Problem Sheet 5

Due date: 22 February 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.

News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. Cartesian and Spherical Multipole Moments, Part II (**10P**): We pick up a problem of the previous HW: Three point-charges are located along a line: $q_1 = -q$ at $\vec{x}_1 = (0,0,a)$, $q_2 = -q$ at $\vec{x}_2 = (0,0,-a)$, and $q_3 = 2q$ at $\vec{x}_3 = (0,0,0)$. Consider now the *spherical* multipole approximation of the same problem.

   a) (**4P**) Determine *all* spherical multipole moments with respect to the origin. Symmetries!

      Fail-safe point: $Q_{lm} = \alpha\, qa^l \delta_{m0}\, [1 + (-1)^l - 2\delta_{l0}]$, with α to be determined.

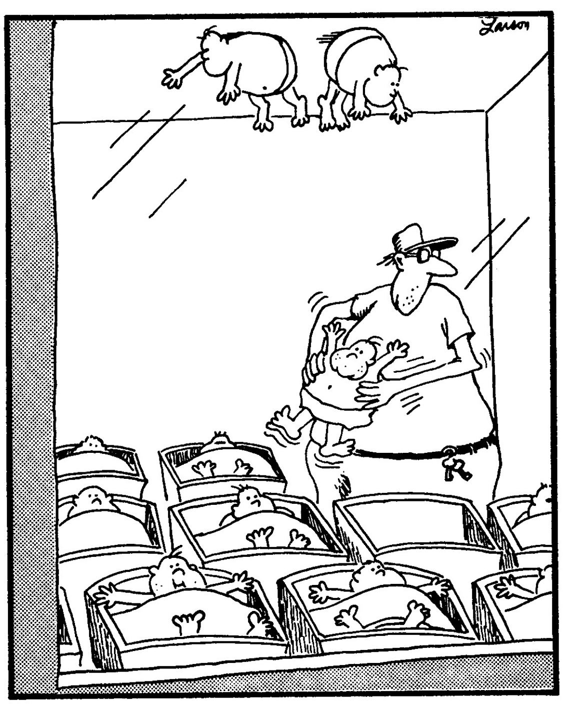
   b) (**2P**) Construct the potential and electric field, also in quadrupole approximation.

   c) (**3P**) Compare your results to the solutions of c) and to the exact solution, a). From which distance on is the relative difference of the solutions less that e.g. 1%, e.g. along the positive z axis? Is you result in line with your expectation from the dimension-less expansion parameter?

   d) (**1P**) We return to problem 4 in HW 12 of the Mathematical Methods course, where the potential on the surface of a sphere with radius R

$$\Phi(\vec{R}) = \frac{\Phi_0}{2R^2}\left(3(\vec{e}_z\cdot\vec{R})^2 - \vec{R}^2\right) \quad \text{led to} \quad \Phi(r,\theta,\phi) = \Phi_0 \left(\frac{R}{r}\right)^3 P_2(\cos\theta) \quad \text{for } r > R \ .$$

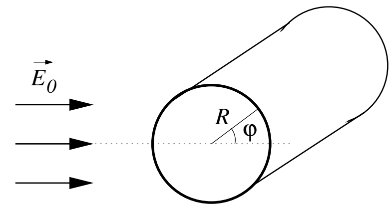
      Determine now the constant Φ_0 such that the field at large distances from the sphere is at leading non-trivial order identical to the one of the charge distribution above.



Late at night, and without permission, Reuben would often enter the nursery and conduct experiments in static electricity.

Please turn over.

2. CYLINDER IN AN EXTERNAL FIELD (**12P**): A uncharged, grounded, infinitely long cylinder of radius R is inserted into an orginally homogeneous electric field $\vec{E}_0$ which is directed perpendicular to the cylinder axis, see figure. Determine the modification of the electric field by constructing a useful Green's function along the following steps.



   a) (**1P**) Determine the elementary solution of the Laplace equation $\Delta\Phi = 0$ in cylindrical coordinates. Show that the ansatz $\Phi(r, \phi, z) = U(r)\,\chi(\phi)$ leads to a separation of the variables and the equation:
   $$\frac{r}{U(r)}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}U(r) = -\frac{1}{\chi(\phi)}\,\frac{\partial^2}{\partial\phi^2}\chi(\phi) = \mu^2$$
   Pay particular attention to the elementary solution for the value $\mu = 0$ of the separation constant. It contains a solution which is irregular at $r = 0$.

   Hint: Re-visit a similar HW of Mathematical Methods last semester.

   b) (**4P**) Determine the potential and electric field outside and inside the cylinder. Sketch both fields!

   Hint: The field at infinity is $\vec{E}_0$ (why?). A partial solution is: $\Phi(r, \phi) = -|\vec{E}_0|\left(r - \frac{R^2}{r}\right)\cos\phi$

   c) (**1P**) How big are Φ and $\vec{E}$ inside the cylinder?

   d) (**2P**) Determine and sketch the induced surface charge density on the cylinder. What is the induced total charge?

   e) (**2P**) Determine the dipole moment, per unit length, induced by the external electric field.

   f) (**2P**) Solve the problem (i.e. calculate the potential) when the cylinder is not grounded but instead carries a charge-density per unit length q, while still $\vec{E}(r \to \infty) = \vec{E}_0$. **Hint:** Superposition principle.

3. CHARGED, ROTATING SPHERE (**8P**): A homogeneous sphere of radius R and total charge Q rotates about a fixed axis with constant angular velocity $\vec{\omega}$.

   a) (**1P**) Show that the current density can in Cartesian coordinates (and with a good choice for the rotation axis) be written as
   $$\vec{j}(\vec{r}) = -\sqrt{\frac{3}{8\pi}}\frac{Qr|\vec{\omega}|}{R^3}\begin{pmatrix} \mathrm{i}[Y_{1,-1}(\Omega) + Y_{1,1}(\Omega)] \\ Y_{1,1}(\Omega) - Y_{1,-1}(\Omega) \\ 0 \end{pmatrix}\theta(R - r)$$

   b) (**2P**) Determine the vector potential $\vec{A}(\vec{x})$ outside the sphere. Show that it is a pure magnetic dipole field.

   c) (**2P**) Determine the magnetic dipole moment. Compute and interpret Landé's *gyromagnetic ratio* g_{L} between magnetic moment and angular momentum $\vec{l}$ of a sphere with mass M,
   $$\vec{m} = g_{\mathrm{L}}\,\frac{Q}{2Mc}\,\vec{l}\ .$$

   d) (**3P**) Determine the gyromagnetic ratio when the mass M is concentrated at the surface of the sphere, while the current is (as before) distributed homogeneously over the whole sphere. Could this serve as model for the proton or neutron? Provide a thorough discussion!