Problem Sheet 3

Due date: 08 February 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. Uniqueness of Electrodynamics (**7P**): Only two gauge-invariant Lagrangeans exist which are at most quadratic in derivatives and gauge fields: $-\dfrac{1}{16\pi c}\,F^{\mu\nu}F_{\mu\nu}$, or

   $$-\frac{1}{4c}\,\epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}F^{\rho\sigma}\quad,\text{ where }\epsilon_{\mu\nu\rho\sigma}\text{ is the totally anti-symmetric unit pseudo-tensor of rank 4.}$$

   a) (**2P**) Express this term via the electric and magnetic field strengths.
   b) (**2P**) Investigate its properties under parity-transformations, time-reversal and their combination.
   c) (**3P**) Show: It can be written as total divergence of a 4-vector. Is it compatible with experiment?

      Hint: Derive the modified Maxwell equations, if you do not see the result immediately.

2. Parallel Electric and Magnetic Fields from Lorentz Boosts (**7P**): Consider arbitrary electric and magnetic fields which are not perpendicular to each other at some point in a given inertial frame **I**. We will show: There exists a reference frame **II** in which the electric and magnetic fields are parallel at that point.

   Hint: The two sub-problems are independent of each other.

   a) (**3P**) Express the field magnitudes in frame **II** in terms of Lorentz invariants.

      Hint: Consider the Lorentz invariants $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ built from the field-strength tensor; see preceeding problem.
   b) (**4P**) Consider the Poynting vector of the electromagnetic field to find the velocity of frame **II** with respect to **I**. The relative velocity depends on the magnitudes of $\vec{E}$ and $\vec{B}$ and the angle between them, all given in frame **I**. Can one always find an inertial frame **II**?

      Hint: You may assume without proof that observer **II** moves perpendicular to both the electric and magnetic fields.

3. Massive Photons with Current-Conservation (**4P**): The photon acquires a mass $m_\gamma=\mu_\gamma\frac{\hbar}{c}$ if one adds the following, obviously (?) not gauge-invariant term to Maxwell's Lagrangean:

   $$+\frac{\mu_\gamma^2}{8\pi\,c}\,A_\mu A^\mu\quad,$$

   Determine the modified Maxwell-equations (now called Proca-equations). Which condition on gauge fields guarantees current conservation?

4. Gauge Fields in the Schrödinger Equation (**2P**): This problem is a precursor to the next one. As familiar from Quantum Mechanics, we can build the Schrödinger equation of a particle in an external electro-magnetic field by *Minimal Substitution*: Using the correspondence principle, the energy and momentum operators are substituted by the following "gauge-covariant" expressions:

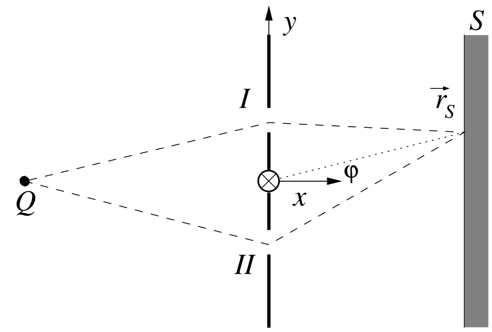
   $$\vec{p}\to\vec{p}-\frac{e}{c}\,\vec{A}(t,\vec{r})\quad,\quad \mathrm{i}\,\hbar\,\frac{\partial}{\partial t}\to\mathrm{i}\,\hbar\,\frac{\partial}{\partial t}-e\Phi(t,\vec{r})$$

   Show: If the wave-function $\Psi(t,\vec{r})$ solves the Schrödinger equation in a gauge potential $A^\mu(t,\vec{r})$, then $\mathrm{e}^{\frac{\mathrm{i}e}{\hbar c}\chi(t,\vec{r})}\,\Psi(t,\vec{r})$ is the solution in a field which is related to $A^\mu(t,\vec{r})$ by a gauge transformation $\chi(t,\vec{r})$.

   Please turn over.

5. AHARONOV-BOHM EFFECT (1959), simplified version (**10P**): In Continuum Mechanics, gauge potentials are "more fundamental" than the field strength tensor. But the only macroscopically observable, i.e. classically treatable, fields are the electromagnetic and gravitational fields. In Quantum Mechanics and Optics, you have nevertheless already encountered e.g. electrons as "matter-waves".

   For the experimental observation of the following effect, Chambers (1960; improved by Tonomura 1983) used electrons. Their spin is irrelevant, so that we can describe them as in wave-mechanics by a complex field/wave-function and argue again using the correspondence principle.

   An infinitely long, infinitesimally thin solenoid ($\otimes$) on the z-axis is located parallel to and in the middle between the two slits of a double-slit experiment. The solenoid is impenetrable to the field $\Psi(\vec{r})$ which describes the electrons moving from the source Q to the screen S. A magnetic field with flux Φ_{mag} is confined to the interior of the solenoid, see figure. The problem is static.

   The vector potential in cylindrical coordinates

$$\vec{A}(\vec{r}) = \vec{e}_\varphi \, \frac{\Phi_{\text{mag}}}{2\pi r}$$

   describes the magnetic field: no magnetic flux outside, but Φ_{mag} inside the solenoid.

   a) (**2P**) Show that the vector potential outside can at each point be written as gradient of a scalar function $\chi(\vec{r}) = \dfrac{\Phi_{\text{mag}}\varphi}{2\pi}$. Under which conditions is $\chi(\vec{r})$ outside the solenoid a function of the coordinates which is well-defined (unique and continuously differentiable) *at all points* outside the solenoid simultaneously? Recall as analogy "cuts" in Complex Analysis.

   b) (**1P**) Determine the force exerted by the magnetic field on *point-like* electrons.

   We now consider the electrons to be waves and first switch off $\vec{A}$. As you know, an interference pattern appears at the point $\vec{r}_S$ of the screen S, coming from a linear superposition of the waves originating from slits I and II, $\Psi_{I,\Phi_{\text{mag}}=0}(\vec{r}_S)$ and $\Psi_{II,\Phi_{\text{mag}}=0}(\vec{r}_S)$.

   c) (**3P**) Let $\Psi_{II,\Phi_{\text{mag}}=0}(\vec{r}_S) = e^{i\alpha(\vec{r}_S)}\, \Psi_{I,\Phi_{\text{mag}}=0}(\vec{r}_S)$. Show that the interference pattern has the form

$$\text{Intensity } I \propto 1 + \cos\alpha(\vec{r}_S) \ .$$

   d) (**4P**) Compute now the change of the pattern for $\Phi_{\text{mag}} \neq 0$, using the result from the previous problem. Interpretation! Are gauge fields observable? Consider in particular $\dfrac{e\Phi_{\text{mag}}}{\hbar c} \in 2\pi\mathbb{Z}$.

   Hint: The angle φ is different for the two paths by which an electron can hit a point on the screen, via slit I or slit II.

   Note: In SQUIDS, one uses this principle to measure even the tiniest magnetic fields.