyn17/edyn17.html. }}$

1. Magnetic Field of a Superconducting Ring (9P): A constant, electric current $I$ circles in the mathematically positive sense inside a closed, infinitesimally thin, circular conductor of radius $R$. The ring is centered in the $x y$-plane at the origin, see figure. Determine in the following some characteristics of the magnetic field.


Hint: Item a) forms the basis for each of the sub-sequent questions, each of which can be treated independently of the others. You may use spherical or Cartesian coordinates as you please; an algebraïc manipulation programme might come in handy.
a) ( $\mathbf{3 P}$ ) Reduce the problem to a two-dimensional one by symmetry considerations. Determine the vector potential $\vec{A}$ as integral over the angle $\phi^{\prime}$ parameterising the circumference of the ring.
b) ( $\mathbf{2 P}$ ) Calculate from this the magnetic field everywhere on the $z$-axis. You can speed up the calculation enormously by first identifying its direction.
c) ( $\mathbf{3} \mathbf{P}$ ) Calculate for each of the three components the leading non-vanishing term of the field $\vec{B}(x, 0, z)$ close to the origin.
d) $\mathbf{( 1 P )}$ From these results, determine the magnetic dipole moment of this arrangement.
2. Atomic Dipole- and Quadrupole-Radiation, Part I (21P): Electric quadrupole radiation in atoms is considerably weaker than electric dipole-radiation. It can therefore be neglected - except of course if electric dipole radiation is forbidden e.g. because of selection rules. We now discuss a classical analogon to this phenomenon.
Point-charges move in the $x y$-plane on a circle with radius $l$ in the mathematically positive sense with constant, non-relativistic angular velocity. The atomic nucleus rests at the centre, making the atom as a whole electrically neutral. The electric dipole is then described by one point-charge $q$ which rotates with angular velocity $\omega_{0}$. The quadrupole can be thought of as two point-charges $q / 2$ which are always opposite to each other and rotate with angular velocity $\omega_{0} / 2$; see figures. We consider only the far-zone, and the lowest non-vanishing multipole approximation.


## Please turn over.

We first solve the dipole, and next week the quadrupole: Concentrate now on the left figure.
Hint: You can of course just copy equations from textbooks. Or you use the opportunity to start from the wave-equation and derive step-by-step the solution by yourself, understanding radiation theory in this simple example. In sub-problems b), d) and f), arguments can substitute calculations.
a) ( $\mathbf{2 P}$ ) Show that the position of the charge $q$ at time $t$ can be written as the complex vector $\vec{R}(t)=l\left(\overrightarrow{\mathrm{e}}_{x}+\mathrm{i}_{y}\right) \mathrm{e}^{-\mathrm{i} \omega_{0} t}$. Its real part is the physically relevant component.
b) ( $\mathbf{3 P} \mathbf{P})$ Determine the time-dependent, Cartesian dipole and quadrupole moments in the left figure. Show that your result implies that you can neglect the quadrupole radiation of the single charge.
c) (1P) Determine the frequency of the emitted dipole and quadrupole radiations, respectively.
d) ( $\mathbf{2 P}$ ) Determine the retarded vector-potential $\vec{A}_{\text {ret }}$ in the far-zone. A possible answer in $\omega$-space:

$$
\vec{A}^{(E 1)}(\omega, \vec{r})=\frac{\mathrm{e}^{\mathrm{i} k r}}{r} \frac{\omega_{0}}{c} l q\left(\overrightarrow{\mathrm{e}}_{y}-\mathrm{i} \overrightarrow{\mathrm{e}}_{x}\right) \delta\left(\omega-\omega_{0}\right)
$$

e) ( $\mathbf{1 P}$ ) Is the retarded scalar potential $\Phi_{\text {ret }}$ an independent quantity? Is knowing it imperative to determine the electric and magnetic fields of the radiation?
f) (5P) Determine the time-dependent electric and magnetic field in the far-zone. A possible answer:

$$
\vec{E}^{(E 1)}(t ; r, \theta, \phi)=\frac{l q \omega_{0}^{2}}{c^{2}} \frac{1}{r} \operatorname{Re}\left[\exp \left[\mathrm{i}\left(k r+\phi-\omega_{0} t\right)\right]\left(\overrightarrow{\mathrm{e}}_{\theta} \cos \theta+\mathrm{i} \overrightarrow{\mathrm{e}}_{\phi}\right)\right]
$$

Hint: You might have to derive

$$
\overrightarrow{\mathrm{e}}_{r} \times\left(\overrightarrow{\mathrm{e}}_{x}+\mathrm{i} \overrightarrow{\mathrm{e}}_{y}\right)=\mathrm{e}^{\mathrm{i} \phi}\left(\overrightarrow{\mathrm{e}}_{\phi} \cos \theta-\mathrm{i} \overrightarrow{\mathrm{e}}_{\theta}\right)
$$

g) ( $\mathbf{2 P}$ ) Discuss the polarisation of the radiation, depending on the observer's position. The polar angles $\theta=0, \frac{\pi}{2}, \pi$ are particularly interesting.
h) (5P) Show that the time-averaged power radiated into an arbitrary solid angle is

$$
\frac{\mathrm{d} P^{(E 1)}}{\mathrm{d} \Omega}=\frac{q^{2} l^{2} \omega_{0}^{4}}{8 \pi c^{3}}\left(\cos ^{2} \vartheta+1\right)
$$

Determine also the total radiated power. Discuss and sketch the radiation characteristics (angular distribution) of this kind of dipole radiation. Compare in particular to Hertz' dipole discussed in the lecture. Could you use one or more Hertz' dipoles to get the same characteristics?


