Problems in Lecture

2 March 2017

You will develop solutions to the following problem in class. It is helpful if you have looked at it before class and have come up with a rough strategy what to do.

1. SEMI-CLASSICAL RADIATION FROM AN ATOM : Consider a radiative transition between the $m = 0$ ^{2}P-state and the ground state in atomic hydrogen. At a certain moment, the electron wave-function is therefore described by the superposition of the initial and final states Ψ_i and Ψ_f:

$$\Psi(t;\vec{r}) = \Psi_i(t;\vec{r}) + \Psi_f(t;\vec{r})$$

In the semi-classical treatment, one first identifies the part of the corresponding probability density which describes the transition. This is then interpreted as the probability density with which one can find the electron at a certain location during the transition.

In the natural system of units, the spatial dependence of the first few normalised energy eigenfunctions to energy eigenvalues $E_n = -\frac{q^2}{2a\,n^2}$ of the electron with charge q in the (n,l,m) states of the hydrogen atom is in spherical coordinates:

$$\Psi(^1\mathrm{S},m) = \frac{2}{a^{3/2}}\,Y_{0m}(\theta,\phi)\ \exp-\frac{r}{a} \quad ; \quad \Psi(^2\mathrm{P},m) = \frac{1}{(2a)^{3/2}}\,Y_{1m}(\theta,\phi)\ \frac{r}{\sqrt{3}\,a}\ \exp-\frac{r}{2a} \quad .$$

Here, $Y_{lm}(\theta,\phi)$ are the Spherical Harmonics, $a = 1/(m_\mathrm{e}q^2)$ the Bohr radius, m_e the electron mass.

a) Construct the time-dependent transition charge density $\rho(t;\vec{r})$. It can be cast into the form

$$\rho(t;\vec{r}) = \Gamma\ \frac{q}{a^4}\ r\ \mathrm{e}^{-\gamma r/a}\cos\theta\ \mathrm{e}^{-\mathrm{i}\omega t} \quad ,$$

with the constants $\Gamma \approx 1/20$, $\gamma \approx 10^0$ pure numbers and ω a frequency you need to determine.

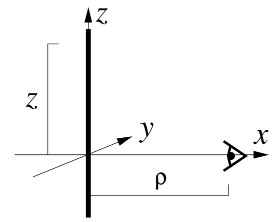
b) Calculate the leading contribution to the total time-averaged power emitted during the transition. Your result should only depend on a and q.

c) Your colleague measures the typical lifetime of another transition between the same two states as $[1.36 \pm 0.02] \times 10^{-4}$s. Given your results, speculate on the photon multipolarity in her experiment.

d) By how much does the lifetime of the ^{2}P state change if the electron is replaced by a muon, a particle with the same charge but 200 times the electron mass.

Please turn over.

2. Radiation from Sudden Current in a Wire: Here is a rare case in which retardation effects can be taken into account easily and an exact solution for the exact radiation formula exists. An infinitely long, infinitesimally thin, straight wire is positioned along the z-axis. The wire is always neutral. For $t < 0$, it is also current-free, while a constant current I is applied at time $t = 0$ over the whole length of the wire, simultaneously. At later times, the current continues to flow at a constant rate, without loss, $\vec{j}(t) = I\vec{e}_z$ for $t > 0$. We now consider the resulting fields measured by an observer which is a distance ρ away from the wire on the x-axis, see figure.



   a) At which time is the observer at ρ reached by news that the current has been switched on at the point z on the wire? Describe the electric and magnetic fields measured by the observer for $ct < \rho$.

   b) Show that the time-dependent vector-potential has the form:

$$\vec{A}(t;\rho,z) = \Gamma\ \ln\frac{ct+\sqrt{c^2t^2-\rho^2}}{ct-\sqrt{c^2t^2-\rho^2}}\ \vec{e}_A \quad \text{for } ct > \rho\ \text{, but zero otherwise.}$$

   Find the constant Γ and the direction $\vec{e}_A$ of the vector potential.

   c) Determine the time-dependence of the electric and magnetic fields and check that they meet your expectations for $t \to \pm\infty$.

   d) Determine in detail the polarisation of the resulting radiation (circular, elliptic, linear), including its helicity or polarisation axis, as appropriate.

   e) Find the total power radiated per unit wire length into a circle of radius ρ around the wire at a given time t. Compare to your expectation for $t \to \infty$.