Problem Sheet 2

Due date: 01 February 2017 **16:00**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at http://home.gwu.edu/~hgrie/lectures/edyn17/edyn17.html.

1. Relativistic Point-Particle (**6P**): A charged particle moves along the x-axis with a velocity close to the speed of light. It enters a large region in which there is a uniform electric field along the y-axis.

   a) (**2P**) Formulate the relativistic equations of motion.
   b) (**2P**) Determine whether the x-component of the particle's momentum is increasing, decreasing or remains constant.
   c) (**2P**) Determine whether the x-component of the particle velocity is increasing, decreasing or remains constant.

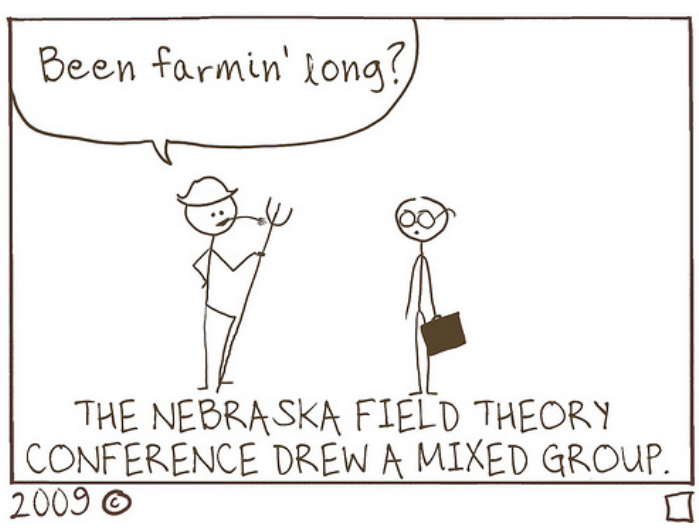
   Bonus Question (up to 7 extra P): Determine the trajectory $x(y)$ of a particle with initial momentum $\vec{p}_0 = p_0\,\vec{e}_x$ and show that you recover the expected result in the non-relativistic limit.

2. Lorentz and Gauge Transformations (**3P**): An arbitrary gauge field A^μ obeys in inertial system **I** the Coulomb gauge $\vec{\nabla}\cdot_{\mathbf{I}}\vec{A}_{\mathbf{I}} = 0$. Recall that ∂_μ is a four-vector, i.e. temporal- and spatial derivatives mix under Lorentz-transformations and the spatial derivatives in two coordiante systems are different, $\vec{\nabla}_{\mathbf{I}} \neq \vec{\nabla}_{\mathbf{II}}$.

   Show that the field does not obey the Coulomb gauge condition in an inertial frame **II** which moves relative to **I**.

   Hint: A possible solution uses that a vector can be decomposed into components which are longitudinal and transversal with respect to the relative velocity $\vec{v}$ between the inertial frames. The Lorentz-transformation of these components was exemplified in the lecture by considering the vector $\vec{r}$.

3. Faraday's Law of Induction (**6P**): Since no magnetic monopoles exist in any coordinate system, $\vec{\nabla}\cdot\vec{B} = 0$ always. Derive from this the law of induction, $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\dot{\vec{B}}$, by Lorentz-transformations. You may *not* use that both are results of the the relativistic version $\epsilon_{\mu\nu\sigma\rho}\partial^\mu F^{\nu\sigma} = 0$ of the homogeneous Maxwell equations.



Please turn over.

4. Moving Electron (**9P**): This and the next problem are very closely related. It is simple to write down the electric and magnetic field generated by an electron at rest (inertial system **II**). The electron moves now with constant direction and speed v, passing at time $t = 0$ an observer who is at rest in an inertial frame **I** at the point $(0, b, 0)$. At that moment $t_{\mathbf{I}} = 0 = t_{\mathbf{II}}$, the coordinate systems of both frames coincide. Thus, the minimum distance between observer and electron is b.

   a) (**2P**) Determine the electric and magnetic field measured by the observer: It will also be needed in the next problem.

   Hint: There are (at least) two ways: One can transform directly the electric and magnetic fields using the formulae from the lecture. Or (and more elegantly) one can first determine the four-vector $A^{\mu}_{\mathbf{II}}$ which describes the field generated by the charge in frame **II**, and transform it into frame **I**. Either way: do not forget to transform also the coordinates x^{μ} at which the electric and magnetic fields are to be calculated.

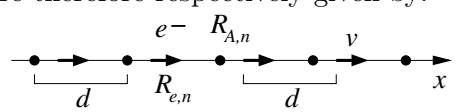
   b) (**2P**) In which inertial frame **I** can the observer measure a magnetic field which is larger in magnitude than the electric field he/she measures?

   c) (**3P**) Calculate and interpret the electromagnetic force exerted by the electron on a test charge held by the observer.

   d) (**2P**) Sketch the temporal evolution of the fields seen by the observer for different values of v. The case $v \to c$ is particularly interesting.

5. Current in a Wire (**6P**): A simple model for the current in a wire is depicted below: The atomic ions are aligned on the x-axis like pearls on a string at fixed positions, separated by d. All electrons are also separated by the distance d, but move with the same, constant velocity v along the x-axis. The position-four-vector for the nth ion and nth electron are therefore respectively given by:

   $R_{A,n} = (ct, nd, 0, 0)$ and $R_{e,n} = (ct, nd + vt, 0, 0)$



   a) (**3P**) Transform these four-vectors into the electron's rest-system. Determine from this the new separations d'_A and d'_e of ions and electrons, respectively. Is the electron density in this inertial system smaller or larger than in the original system? Is it smaller or larger than the ion density? What does that mean for the net charge of the wire? Interpretation!

   b) (**3P**) It is simple to compute the electric and magnetic field generated by the electrons in their rest-frame; see the previous problem. Determine now the strength of the electric and magnetic field generated by the electrons in the rest frame of the ions. Do not forget to transform also the co-ordinates at which the electric and magnetic fields are computed.

   c) (**Optional for 5 more P**) As cross-check, compare the fields to the well-known result in the limit of a finite, continuous current I of electrons ($d \to 0$ with $\mu = -e/d =$const.) in an infinitesimally thin, infinitely long wire.

   Hint: Current = velocity $\times$ charge density