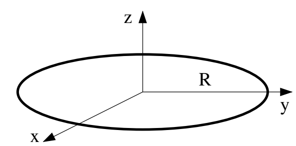
Problem Sheet 4

Due date: 15 February 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. ELECTROSTATIC MULTIPOLE MOMENTS (**10P**): An infinitesimally thin, conducting, circular ring of radius R carries the homogeneous line charge density μ. It is centred at the origin in the xy plane, see figure.



   Hints: Determine all spherical multipole moments of the charge distribution with respect to the origin.
   If multipole moments vanish, you can substitute a calculation by a short argument.

   Hint: Recall that $Y_{l0}(\theta,\phi) \propto P_l(\cos\theta)$, so that some results can also be written using Legendre polynomials at a given value, which you need to look up.

   a) (**1P**) Convince yourself that the charge density can take the form $\rho(r,\theta,\phi) = \frac{\mu}{r}\,\delta(r-R)\,\delta(\cos\theta)$.
   Is this a Dirichlet or von-Neumann boundary problem?

   b) (**5P**) Determine all spherical multipole moments for $r \gg R$. Give explicit answers for the monopole, dipole and quadrupole moments.

   c) (**1P**) Determine the scalar potential far from the origin as expansion in a suitable, small parameter.

   d) (**3P**) Determine the scalar potential close to the origin as expansion in a suitable, small parameter.

2. ELECTRIC FIELD OF THE HYDROGEN ATOM (**8P**): The charge density of the hydrogen atom (Bohr radius a) in its ground state is:

$$\rho(\vec{r}) = q\,\delta^{(3)}(\vec{r}) - \frac{q}{\pi a^3}\,\mathrm{e}^{-\frac{2r}{a}}$$

   a) (**1P**) Motivate from your QM experience that the this charge density is reasonable.
   Which symmetries should the electric field (and the electrostatic potential Φ) have?

   b) (**4P**) Show that $\Phi(\vec{r}) = q\mathrm{e}^{-2r/a}\left(\frac{1}{r}+\frac{1}{a}\right)$ and calculate $\vec{E}$.

   Hint: There are many ways to do this. When you solve directly for Φ, pay particular attention to integration constants and recall that $\frac{1}{r}$ is the Green's function to the Laplace operator. Better check that your Φ indeed solves the Poisson equation.

   c) (**3P**) Calculate the *numerical value* of the magnitude of the electric field at a distance of one Bohr radius, independently in SI- and Gauss' cgs-units. Check that the two results are consistent.

Please turn over.

3. DIPOLE MOMENT OF THE HCl MOLECULE (**5P**): In a simplified model, the HCl molecule consists of a hydrogen atom (charge number $Z = 1$) with a chlorine atom, $Z = 17$, where the H-atom transfers its electron to the chlorine atom. The 18 electrons of Cl^- are then approximated by a spherically symmetric cloud around the Cl nucleus. Both nuclei are separated by $l = 1.28$ Å. We now compare the dipole moment of this molecule with the measured value, $d_{\text{exp}} = 1.03 \times 10^{-18}$ esu cm (Gauß system).

   a) (**1P**) Prove that the dipole moment of a neutral object is independent of the choice of the coordinate origin.

   b) (**2P**) Calculate the dipole moment of the HCl molecule assuming that it consists of two point-like ions H^+ and Cl^- separated by a distance l. Given what you know about ionic binding, how can you explain the discrepancy to experiment?

   c) (**2P**) How far does one have to shift the centre of negative charge from the Cl nucleus towards the proton in order to reproduce the measured value?

4. CARTESIAN AND SPHERICAL MULTIPOLE MOMENTS, PART I (**7P**): Three point-charges are located along a line: $q_1 = -q$ at $\vec{x}_1 = (0, 0, a)$, $q_2 = -q$ at $\vec{x}_2 = (0, 0, -a)$, and $q_3 = 2q$ at $\vec{x}_3 = (0, 0, 0)$.

   a) (**1P**) Calculate the exact solution for the potential $\Phi(\vec{x})$.

   We now consider the *Cartesian* multipole approximation of this problem.

   b) (**4P**) Determine the first three *Cartesian* multipole moments (monopole, dipoles, quadrupoles) with respect to the origin. Symmetries help to avoid some calculations!

   c) (**2P**) Construct the potential and electric field in that approximation.

   More next Week.