

Pictures of Dynamic Electric Fields

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The electric field of a point charge moving at relativistic velocities along a single plane of motion may be visualized in a physically intuitive way with the aid of pictures of the electric lines of force. A general derivation of exact parametric equations for these lines of force is described and applied to create the pictures presented here of synchrotron radiation, low-angle high-energy Coulomb scattering, abrupt linear acceleration of a charge, and a charge undergoing simple harmonic motion.

The electromagnetic fields of accelerated charges are generally discussed in terms of the angular and frequency distributions of the radiation, the total power radiated, and other parameters of direct interest to the experimentalist.¹ Approximations are usually made, for example, that only the far-field radiation proportional to r^{-1} is considered, not the r^{-2} Coulomb field; or that the excursion of the charge is small compared to the wavelength of radiation (i.e., that the charge moves at non-relativistic velocities). The results thus obtained are easily confirmed experimentally but do not readily yield any intuitive feeling for how the fields themselves are interacting and developing in time. In this article, electric field maps are presented which may help in the visualization of the field patterns. The particular form of graphical representation chosen is lines of force. This representation yields clear, uncluttered visual patterns at the cost of throwing away quantitative information about the magnitude of the field, but that magnitude in these problems has such a wide range that no other simple means of depiction could do much better.

The electric and magnetic fields \mathbf{E} and \mathbf{B} of an arbitrarily moving charge are²

$$\mathbf{E} = q \left(\frac{1 - \beta^2}{R^2} \frac{(\hat{n} - \boldsymbol{\beta})}{(1 - \hat{n} \cdot \boldsymbol{\beta})^3} + \frac{\hat{n} \times [(\hat{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{R(1 - \hat{n} \cdot \boldsymbol{\beta})^3} \right);$$

$$\mathbf{B} = \hat{n} \times \mathbf{E}, \quad (1)$$

where the units are cgs Gaussian with $c=1$; q is the charge in esu, $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ the velocity and acceleration of the charge at the retarded time, R and \hat{n} the scalar distance and unit vector from the retarded position to the field point.

The retarded time t' is defined by $t' = t_0 - R$, t_0 being the time of observation. Once the trajectory of the charge and t_0 have been specified, \mathbf{E} is a function only of \mathbf{r} , the position vector of the field point. The line of force is then determined by the differential equation

$$d\mathbf{r} = \mathbf{E}(\mathbf{r}) ds / |E(\mathbf{r})|, \quad (2)$$

where ds is the (scalar) differential element of arc length. Two methods for solving this differential equation were used. One approach is to do a direct numerical integration of the differential equation in the form $\Delta \mathbf{r} = \mathbf{E}(\mathbf{r}) \Delta s / |E(\mathbf{r})|$ with a small but finite Δs , then plot the successive points so obtained. An IBM 360/65 computer programmed in FORTRAN IV and a California Computer Products model 665 11-in. drum plotter were the machines used in all this work. A second method, which works for quite a number of interesting types of particle motion as long as that motion is restricted to a single plane, is the explicit integration of the differential equation to yield a parametric equation for \mathbf{r} as a function of R , the retarded distance. Once the parametric equation is obtained, it is still convenient to use a computer program to plug in successive increasing values for R and grind out x and y coordinates, which are fed to the plotter as before.

Both methods require initial conditions on the differential equation to be supplied, of course. On a two-dimensional sheet of paper the spacing

of field lines has no strict quantitative physical meaning; therefore one may impose the arbitrary but reasonable initial condition that in the immediate vicinity of the charge, the field lines are distributed like the familiar Lorentz-contracted “pincushion” pattern of a charge moving in a straight line with the same instantaneous velocity. Thus if a total of n lines of force are to spread out from the charge, the m th line starts out at an angle ϕ_m counterclockwise from the direction of the particle’s motion such that $\tan\phi_m = \gamma \tan(2\pi m/n)$, where γ denotes $(1-\beta^2)^{-1/2}$ as usual.³

To obtain the parametric equation for the line of force, let the trajectory of the charge be specified in Cartesian coordinates $x(t')$ and $y(t')$. Then define the following functions of t' :

$$\begin{aligned} \beta_x(t') &= \dot{x}(t'), \\ \beta_y(t') &= \dot{y}(t'), \\ |\beta(t')| &= (\beta_x^2 + \beta_y^2)^{1/2}, \\ \theta(t') &= \tan^{-1}(\beta_y/\beta_x), \\ R &= t_0 - t'; \end{aligned}$$

t_0 , the time at which the “snapshot” of \mathbf{E} is to be taken, is a constant for the problem. Let u and v be the Cartesian coordinates of any field point. u and v are just the x and y components of \mathbf{r} , the observer’s position vector. Let α be the angle from $\beta(t')$ to \hat{n} , where \hat{n} once again is the unit vector pointing from the retarded position (x, y) towards (u, v) . Then, from Fig. 1,

$$\begin{aligned} u &= x(t') + R \cos[\theta(t') + \alpha], \\ v &= y(t') + R \sin[\theta(t') + \alpha]. \end{aligned} \quad (3)$$

Finally, define $\hat{\alpha}$ to be $\hat{z} \times \hat{n} =$ the unit vector pointing in the direction of increasing α . (α is measured counterclockwise.) In all the above it has been assumed of course that the particle’s motion is confined to a single plane and that the observer is in that same plane. While this may seem to be a very restrictive assumption, quite a few of the interesting radiative systems do satisfy it, and for such motions the radiation is mostly in that same plane.⁴ From Eq. (3) and the relation $t' = t_0 - R$, it is clear that R and α uniquely determine u and v . Conversely, for any field

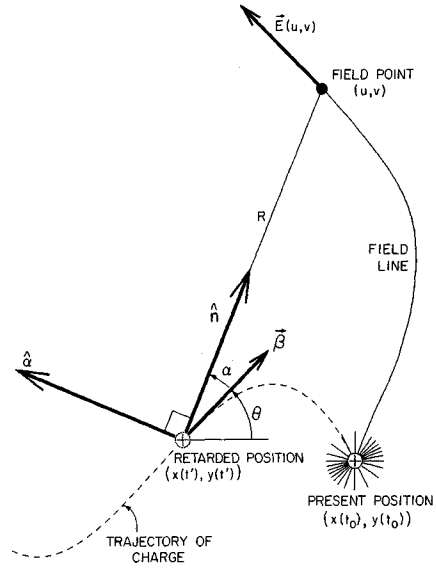


FIG. 1. Geometry of the special two-dimensional curvilinear coordinate system within which exact parametric formula for electric field lines may be derived. Every field point of Cartesian coordinates (u, v) corresponds to a unique retarded distance R and angle α .

point (u, v) , there is only one retarded position $[x(t'), y(t')]$ so long as the source charge moves more slowly than the waves it produces since, by the triangle inequality, two photons that emanate from different retarded positions cannot strike the observer simultaneously. The upshot is that u and v uniquely determine R and the principal value of α . Since there is a one-to-one differentiable map connecting (u, v) and (R, α) , the latter coordinates form a perfectly legitimate curvilinear coordinate system that turns out, not surprisingly, to be the natural coordinate system for the problem.

Use of the above definitions to resolve Eq. (1) for \mathbf{E} into radial and transverse components gives

$$\begin{aligned} \mathbf{E} \cdot \hat{n} &= q(1-\beta^2)R^{-2}(1-\beta \cos\alpha)^{-2}, \\ \mathbf{E} \cdot \hat{\alpha} &= q(1-\beta^2)(\beta \sin\alpha)R^{-2}(1-\beta \cos\alpha)^{-3} \\ &\quad + q[(\beta \sin\alpha)(\hat{n} \cdot \hat{\beta}) - (\hat{\beta} \cdot \hat{\alpha})(1-\beta \cos\alpha)]R^{-1} \\ &\quad \times (1-\beta \cos\alpha)^{-3}. \end{aligned} \quad (4)$$

An increment along the line of force has components

$$\begin{aligned} d\mathbf{r} \cdot \hat{n} &= (\mathbf{E}/|E|) \cdot \hat{n} ds, \\ d\mathbf{r} \cdot \hat{\alpha} &= (\mathbf{E}/|E|) \cdot \hat{\alpha} ds. \end{aligned}$$

When \mathbf{r} changes in this manner, how do the scalar coordinates R and α change? By a brute force calculation using the Cartesian coordinates u and v as intermediaries, one finds

$$\begin{aligned} du &= (d\mathbf{r} \cdot \hat{\mathbf{n}}) \cos(\theta + \alpha) - (d\mathbf{r} \cdot \hat{\boldsymbol{\alpha}}) \sin(\theta + \alpha), \\ dv &= (d\mathbf{r} \cdot \hat{\mathbf{n}}) \sin(\theta + \alpha) + (d\mathbf{r} \cdot \hat{\boldsymbol{\alpha}}) \cos(\theta + \alpha), \end{aligned}$$

and

$$\begin{aligned} dR &= \frac{(\partial v / \partial \alpha) du - (\partial u / \partial \alpha) dv}{\partial(u, v) / \partial(R, \alpha)}, \\ d\alpha &= \frac{(\partial u / \partial R) dv - (\partial v / \partial R) du}{\partial(u, v) / \partial(R, \alpha)}, \end{aligned}$$

so that

$$\begin{aligned} dR &= (1 - \beta \cos \alpha)^{-1} d\mathbf{r} \cdot \hat{\mathbf{n}}, \\ d\alpha &= R^{-1} [d\mathbf{r} \cdot \hat{\boldsymbol{\alpha}} + (1 - \beta \cos \alpha)^{-1} \\ &\quad \times (R\dot{\theta} - \beta \sin \alpha) d\mathbf{r} \cdot \hat{\mathbf{n}}]. \end{aligned} \quad (5)$$

After substituting the above equations one by one into the expressions for dR and $d\alpha$, the result is

$$\begin{aligned} dR &= (1 - \beta^2) R^{-2} (1 - \beta \cos \alpha)^{-3} (q/|E|) ds, \quad (6) \\ d\alpha &= [(1 - \beta^2)\dot{\theta} + (\beta \sin \alpha) (\hat{\mathbf{n}} \cdot \dot{\hat{\boldsymbol{\beta}}}) \\ &\quad - (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}}) (1 - \beta \cos \alpha)] R^{-2} \\ &\quad \times (1 - \beta \cos \alpha)^{-3} (q/|E|) ds. \end{aligned} \quad (7)$$

Now divide Eq. (7) by Eq. (6):

$$\begin{aligned} (d\alpha/dR) &= \dot{\theta} + (1 - \beta^2)^{-1} \\ &\quad \times [(\beta \sin \alpha) (\hat{\mathbf{n}} \cdot \dot{\hat{\boldsymbol{\beta}}}) + (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}}) (\beta \cos \alpha) \\ &\quad - (1 - \beta^2)^{-1} (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}})]. \end{aligned}$$

To simplify the middle term, note that

$$\begin{aligned} (\beta \sin \alpha) (\hat{\mathbf{n}} \cdot \dot{\hat{\boldsymbol{\beta}}}) + (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}}) (\beta \cos \alpha) &= -(\boldsymbol{\beta} \cdot \hat{\boldsymbol{\alpha}}) (\hat{\mathbf{n}} \cdot \dot{\hat{\boldsymbol{\beta}}}) + (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}}) (\hat{\mathbf{n}} \cdot \boldsymbol{\beta}) \\ &= (\boldsymbol{\beta} \times \dot{\hat{\boldsymbol{\beta}}}) \cdot (\hat{\mathbf{n}} \times \hat{\boldsymbol{\alpha}}) \\ &= \beta_x \dot{\beta}_y - \beta_y \dot{\beta}_x \\ &= \beta^2 (d/dt') \tan^{-1}(\beta_y/\beta_x) \\ &= \beta^2 \dot{\theta}. \end{aligned}$$

Hence

$$d\alpha/dR = \dot{\theta} + \beta^2 \dot{\theta} / (1 - \beta^2) - (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}}) / (1 - \beta^2),$$

or

$$\begin{aligned} d\alpha/dR &= [\dot{\theta} - (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}})] (1 - \beta^2)^{-1} \\ &= \gamma^2 [\dot{\theta} - (\dot{\hat{\boldsymbol{\beta}}} \cdot \hat{\boldsymbol{\alpha}})]. \end{aligned} \quad (8)$$

This is the boiled-down differential equation for the line of force. $\dot{\theta}$, $\dot{\hat{\boldsymbol{\beta}}}$, and β^2 are all functions of $t' = t_0 - R$ for which specific expressions may now be substituted. Assuming this differential equation can be solved (as it typically can for various simple trajectories discussed below) to give α as a function of R , then Eqs. (3) may be used to regenerate u and v for successive increasing values of R ; u and v are then fed to the digital plotter. As stated above, the initial condition for the m th line of force out of a total of n lines is that

$$\gamma \tan(2\pi m/n) = \tan \phi_m = \tan[(\phi_m + \theta) - \theta];$$

but

$$\begin{aligned} \tan(\phi_m + \theta) &= (dv/du)_{R=0} \\ &= \frac{-\beta \sin \theta + \sin(\theta + \alpha_m)}{-\beta \cos \theta + \cos(\theta + \alpha_m)} \end{aligned}$$

and α_m = the value of α when $R = 0$ for the m th line [the constant of integration for Eq. (8)]. Application of the formula for the tangent of the difference of two angles yields $\gamma \tan(2\pi m/n) = \tan \phi_m = (\sin \alpha_m) / (-\beta + \cos \alpha_m)$. Solving this equation in any direct way for α_m is an algebraic morass that finally reduces to the simple and elegant formula

$$\tan \frac{1}{2} \alpha_m = (1 - \beta)^{1/2} (1 + \beta)^{-1/2} \tan(\pi m/n), \quad (9)$$

which is much easier to verify than to derive from scratch.

Specific applications will now be discussed.

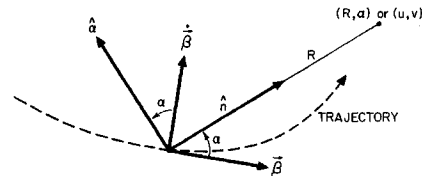


FIG. 2. Relationship between $\boldsymbol{\beta}$, $\dot{\hat{\boldsymbol{\beta}}}$, $\hat{\mathbf{n}}$, and $\hat{\boldsymbol{\alpha}}$ for a charge whose acceleration is purely transverse.

CHARGE UNDERGOING TRANSVERSE ACCELERATION ONLY

Here β and $\dot{\beta}$ are perpendicular. $\dot{\beta}$ has magnitude $|\beta\dot{\theta}|$ and by considering a diagram such as Fig. 2 one sees that $\dot{\beta} \cdot \hat{a} = \beta\dot{\theta} \cos\alpha$, so that Eq. (8) becomes

$$(1 - \beta \cos\alpha)^{-1} d\alpha = (1 - \beta^2)^{-1/2} (dR/dt') dt' = -\gamma^2 d\theta.$$

Integration and substitution of Eq. (9) for α_m yields

$$\tan \frac{1}{2}\alpha = (1 - \beta)^{1/2} (1 + \beta)^{-1/2} \times \tan \left[(\pi m/n) - \frac{1}{2}\gamma\theta(t_0 - R) + \frac{1}{2}\gamma\theta(t_0) \right]. \quad (10)$$

Equation (3) requires $\cos\alpha$ and $\sin\alpha$, which are then obtained from $\tan \frac{1}{2}\alpha$ by standard trigonometric identities.

Two examples will now be demonstrated. First, take the standard problem of synchrotron radiation. Suppose the charge is in an exact circular orbit of radius a ; $x(t') = a \sin\omega t'$, $y(t') = a \cos\omega t'$ (clockwise motion), where $\omega =$ tangential velocity/radius $= \beta/a$. Thus $\theta(t') = -\omega t'$. Substituting into Eq. (10) above, the equation in the (R, α) coordinates for the m th of n lines of force is

$$\tan \frac{1}{2}\alpha = (1 - \beta)^{1/2} (1 + \beta)^{-1/2} \times \tan(\pi m/n - \gamma\beta R/2a). \quad (11)$$

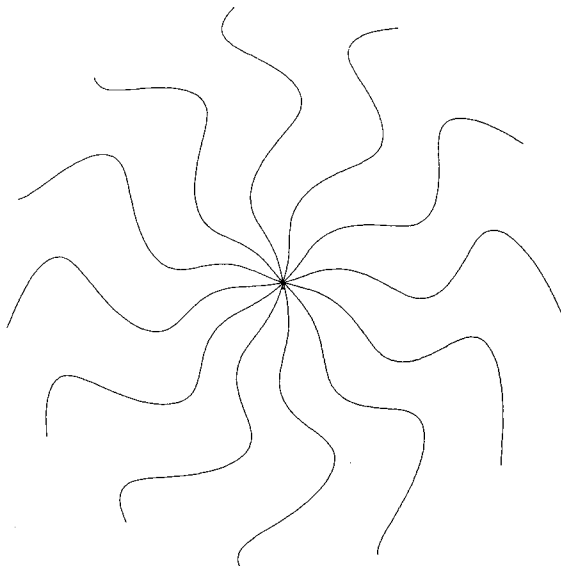


FIG. 3. Electric field lines for a charge moving at tangential speed $\beta=0.20$ on a circular path centered on the \times .

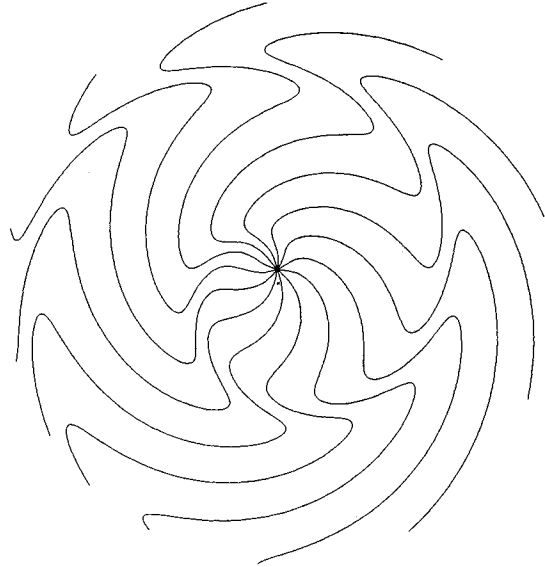


FIG. 4. Electric field lines for a charge moving at tangential speed $\beta=0.50$ on a circular path centered on the \times .

By a historical accident, the FORTRAN program to plot Eq. (11) used $(m - \frac{1}{2})$ in place of m itself. Clearly this substitution really makes no difference. The resulting electric field maps for various values of β are presented in Figs. 3-6. t_0 is 0 in each case. The center of the circular orbit is always marked by the small \times ; notice that the scale of representation differs in each diagram.

Before the above exact solution was devised, this problem was solved by numerical integration. For any (u, v) , the retarded time t' is obtained by using Newton's iterative method to solve the transcendental equation

$$[(u - \sin\omega t')^2 + (v - \cos\omega t')^2]^{1/2} = -t'.$$

Once an approximate t' is found, then $\mathbf{E}(u, v)$ comes out of Eq. (1); u and v are then incremented by $\Delta u = (E_x/|E|)\Delta s$ and $\Delta v = (E_y/|E|)\Delta s$ and the process repeated. The diagrams thus produced are almost exactly superimposable on those of the (R, α) explicit solution and thereby provide independent confirmation of the latter derivation. However, numerical integration requires a much longer program and up to fifteen times as much execution time, partly because the field lines have sharp kinks that force one to set Δs very small.

It should be physically obvious that an increase in t_0 , the moment of observation, corresponds

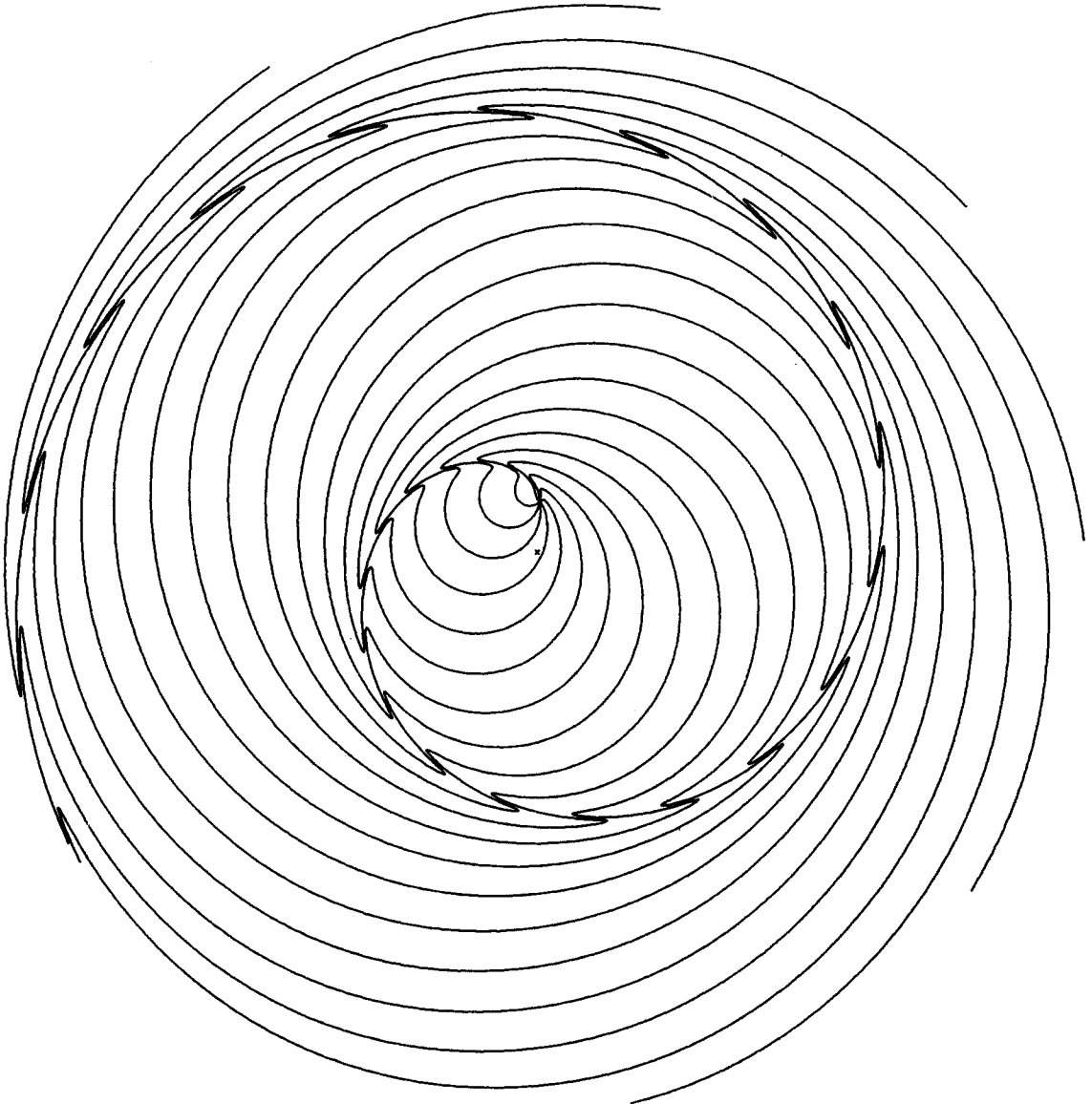


FIG. 5. Electric field lines for a charge moving at tangential speed $\beta = 0.90$ on a circular path centered on the \times .

merely to a rigid clockwise rotation of the entire diagram since the only preferred axis in the physical situation is the vector from the center of the orbit to the instantaneous position of the charge. Alternatively, simply note that Eq. (11) is independent of t_0 . Thus, to show how the field evolves in time, simply rotate the entire diagram clockwise. A convenient way to do this is to photocopy the picture, punch a hole through the \times , and place it on a spinning phonograph turntable. (Incidentally, that is why θ was

chosen negative rather than positive: Phonographs run clockwise.) The objection might be raised that beyond a radius of a/β , the field lines are moving faster than the speed of light, which is impossible. The answer is that an individual field line has no physical significance by itself; what really matters is the density of lines, *changes* in which must propagate at speed c . As may be verified by inspection, the actual radiation wavefronts (the kinks in the lines) really do travel radially outwards at that speed.

For low values of β , the radiation is basically isotropic and sinusoidal with angular frequency β/a , the same as the mechanical rotation frequency. (In conventional units $\beta c/a$). Near the charge, the field is clearly dominated by the quasistatic R^{-2} Coulomb field; further out, the transverse R^{-1} radiation takes over, causing greater curvature in the lines of force. As β is increased, the radiation tends to bunch into a spiral pattern with sharp kinks. These kinks are

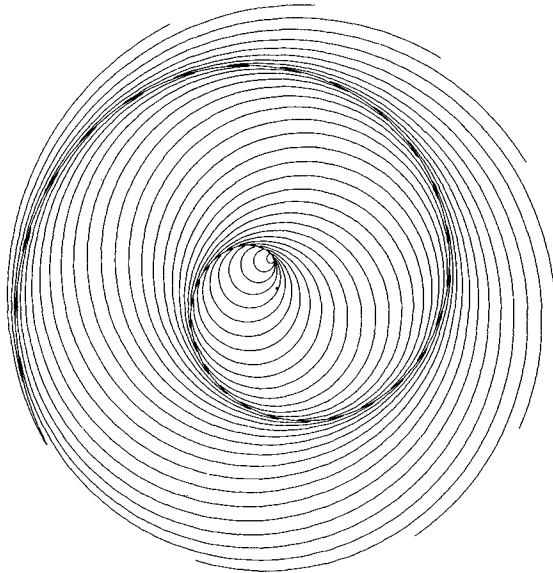


FIG. 6. Electric field lines for a charge moving at tangential speed $\beta=0.95$ on a circular path centered on the \times .

responsible for the rich harmonic content of relativistic synchrotron radiation. The spiral pattern centers around a curve whose equation is $\alpha=0$ or equivalently

$$u = -a \sin(\beta R/a) + R \cos(\beta R/a),$$

$$v = a \cos(\beta R/a) + R \sin(\beta R/a);$$

in the limit $\beta \rightarrow 1$, this becomes the evolute of the circle. When α is small and β close to 1, $1 - \hat{n} \cdot \beta = 1 - \beta \cos \alpha$ becomes small and makes $|E|$ large and $E \cdot \hat{\alpha}$ dominant over $E \cdot \hat{n}$. The width of the radiation pulse may be estimated with the help of Fig. 7. The points 1 and 2 in Fig. 7 that mark the edges of the pulse are characterized by $E \cdot \hat{\alpha} = 0$ there. As long as $R \gg (1 - \beta^2)a$, which it certainly is in the region of interest, $E \cdot \hat{\alpha}$ in Eq. (4) may be

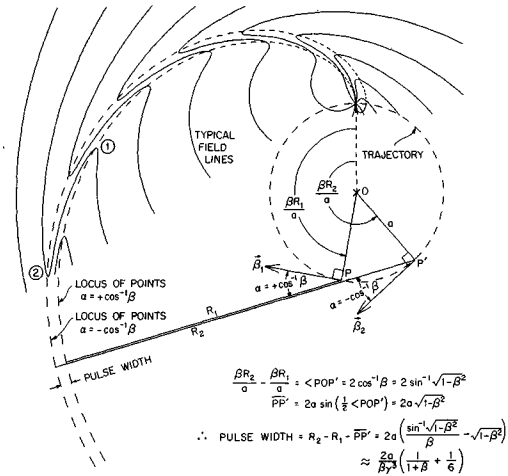


FIG. 7. A selective enlargement of Fig. 5 to show how to calculate the width of the synchrotron radiation pulse. The lines labeled R_1 and R_2 actually coincide but have been separated in this drawing for clarity.

approximated by just the second term on the right, the acceleration field. The condition that the latter is zero implies that at points 1 and 2, $\cos \alpha = \beta$, so that the pulse zone is bounded by the two curves $\alpha = \pm \cos^{-1} \beta$ that are drawn as dotted spirals in Fig. 7. A simple geometrical argument shows that the distance between these two curves (the pulse width) is

$$(2a/\beta) [\sin^{-1}(1 - \beta^2)^{1/2} - \beta(1 - \beta^2)^{1/2}]$$

$$\approx (2a/\beta \gamma^3) [\frac{1}{6} + (1 + \beta)^{-1}],$$

which is basically the same as the standard result⁵ though marginally more precise.

A second example of purely transversely accelerated motion, to a good approximation, is the small-angle scattering of a fast-moving charge off a stationary Coulomb potential. An energetic

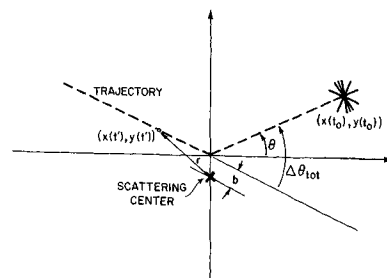


FIG. 8. Kinematics of small-angle high-energy repulsive Coulomb scattering.

particle of mass m , charge e_1 , and speed β encounters a fixed electrostatic potential, which might arise from an infinitely massive charge e_2 or just be the effective potential in the center-of-mass system. The incoming particle is assumed to have a sufficiently high energy γm and to strike at a sufficiently large impact parameter b so that at all times kinetic energy far exceeds potential energy in magnitude and so that the angular deflection is small. The kinematics of the collision are shown in Fig. 8; the total angular deflection $\Delta\theta_{\text{tot}}$ has been much exaggerated for clarity. Under the assumptions made above, the components of the central force parallel and perpendicular to the velocity are

$$F_{\parallel} = dp_{\parallel}/dt \approx e_1 e_2 (r^2 - b^2)^{1/2} r^{-3},$$

$$F_{\perp} = dp_{\perp}/dt \approx e_1 e_2 b r^{-3}.$$

One now makes the customary approximation that the total effect of F_{\parallel} on the electric field is negligible compared to the influence of F_{\perp} . This approximation is valid if γ is very large,⁶ or if kinetic energy always far exceeds potential energy, or if the observer is far enough away so that practically all he can see is a sudden impulsive angular deflection of the charge. Once F_{\parallel} is neglected, the rest is straightforward. β and γ are now constants:

$$d\theta = dp_{\perp}/p_{\parallel}$$

$$= e_1 e_2 b (\gamma m \beta)^{-1} r^{-3} dt$$

$$= e_1 e_2 b (\gamma m \beta)^{-1} (\beta^2 t'^2 + b^2)^{-3/2},$$

where t' is defined to be 0 at the point of closest approach. The result after integrating is

$$\theta = e_1 e_2 (\gamma m \beta b)^{-1} (\beta^2 t'^2 + b^2)^{-1/2} t' + C;$$

it is convenient then to take $C=0$ so that $\theta=0$ (so that the charge is moving exactly horizontally) at closest approach. Then $dx = \beta \cos\theta dt' \approx \beta dt'$ since θ is always small; $dy = \beta \sin\theta dt' \approx \beta \theta dt'$ similarly. Setting the origin of coordinates at the point of closest approach as well, one obtains the trajectory

$$x = \beta t', \quad y = e_1 e_2 (\gamma m b)^{-1} \beta^{-2} [(\beta^2 t'^2 + b^2)^{1/2} - |b|].$$

Since the charge is at every moment undergoing instantaneously circular motion with β and γ as constants, Eq. (10) may be applied directly to obtain

$$\tan \frac{1}{2} \alpha = (1 - \beta)^{1/2} (1 + \beta)^{-1/2} \tan \left(\frac{1}{2} e_1 e_2 (m \beta b)^{-1} \right. \\ \left. \times \{ (R - t_0) [\beta^2 (R - t_0)^2 + b^2]^{-1/2} \right. \\ \left. + t_0 (\beta^2 t_0^2 + b^2)^{-1/2} \} + \pi m / n \right).$$

Two examples, corresponding to medium and high velocities, are shown in Figs. 9 and 10. The dashed line indicates the trajectory of the particle up to the moment of observation t_0 , except that to the right of the scattering center (which is marked by the small \times) the trajectory is unfortunately covered up by one of the lines of force. The \times really represents only a repulsive potential; for an attractive potential, the \times should have been placed on the other side of the dashed line at the same impact parameter. Figure 9 may be taken to represent a 100-keV electron grazing an aluminum nucleus ($Z=13$) at an impact parameter of 2.0×10^{-10} cm; Fig. 10 was set arbitrarily.

Just as one would expect, the sudden deflection of the charge creates an expanding spherical shell containing a pulse of transverse radiation. As the kinks propagate outward, they become bigger and sharper as the acceleration field begins to dominate the velocity field. The peak intensity

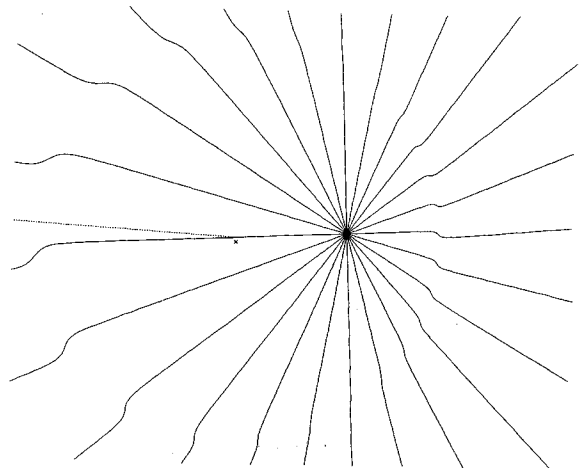


FIG. 9. Electric field lines for a charge of speed $\beta=0.547$ ($\gamma=1.195$) scattered through a total angular deflection $\Delta\theta_{\text{tot}}=3.0^\circ$.

of the radiation, which is crudely reflected in the density of the lines in the picture, goes up sharply as β approaches unity. To understand the angular distribution of the radiation, remember that at any moment t' on its trajectory, the charge is radiating most strongly dead ahead, just as if a headlamp were mounted on it; furthermore, the charge underwent its maximum acceleration at the point of closest approach to the scattering center, at which moment it was moving exactly horizontally. Thus the radiated power ought to be maximum directly horizontal from the point of closest approach (which happens to be the origin of coordinates). This direction is halfway between the line the charge would have followed

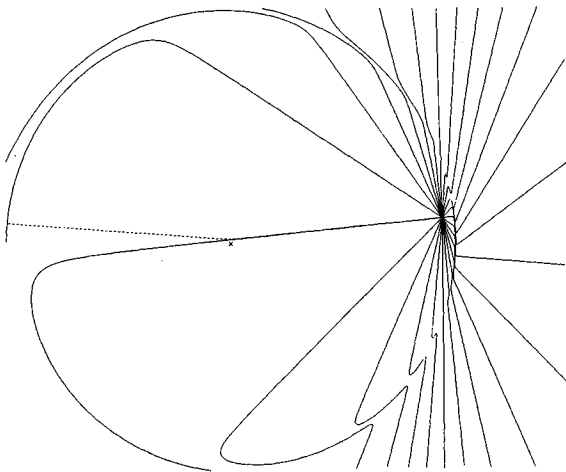


FIG. 10. Electric field lines for a charge of speed $\beta=0.943$ ($\gamma=3.00$) scattered through $\Delta\theta_{tot}=5.0^\circ$.

in the absence of the scatterer and the line it actually did take. As for the minimum, examine Eq. (4). The radiation due to acceleration corresponds to the second term in the expression for $\mathbf{E} \cdot \hat{\alpha}$; if that term is to be zero, then

$$(\beta \sin \alpha)(\hat{n} \cdot \hat{\beta}) = (\hat{\beta} \cdot \alpha)(1 - \beta \cos \alpha),$$

which leads (just as for Fig. 7) to the condition that $\cos \alpha = \beta$. Indeed, if one tries drawing a pair of lines which stick out from the origin to the right at angles $\pm \cos^{-1} \beta$, it is evident that along these lines there are no radiation kinks. This observation is confirmed by the conventional formula for the power radiated per unit solid angle, along the plane of motion, at angle α away from the velocity,

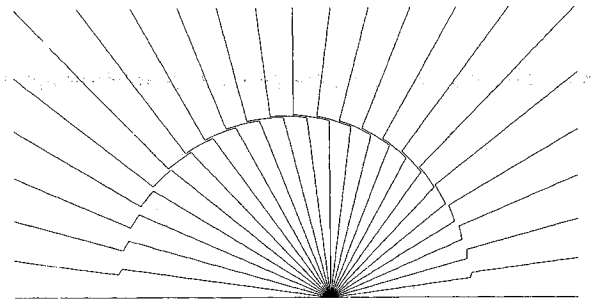


FIG. 11. Electric field lines of a charge accelerated from rest by a uniform force to a final velocity $\beta=0.20$, $t_0=32\Delta t$.

for a charge e in instantaneously circular motion⁷:

$$dP/d\Omega = (e^2\beta^2/4\pi c^3)(\beta - \cos \alpha)^2(1 - \beta \cos \alpha)^{-5},$$

which clearly has a null at $\cos \alpha = \beta$.

CHARGE MOVING IN ONE DIMENSION

Now $\hat{\beta}$ and $\hat{\beta}$ are parallel. Since the charge moves along a straight line, $\theta=0$. Therefore Eq. (8) becomes

$$d\alpha/dR = -\gamma^2(\hat{\beta} \cdot \hat{\alpha}) = \gamma^2\beta \sin \alpha$$

since the angle between $\hat{\beta}$ and $\hat{\alpha}$ is $\alpha + \frac{1}{2}\pi$; thus

$$\csc \alpha d\alpha = \gamma^2\beta dR = -\gamma^2\beta dt' = -(1-\beta^2)^{-1}d\beta,$$

where β is a function of $t' = t_0 - R$. Integrating,

$$\ln |\tan \frac{1}{2}\alpha| = -\frac{1}{2} \ln |(1+\beta)/(1-\beta)| + C;$$

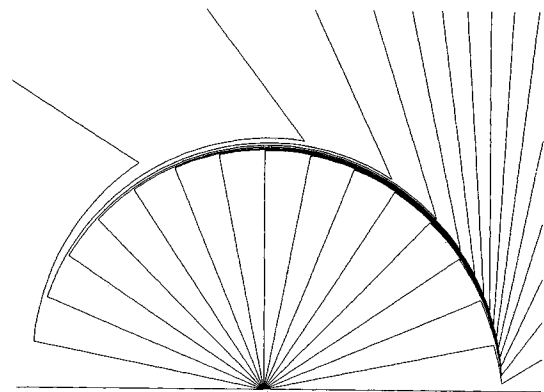


FIG. 12. Electric field lines of a charge decelerated from an initial velocity $\beta=0.95$ to rest by a uniform force, $t_0=16\Delta t$.

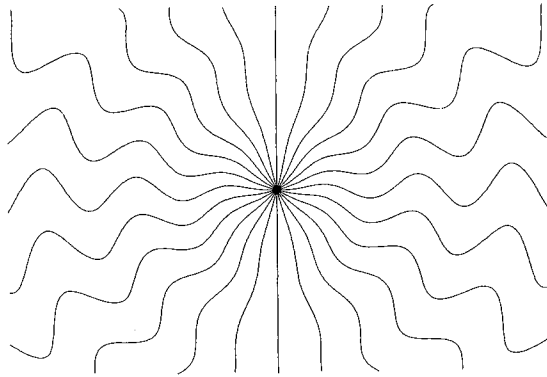


FIG. 13. Electric field lines of a charge undergoing one-dimensional simple harmonic motion with $\beta_{\max}=0.10$, $t_0=0$.

C is then adjusted to fit the initial condition, Eq. (9), that when $R=0$,

$$\tan \frac{1}{2}\alpha = (1-\beta)^{1/2}(1+\beta)^{-1/2} \tan(\pi m/n);$$

the net result is the equation for the line of force:

$$\tan \frac{1}{2}\alpha = [1-\beta(t_0-R)]^{1/2} \times [1+\beta(t_0-R)]^{-1/2} \tan(\pi m/n). \quad (12)$$

As a first application, consider a charge of mass m , moving along the x axis, accelerated from or decelerated to rest by a constant force F applied for a limited period Δt . The equation of motion is $F = (d/dt')(m\gamma\beta)$. Choosing the constants of integration to make the results as simple

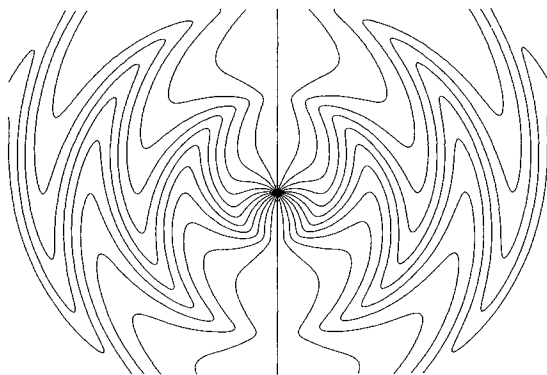


FIG. 14. Electric field lines of a charge undergoing one-dimensional simple harmonic motion with $\beta_{\max}=0.50$, $t_0=0$.

as possible, one finds

$$\begin{aligned} x(t') &= \pm [(m/F)^2 + t'^2]^{1/2} \mp |m/F| \\ \beta(t') &= \pm t' [(m/F)^2 + t'^2]^{-1/2} \quad \text{for } 0 < \pm t' < \Delta t, \\ \beta(t') &= \Delta t [(m/F)^2 + (\Delta t)^2]^{-1/2} \quad \text{for } \pm t' > \Delta t, \end{aligned}$$

where the upper and lower signs refer to acceleration ($+F$) and deceleration ($-F$), respectively. To convert these expressions to ordinary cgs units, replace m by mc^2 , t' by ct' , and Δt by $c\Delta t$. Now that $\beta(t')$ and $x(t')$ are known, Eqs. (12) and (3) generate Cartesian coordinates for the line of force as usual (Figs. 11-12). The field below the x axis has been omitted since obviously the field is cylindrically symmetrical about that axis.

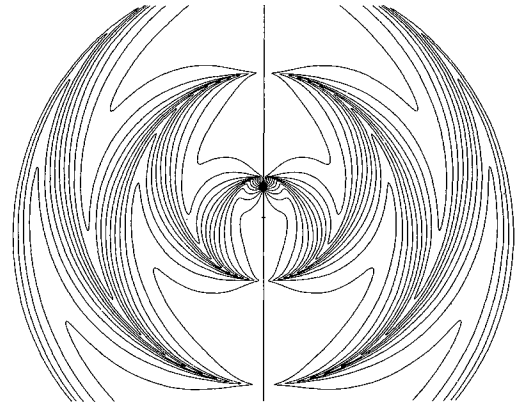


FIG. 15. Electric field lines of a charge undergoing one-dimensional simple harmonic motion with $\beta_{\max}=0.90$, $t_0 = \frac{1}{2}\pi/\omega$ so that $y(t_0) = a$.

The basic qualitative features of these diagrams confirm the discussion in elementary texts such as the Berkeley Physics Course.⁸ One can in addition get a crude idea of the angular distribution of the radiation by looking for where the field lines are bunched most tightly. For very low β 's the angle of maximum bunching is around $\frac{1}{2}\pi$, but it decreases as the accelerations and velocities increase in magnitude, so that when those get to be large, the radiation is beamed into a narrow cone surrounding the forward direction.

As a second example, let a charge be constrained to follow the path $x(t')=0$, $y(t')=a \sin \omega t'$, where $\omega = \beta_{\max}/a$ (simple harmonic motion). Notice that Eqs. (4)-(9) allow β to be negative so long as the basic relations $\beta_x = \beta \cos \theta$ and

$\beta_y = \beta \sin\theta$ are still satisfied, so that one may set $\beta = \beta_{\max} \cos\omega t'$ and $\theta = \frac{1}{2}\pi$. Correspondingly, α is an angle measured counterclockwise from the positive y axis even if the charge is momentarily moving in the negative y direction. From Eq. (11),

$$\tan\frac{1}{2}\alpha = [1 - \beta_{\max} \cos(\omega t_0 - \omega R)]^{1/2} \times [1 + \beta_{\max} \cos(\omega t_0 - \omega R)]^{-1/2} \tan(\pi m/n).$$

Like the circular orbit problem discussed above, this problem of one-dimensional simple harmonic motion was solved both by exact integration and numerical integration; as before, the two methods produced identical pictures but the analytical solution took much less computer time. The graphs presented here (Figs. 13–15) assume $t_0 = 0$ or $(\frac{1}{2}\pi/\omega)$, various β_{\max} values, and a scale of presentation adjusted for each picture so that two or so wavelengths would fit inside the frame. The two tic marks on the y axis (which unfortunately are sometimes hidden by the dense cluster of field lines) mark the limits of the sinusoidal excursion, $y = \pm a$. The physical interpretation of these pictures is a straightforward matter that is left to the interested reader as an exercise. A hint concerning the ultrarelativistic case (Fig. 15) is to look for areas of similarity to Figs. 5 and 12.

CHARGE UNDERGOING ARBITRARY MOTION IN THE PLANE

What can one say about the field lines if β and $\dot{\beta}$ are neither exactly parallel nor exactly perpendicular? Any arbitrary motion in the plane is in some sense a superposition of instantaneously circular and linear motions; the similarity of Eqs. (10) and (12) leads one to hope that an equation of the same general form might solve Eq. (8) in general. In fact, a little experimentation yields the desired general solution

$$\tan\frac{1}{2}\alpha = [1 - \beta(t_0 - R)]^{1/2} [1 + \beta(t_0 - R)]^{-1/2} \tan\psi,$$

where

$$\psi = -\frac{1}{2} \int_{t_0}^{t_0 - R} \gamma(\tau') \dot{\theta}(\tau') d\tau' + \frac{\pi m}{n}. \quad (13)$$

To verify this, differentiate with respect to $t' = t_0 - R$:

$$\begin{aligned} & (\sec^2\frac{1}{2}\alpha) \frac{1}{2} d\alpha/dt' \\ &= (1 - \beta)^{1/2} (1 + \beta)^{-1/2} (\sec^2\psi) d\psi/dt' \\ & \quad + \tan\psi (d/dt') [(1 - \beta)^{1/2} (1 + \beta)^{-1/2}]. \end{aligned}$$

Carrying out the differentiations indicated and solving for $d\alpha/dt'$ one finds

$$\begin{aligned} d\alpha/dt' &= 2(1 + \tan^2\frac{1}{2}\alpha)^{-1} (1 - \beta)^{1/2} (1 + \beta)^{-1/2} \\ & \quad \times [-\frac{1}{2}\gamma\dot{\theta}(1 + \tan^2\psi) - \gamma^2 \tan\psi (d\beta/dt')], \\ \frac{d\alpha}{dR} &= -\left(\frac{d\alpha}{dt'}\right) = \frac{\dot{\theta}(1 - \beta \cos\alpha) + (d\beta/dt') \sin\alpha}{(1 - \beta^2)}. \end{aligned}$$

In the above,

$$d\beta/dt' = (d/dt') (\beta \cdot \beta)^{1/2} = \dot{\beta} \cdot (\beta/\beta).$$

Also using the relations

$$\cos\alpha = \hat{n} \cdot \beta/\beta = (\hat{\alpha} \times \hat{z}) \cdot (\beta/\beta), \quad \sin\alpha = -\hat{\alpha} \cdot (\beta/\beta),$$

and

$$\dot{\theta} = (\beta \times \dot{\beta}/\beta^2) \cdot \hat{z},$$

one obtains

$$\begin{aligned} \frac{d\alpha}{dR} &= \gamma^2 \left\{ \dot{\theta} - \left[\dot{\beta} \cdot \left(\hat{z} \times \frac{\beta}{\beta} \right) \right] \left[\left(\hat{z} \times \frac{\beta}{\beta} \right) \cdot \hat{\alpha} \right] \right. \\ & \quad \left. - \left[\dot{\beta} \cdot \left(\frac{\beta}{\beta} \right) \right] \left[\left(\frac{\beta}{\beta} \right) \cdot \hat{\alpha} \right] \right\}. \end{aligned}$$

Since $(\hat{z} \times \beta/\beta)$ and (β/β) form a complete orthonormal basis for all vectors in the plane, then $d\alpha/dR = \gamma^2(\dot{\theta} - \dot{\beta} \cdot \hat{\alpha})$ as desired. Now that Eq. (13) has been proven, all that remains in principle is to evaluate ψ . For the four specific cases treated in this article, that evaluation was trivial. In the general case it will not be so; one might be forced to do the integral numerically. Then why not abandon the (R, α) method altogether and switch back to direct numerical integration of Eq. (1)? After all, the latter has the additional advantage that quantitative values for \mathbf{E} , which are often nice to have, may easily be printed out as the plot is being generated, whereas

a special calculation has to be done in the (R, α) method to get \mathbf{E} itself. On the other hand, the numerical integral for ψ is likely to be easier to do than the direct integration of E_x and E_y . More importantly, the (R, α) method completely bypasses the need to solve the retarded time equation

$$\{[u-x(t')]^2 + [v-y(t')]^2\}^{1/2} - R = 0.$$

This equation, usually transcendental, is a nuisance to solve, especially if x and y are available only as numerical tabulations since any small errors in R tend to accumulate rapidly into large errors in u and v .

Recently Eq. (13) has been applied successfully to depict the field from a free electron oscillating in response to an incident plane-polarized monochromatic electromagnetic wave of arbitrary intensity. In this system, β and $\dot{\beta}$ are neither

exactly parallel nor perpendicular, yet the integration for ψ may be performed explicitly.

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¹ J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chap. 14.

² Reference 1, p. 467, Eqs. (14.14) and (14.12).

³ E. M. Purcell, *Berkeley Physics Course* (McGraw-Hill, New York, 1965), Vol. II, p. 167.

⁴ Reference 1, p. 484.

⁵ Reference 1, p. 476, Eq. (14.49).

⁶ Reference 1, p. 476.

⁷ Reference 1, p. 474, Eq. (14.44) after substituting $\cos^2\phi = 1$ in the original formula.

⁸ Reference 3, pp. 164-167.

Misinterpretation of the Aharonov-Bohm Effect

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It is pointed out that accounts in the literature sometimes misinterpret the Aharonov-Bohm effect, involving the shift in interference pattern for electrons passing a long solenoid. The quantum description and experimental verification of the effect are reviewed, and it is emphasized that both theory and experiment indicate that there is no deflection of the average momentum of the electrons passing the solenoid. There is no average force on the electrons due to the solenoid. It is remarked that the Aharonov-Bohm effect involves the shift of double-slit interference fringes which is independent of the quantum unit of action \hbar and yet allows a natural classical limit.

INTRODUCTION

When a long solenoid is placed between the slits of a double-slit electron interference pattern, the pattern is shifted even though the external magnetic field of the solenoid is arbitrarily small. This effect, predicted by Aharonov and Bohm¹ in 1959, has been confirmed experimentally.²⁻⁵ The quantum analysis for the effect has fueled a lively controversy as to the concept of force in physics, and as to whether or not one should regard the electromagnetic potentials V and \mathbf{A} as holding a new and dominant role in quantum theory in contrast to their place subordinate to the fields \mathbf{E} and \mathbf{B} in classical electromagnetic theory.

The Aharonov-Bohm effect and also these authors' interpretation of a new role for the potentials seems to be accepted in the literature, appearing even in that primer for graduate students and professors, *The Feynman Lectures on Physics*.^{6,7} Both among physicists and in the accounts in the literature, however, there are