EFFECTIVELY NOWHERE SIMPLE RELATIONS ON COMPUTABLE STRUCTURES

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ABSTRACT. Let $\mathcal{A}$ be a computable structure and let R be an additional relation on its domain. The notion of "quasi-simplicity" of R on $\mathcal{A}$, first studied by G. Hird, is analogous to the computability-theoretic notion of simplicity, given the definability of various subrelations of $\neg R$. In the present paper, we define corresponding versions of the notions "nowhere simple" and "effectively nowhere simple." We establish a sufficient condition for existence of noncomputable effectively nowhere simple relations on a restricted class of computable structures.

1. INTRODUCTION AND NOTATION

We will denote structures by script letters, and their domains by the corresponding capital Latin letters. Let $\mathcal{A}$ be a structure for L. For $X \subseteq A$, let L_X be the language $L \cup \{\mathbf{a} : a \in X\}$, L expanded by adding a constant $\mathbf{a}$ for every $a \in X$. Let $\mathcal{A}_X = (\mathcal{A}, a)_{a \in X}$ be the expansion of $\mathcal{A}$ to the language L_X such that for every $a \in X$, $\mathbf{a}$ is interpreted by a. The *atomic diagram* of $\mathcal{A}$ is the set of all atomic and negated atomic sentences of L_A which are true in $\mathcal{A}_A$. A structure $\mathcal{A}$ is computable if its domain A is a computable subset of ω and its relations and operations are uniformly computable. That is, $\mathcal{A}$ is computable if its atomic diagram is computable.

A structure $\mathcal{B}$ isomorphic to a computable structure $\mathcal{A}$ is not necessarily computable. However, even if $\mathcal{B}$ is computable, it can still lose many of the computable properties of $\mathcal{A}$. One of the important and interesting questions in computable model theory is how a specific aspect of a computable structure may change if the structure is isomorphically transformed so that it remains computable.

Let $\mathcal{A}$ be a fixed computable structure. A computable property of $\mathcal{A}$ which Ash and Nerode considered is given by a new computable relation R on the domain A of $\mathcal{A}$. Here, we call R *new* if R is not named in the language of $\mathcal{A}$. By $Im_{\mathcal{A}}(R)$ we denote the set of images of R under all isomorphisms from $\mathcal{A}$ to other computable structures. For example, Ash and Nerode [2] investigated the conditions under which every relation in $Im_{\mathcal{A}}(R)$ must be c.e. Such R is called *intrinsically* *c.e.* on $\mathcal{A}$. In [7] we found conditions for $(\mathcal{A}, R)$ under which there is a c.e. relation in $Im_{\mathcal{A}}(R)$ of an arbitrary c.e. degree.

A sequence of variables displayed after a formula contains a subsequence of all free variables occurring in the formula. An $L_{\omega_1\omega}$ formula $\alpha(\overrightarrow{x})$ is a Σ_1 formula if it is equivalent to a formula of the form

$$\bigvee_{i\in I} \exists \overrightarrow{y_i}\theta_i(\overrightarrow{x}, \overrightarrow{y_i}),$$

where for every $i \in I$, $\theta_i(\overrightarrow{x}, \overrightarrow{y_i})$ is a finitary quantifier-free formula. If the index set I is c.e., then $\alpha(\overrightarrow{x})$ is a *computable* Σ_1 formula. If I is finite, then $\alpha(\overrightarrow{x})$ is a finitary Σ_1 formula.

For simplicity, we shall assume throughout the text that all new relations on computable structures are unary. We fix a new computable relation R on the domain of a computable structure $\mathcal{A}$ for a language L. Ash and Nerode [2] introduced a computable syntactic condition for R on $\mathcal{A}$, called being a formally c.e. relation. The relation R is *formally c.e.* (on $\mathcal{A}$) if R can be defined by a computable Σ_1 formula with parameters. That is, there is a sequence $\overrightarrow{c} \in A^{<\omega}$ and a computable Σ_1 formula $\alpha(\overrightarrow{\mathbf{c}}, x)$ such that the following equivalence holds for every $a \in A$:

$$R(a) \Leftrightarrow \mathcal{A}_A \models \alpha(\overrightarrow{\mathbf{c}}, \mathbf{a}).$$

Clearly, if R is formally c.e. on $\mathcal{A}$, then R is intrinsically c.e. on $\mathcal{A}$. Ash and Nerode [2] also established the converse for a computable relation R, under an additional decidability condition on $(\mathcal{A}, R)$.

Let $\mathcal{B}$ be a computable structure for L, and let X be a new relation the domain B. The complement of X with respect to B is $\overline{X}$. Let $\mathbf{R}$ be a symbol for X. Since we are interested in the case when X is c.e., certain first-order formulae with positive occurrences of $\mathbf{R}$ in the expanded language $L \cup \{\mathbf{R}\}$ play a special role (see [3], [7] and [10]). A Σ_1 formula in $L \cup \{\mathbf{R}\}$, possibly with individual constants (parameters), in which $\mathbf{R}$ occurs only positively is also called a Σ_1^Γ formula. This notation was introduced in [1], where a hierarchy of infinitary formulae was defined in a general setting in which Γ is a function assigning computable ordinals to relation symbols. A unary relation F on B is definable by a computable Σ_1^Γ formula $\alpha(\overrightarrow{d}, x)$ (with parameters $\overrightarrow{d} \in B^{<\omega}$) if for every $b \in B$:

$$F(b) \Leftrightarrow (\mathcal{B}_B, X) \models \alpha(\overrightarrow{\mathbf{d}}, \mathbf{b}).$$

The subsets of $\overline{X}$ which are definable by computable Σ_1^Γ formulae, and their subsets, play the role of finite sets to some extent.

We say that the relation X is *quasi-simple* on $\mathcal{B}$ if X is c.e., $\overline{X}$ is not definable by a computable Σ_1^Γ formula, and for every c.e. $W \subseteq \overline{X}$, there is a unary relation F definable by a computable Σ_1^Γ formula such that

$$W \subseteq F \subseteq \overline{X}.$$

Quasi-simplicity was first investigated by Hird [8], [10]. He proved that, under certain decidability conditions on $(\mathcal{A}, R)$, $Im_{\mathcal{A}}(R)$ contains a quasi-simple relation. Ash, Knight and Remmel [3] gave conditions on $(\mathcal{A}, R)$ which are sufficient for obtaining a quasi-simple relation of an arbitrary c.e. degree in $Im_{\mathcal{A}}(R)$.

For a formula θ, let $\theta^1 =_{def} \theta$ and $\theta^0 =_{def} \neg\theta$. If f is a partial function, then $dom(f)$ is the domain of f, $rng(f)$ is the range of f, and $f(a)\downarrow$ denotes that $a \in dom(f)$. The length of a sequence $\overrightarrow{x}$ is denoted by $lh(\overrightarrow{x})$. If $\overrightarrow{x} = (x_0, \dots, x_{m-1})$ and f is a function, then $f(\overrightarrow{x}) =_{def} (f(x_0), \dots, f(x_{m-1}))$. The concatenation of sequences is denoted by $\hat{}$.

We fix $\langle \cdot, \cdot \rangle$ to be a computable bijection from ω^2 onto ω, which is strictly increasing with respect to both coordinates. Let $W_0, W_1, \dots$ be a standard computable enumeration of all c.e. sets. By $\equiv_T$ we denote Turing equivalence of sets.

2. Nowhere simple relations on structures

Shore [15] introduced the concepts of nowhere simple and effectively nowhere simple sets in computability theory. Let $X \subseteq \omega$. We say that X is *nowhere simple* if X is c.e., and for every c.e. set W such that $W - X$ infinite, there is an infinite c.e. set W' such that $W' \subseteq W - X$. Thus, nowhere simplicity is definable in the lattice of all c.e. sets. Similarly, X is *effectively nowhere simple* if X is c.e., and there is a unary computable function g such that for every $e \in \omega$, $W_{g(e)} \subseteq W_e - X$ and

$$W_e - X \text{ is infinite} \Rightarrow W_{g(e)} \text{ is infinite.}$$

The function g is called a *witness function* for X. Clearly, every computable set is effectively nowhere simple. Miller and Remmel [12] established that X is effectively nowhere simple if there is a c.e. set W such that $W \cap X = \emptyset$ and for every $e \in \omega$,

$$W_e - X \text{ is infinite} \Rightarrow W_e \cap W \text{ is infinite.}$$

The set W is called a *witness set* for X. Hence, effective nowhere simplicity is definable in the lattice of all c.e. sets. Shore [15], and

Miller and Remmel [12] proved that every c.e. Turing degree contains an effectively nowhere simple set.

Let $\mathcal{V}_\infty$ be a computable infinite dimensional vector space over a computable field F. We also assume that $\mathcal{V}_\infty$ has a dependence algorithm. We consider only subspaces of $\mathcal{V}_\infty$. For any subset $S \subseteq V_\infty$, by $\mathcal{S}^*$ we denote the subspace (with the domain S^*) generated by S. Let $V_e = W_e^*$. Then $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \ldots$ is a computable list of all c.e. subspaces of $\mathcal{V}_\infty$. A subspace $\mathcal{V}$ of $\mathcal{V}_\infty$ is *nowhere simple* (see [14], [4] and [5]) if $\mathcal{V}$ is c.e., and for every $e \in \omega$, there is a c.e. subspace $\mathcal{V}'_e$ such that $V'_e \subseteq V_e$, $V'_e \cap V = \{0\}$, and

$$dim(\mathcal{V}_e/\mathcal{V}) = \infty \Rightarrow dim(\mathcal{V}'_e) = \infty.$$

If, in addition, there is a unary computable function g such that for every $e \in \omega$, $V'_e = V_{g(e)}$ then $\mathcal{V}$ is called *effectively nowhere simple*. Nerode and Remmel [14] proved that a nowhere simple subset of a computable basis of $\mathcal{V}_\infty$ generates a nowhere simple subspace of $\mathcal{V}_\infty$. Similarly, an effectively nowhere simple subset of a computable basis generates an effectively nowhere simple subspace. Let $\mathbf{c}$ be a nonzero c.e. Turing degree. Downey and Remmel [5] established that $\mathbf{c}$ contains a simple subset of a given computable basis of $\mathcal{V}_\infty$, which generates an effectively nowhere simple subspace $\mathcal{V}$ of $\mathcal{V}_\infty$. Moreover, $\mathcal{V}$ has a *witness space* $\mathcal{W}$, where $\mathcal{W}$ is c.e., $W \cap V = \{0\}$ and for every $e \in \omega$,

$$dim(\mathcal{V}_e/\mathcal{V}) = \infty \Rightarrow dim([\mathcal{V}_e \cap (\mathcal{W} \cup \mathcal{V})^*]/\mathcal{V}) = \infty.$$

Downey and Remmel [5] proved that the existence of a witness space guarantees effective nowhere simplicity. It is not known whether every effectively nowhere simple subspace has a witness space. It is not even known for an effectively nowhere simple subspace $\mathcal{V}$ and $v \in V_\infty - V$, whether $V \cup \{v\}$ generates an effectively nowhere simple subspace.

We consider the following general definition of nowhere simplicity and effective nowhere simplicity of relations on computable structures.

Definition 2.1. *Let $\mathcal{B}$ be a computable structure and let X be a new unary relation on B. Then X is nowhere simple on $\mathcal{B}$ if X is c.e., and for every c.e. set W, there is a c.e. set $W' \subseteq W - X$ such that if $W - X$ is not contained in any subset F of $\overline{X}$ definable by a computable Σ_1^Γ formula, then the same is true of W'.*

The relation X is effectively nowhere simple on $\mathcal{B}$ if there is a computable unary function g such that for every $e \in \omega$, $W_{g(e)} \subseteq W_e - X$ and if $W_e - X$ is not contained in any subset F of $\overline{X}$ definable by a computable Σ_1^Γ formula, then the same is true of $W_{g(e)}$.

3. Main result

We fix a computable structure $\mathcal{A}$ for a language L, and a new computable unary relation R on A. Let $\overrightarrow{c} \in A^{<\omega}$ and $a \in A$. The property of $\overrightarrow{c}$ and a, termed a is free over $\overrightarrow{c}$, played an essential role in the results in [7] and [3]. We say that a is *free over* $\overrightarrow{c}$ if $a \in \overline{R}$ ($\overline{R} = A - R$) and for every finitary Σ_1^Γ formula $\alpha(\overrightarrow{c}, x)$, if

$$(\mathcal{A}_A, R) \models \alpha(\overrightarrow{c}, \mathbf{a})$$

then

$$(\exists a' \in R)[(\mathcal{A}_A, R) \models \alpha(\overrightarrow{c}, \mathbf{a}')].$$

Let the set of all free elements over $\overrightarrow{c}$ be denoted by $fr(\overrightarrow{c})$. Note that $fr(\overrightarrow{c}) \subseteq \overline{R}$. Clearly, if $a \in fr(\overrightarrow{c})$ and $\overrightarrow{d}$ is a subsequence of $\overrightarrow{c}$, then $a \in fr(\overrightarrow{d})$. As in [3], let

$$bd(\overrightarrow{c}) =_{def} \{a \in \overline{R}\colon a \text{ is not free over } \overrightarrow{c}\}.$$

Thus, if $a \in bd(\overrightarrow{c})$ and $\overrightarrow{c}$ is a subsequence of $\overrightarrow{d}$, then $a \in bd(\overrightarrow{d})$. A maximal relation on $\overline{R}$ (with respect to the set-theoretic inclusion) which is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$ is of the form $bd(\overrightarrow{c})$. Conversely, if $bd(\overrightarrow{c})$ is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$, then $bd(\overrightarrow{c})$ is a maximal relation on $\overline{R}$ definable by such a formula.

Theorem 3.1. *Let the following conditions hold for a computable structure $\mathcal{A}$ and a new computable unary relation R on A:*

(1) for every $\overrightarrow{c} \in A^{<\omega}$, there is an element a such that $a \in fr(\overrightarrow{c})$,

(2) for every $\overrightarrow{c} \in A^{<\omega}$, $bd(\overrightarrow{c})$ is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$,

(3) there is an algorithm which for a given $\overrightarrow{c} \in A^{<\omega}$ and $a \in A$, decides whether $a \in fr(\overrightarrow{c})$ (equivalently, whether $a \in bd(\overrightarrow{c})$),

(4) for every $\overrightarrow{c} \in A^{<\omega}$ and $a, v \in \overline{R}$, if $a \in fr(\overrightarrow{c})$, and $v \in bd(\overrightarrow{c})$, then $a \in fr(\overrightarrow{c}\,\hat{}\,v)$,

(5) for every $\overrightarrow{c} \in A^{<\omega}$, $\overrightarrow{u} \in \overline{R}^{<\omega}$ and $a, v \in \overline{R}$, if $a \in fr(\overrightarrow{c}\,\hat{}\,\overrightarrow{u})$, $v \in fr(\overrightarrow{c}\,\hat{}\,a)$, and $v \in fr(\overrightarrow{u})$, then $a \in fr(\overrightarrow{c}\,\hat{}\,\overrightarrow{u}\,\hat{}\,v)$.

Then there is a computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that the image of R on B is noncomputable and effectively nowhere simple on $\mathcal{B}$.

Proof. For simplicity, we assume that $A = \omega$. We will construct a computable structure $\mathcal{B}$ with domain $B = \omega$ and an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ such that $f^{-1}(R)$ is noncomputable and effectively nowhere simple on $\mathcal{B}$. Let s be an arbitrary stage of the construction. We will define a finite set Ψ^s of formulae of the open diagram of $\mathcal{B}$ and certain formulae of the form $\mathbf{R}(\mathbf{b})$ for $b \in B$. We will also define a finite partial isomorphism f_s from $\mathcal{B}$ to $\mathcal{A}$. A finite isomorphism from $\mathcal{B}$ to $\mathcal{A}$ at stage s, is an injective function p with a finite domain such that for every $\theta \in \Psi^s$, if $\theta = \theta(\mathbf{b}_0, \dots, \mathbf{b}_{l-1})$ for some $b_0, \dots, b_{l-1} \in B$, then $p(b_0) \downarrow, \dots, p(b_{l-1}) \downarrow$ and $\mathcal{A} \vDash \theta[p(b_0), \dots, p(b_{l-1})]$, and if $\mathbf{R}(\mathbf{b}) \in \Psi^s$ then $p(b) \in R$. Let $X_s = f_s^{-1}(R)$. Define $\Psi^{-1} = \emptyset$, $f_{-1} = \emptyset$ and $X_0 = \emptyset$. At the end of the construction, we will have that $f = \lim_{s \to \infty} f_s$ exists. Let $X =_{def} f^{-1}(R)$ and $\Psi = \bigcup_{s \geq 0} \Psi^s$. The construction will ensure that X is c.e.

Let $(\theta_e)_{e \in \omega}$ be an effective list of all atomic formulae in L_A. The construction will meet the following requirements for every $e \geq 0$,

$P_e^0 : (\theta_e \in \Psi$ or $\neg\theta_e \in \Psi)$, and $(b \in X \Rightarrow \mathbf{R}(\mathbf{b}) \in \Psi)$;

$P_e^1 : e \in dom(f)$;

$P_e^2 : e \in rng(f)$;

$Q_e : \overline{X} \neq W_e$;

$N_{\langle e,k \rangle}$: If $W_e - X$ is not contained in any subset of $\overline{X}$ definable by a computable Σ_1^Γ formula, we define $u_{e,k}$ such that $u_{e,k} \in W_e - X$ and

$$f(u_{e,k}) \in fr(f(u_{e,0}, \hat{} \dots \hat{} u_{e,k-1})),$$

where for $i \in \{0, \dots, k-1\}$, $u_{e,i}$ is previously defined by $N_{\langle e,i \rangle}$.

The priority ordering of the requirements is:

$$P_0^0, P_0^1, P_0^2, Q_0, N_0, P_1^0, P_1^1, P_1^2, Q_1, N_1, \dots.$$

Let $e, i \in \omega$. We set that at stage -1, $u_{e,i}$ is undefined. To assure effective nowhere simplicity, we will have that the element $u_{e,i}$ once defined will remain unchanged, and $\{u_{e,j} : j \in \omega \wedge (u_{e,j}$ is defined$)\}$ will be a required c.e. subset W_e' of $W_e - X$. Since the construction will be effective uniformly in e, it will follow from the $s-m-n$ theorem that there is a unary computable function g such that $W_{g(e)} = W_e'$.

The strategy for meeting a single requirement Q_e is to chose an element d_e such that

$$d_e \in W_e \Leftrightarrow d_e \in X.$$

We will have $d_e = \lim_{s \to \infty} d_e^s$, where d_e^s is the approximation to d_e at stage s, and d_e^s may be undefined for finitely many s. If d_e^s is defined, then $d_0^s, \dots, d_{e-1}^s$ are also defined. We assume that for every $e \in \omega$, d_e^{-1} is undefined. For $n \in \omega$, let $\overrightarrow{d_{<n}^s} =_{def} (d_0^s, \dots, d_e^s)$, where e is the largest number such that $e < n$ and d_e^s is defined. The construction will ensure that all elements of $\overrightarrow{d_{<n}^s}$ are in $dom(f_s)$.

Let $\overrightarrow{u^s}$ be the vector consisting of all elements $u_{e,i}$, $e, i \in \omega$, which are defined at stage s (listed in the order in which they are first defined). The construction will ensure that all elements of $\overrightarrow{u^s}$ are contained in $dom(f_s)$. For $k \in \omega$, if $u_{e,k}$ is defined at some stage, then $u_{e,o}, \dots, u_{e,k-1}$ are also defined at that stage. Let $\overrightarrow{u_{e,<k}} =_{def} (u_{e,o}, \dots, u_{e,k-1})$. We have the following definitions which will be used in the construction.

Requirement P_e^0 requires attention and can be attacked at s if $\theta_e \notin \Psi^{s-1}$, $\neg\theta_e \notin \Psi^{s-1}$ and all elements of B occurring in θ_e are in the domain of f_{s-1}. Requirement P_e^0 also requires attention if θ is of the form $\mathbf{R}(\mathbf{b})$, $\theta \notin \Psi^{s-1}$, $f_{s-1}(b)\downarrow$ and $f_{s-1}(b) \in R$.

Requirement P_e^1 requires attention and can be attacked at s if $e \notin dom(f_{s-1})$.

Requirement P_e^2 requires attention and can be attacked at s if $e \notin rng(f_{s-1})$.

Requirement Q_e requires attention and can be attacked at s if d_e^{s-1} is undefined at $s-1$, or $d_e^{s-1} \in W_{e,s} - W_{e,s-1}$.

Requirement $N_{\langle e,k\rangle}$ requires attention at s if $u_{e,k}$ is undefined at $s-1$. Such a requirement $N_{\langle e,k\rangle}$ can be attacked at s if $u_{e,o}, \dots, u_{e,k-1}$ are defined at $s-1$, all elements of $\overrightarrow{c}$, where

$$\overrightarrow{c} = (0, \dots, \langle e,k\rangle)\hat{\ }(f_{s-1}^{-1}(0), \dots, f_{s-1}^{-1}(\langle e,k\rangle))\hat{\ }\overrightarrow{d^{s-1}_{<_{\langle e,k\rangle+1}}},$$

are in $dom(f_{s-1})$, and there is $u \in W_{e,s}$ such that $u \in dom(f_{s-1})$, and

$$f_{s-1}(u) \in fr(f_{s-1}(\overrightarrow{c}\hat{\ }\overrightarrow{u_{e,<k}})).$$

Notice that $f_{s-1}(u) \in \overline{R}$.

Requirement P_e^0 is never injured.

Requirement P_e^1 is injured at s if $f_s(e) \neq f_{s-1}(e)$.

Requirement P_e^2 is injured at s if $f_s^{-1}(e) \neq f_{s-1}^{-1}(e)$.

Requirement Q_e is injured at s if d_e^{s-1} is defined, and d_e^s is undefined; or if d_e^s is defined, $d_e^s \in \overline{R}$, and

$$f_s(d_e^s) \in bd(f_s(0, \dots, e)\hat{\ }(0, \dots, e)\hat{\ }f_s(\overrightarrow{d_{<e}^s})\hat{\ }f_s(\overrightarrow{u^s})).$$

Requirement $N_{\langle e,k\rangle}$ is never injured.

Construction

Stage $s \geq 0$

Let Req be the highest priority requirement which requires attention and which can be attacked at stage s.

(i) $Req = P_e^0$

Let $\theta_e = \theta_e(\mathbf{b}_0, \dots, \mathbf{b}_{l-1})$ for some $b_0, \dots, b_{l-1} \in \omega$. Define $\Psi^s = \Psi^{s-1} \cup \{\theta_e^k\}$, where $k \in \{0,1\}$ is such that $\mathcal{A} \vDash \theta_e^k[f_{s-1}(b_0), \dots, f_{s-1}(b_{l-1})]$.

Let $f_s =_{def} f_{s-1}$. The constants associated with Q-requirements are as at stage $s-1$.

(ii) $Req = P_e^1$

We define $f_s = f_{s-1} \cup \{(e,a)\}$, where a is the least unused element in A. If $a \in R$, we set $\Psi^s = \Psi^{s-1} \cup \{\mathbf{R}(\mathbf{e})\}$; otherwise, we set $\Psi^s = \Psi^{s-1}$. The constants associated with Q-requirements are as at stage $s-1$.

(iii) $Req = P_e^2$

We define $f_s = f_{s-1} \cup \{(b,e)\}$, where b is the least unused element in B. If $e \in R$, we set $\Psi^s = \Psi^{s-1} \cup \{\mathbf{R}(\mathbf{b})\}$; otherwise, we set $\Psi^s = \Psi^{s-1}$. The constants associated with Q-requirements are as at stage $s-1$.

(iv) $Req = Q_e$

Let $\overrightarrow{c} = (0,\ldots,e)\hat{\ }(f_{s-1}^{-1}(0),\ldots,f_{s-1}^{-1}(e))\hat{\ }\overrightarrow{d_{<e}^{s-1}}\hat{\ }\overrightarrow{u^{s-1}}$. It follows by construction that all elements of $\overrightarrow{c}$ are in $dom(f_{s-1})$.

Case (a): d_e^{s-1} is undefined.

Let a be the least element in $A - f_{s-1}(W_{e,s})$ such that a is free over $f_{s-1}(\overrightarrow{c})$. This a exists by condition (1) of the theorem. If $a \in ran(f_{s-1})$, we set $d_e^s =_{def} f_{s-1}^{-1}(a)$ and $f_s =_{def} f_{s-1}$. If $a \notin ran(f_{s-1})$, we choose b to be the least new element in $B - W_{e,s}$, and set $d_e^s =_{def} b$ and $f_s =_{def} f_{s-1} \cup \{(b,a)\}$. Notice that in both subcases, $d_e^s \notin X_s$ and $d_e^s \notin W_{e,s}$. For every $e' \neq e$, let $d_{e'}^s =_{def} d_{e'}^{s-1}$. Let $\Psi^s =_{def} \Psi^{s-1}$.

Case (b): $d_e^{s-1} \in W_{e,s} - W_{e,s-1}$.

It will follow by construction that $a = f_{s-1}(d_e^{s-1})$ is free over $f_{s-1}(\overrightarrow{c})$. Hence $a \in \overline{R}$. Let $\psi(\overrightarrow{\mathbf{c}}, x)$ be the Σ_1^Γ formula with parameters $\overrightarrow{\mathbf{c}}$ which captures the part of the diagram of $(\mathcal{B}, X)$, determined by stage $s-1$. That is, $\psi(\overrightarrow{\mathbf{c}}, x) = (\exists \overrightarrow{y})\delta(\overrightarrow{\mathbf{c}}, x, \overrightarrow{y})$, where $\delta(\overrightarrow{\mathbf{c}}, d_e^{s-1}, \overrightarrow{\mathbf{b}})$ is the conjunction of all formulas of Ψ^{s-1}, and $lh(\overrightarrow{y}) = lh(\overrightarrow{b})$. Clearly, $(\mathcal{B}, X) \vDash \psi[\overrightarrow{c}, d_e^{s-1}]$, so $(\mathcal{A}, R) \vDash \psi[f_{s-1}(\overrightarrow{c}), a]$. We choose a' to be the least element such that $a' \in R$ and $(\mathcal{A}, R) \vDash \psi[f_{s-1}(\overrightarrow{c}), a']$. Let $\overrightarrow{a} \in A^{lh(\overrightarrow{b})}$ be the least sequence such that $(\mathcal{A}, R) \vDash \delta[f_{s-1}(\overrightarrow{c}), a', \overrightarrow{a}]$. We set

$$f_s =_{def} (\overrightarrow{c}\hat{\ }d_e^{s-1}\hat{\ }\overrightarrow{b}, f_{s-1}(\overrightarrow{c})\hat{\ }a'\hat{\ }\overrightarrow{a}).$$

For every $e' \leq e$, let $d_{e'}^s =_{def} d_{e'}^{s-1}$, and for every $e' > e$, let $d_{e'}^s$ be undefined. Let $\Psi^s =_{def} \Psi^{s-1} \cup \{\mathbf{R}(\mathbf{b}) : b \in X_s - X_{s-1}\}$.

(v) $Req = N_j$, where $j = \langle e, k \rangle$.

Let $\overrightarrow{c_j} = (0,\ldots,j)\hat{\ }(f_{s-1}^{-1}(0),\ldots,f_{s-1}^{-1}(j))\hat{\ }\overrightarrow{d_{<j+1}^{s-1}}\hat{\ }\overrightarrow{u_{e,<k}}$. It follows by construction that all elements of $\overrightarrow{c_j}$ are in $dom(f_{s-1})$. We choose

the least element $u \in W_{e,s}$ such that $u \in dom(f_{s-1})$ and $f_{s-1}(u) \in fr(f_{s-1}(\overrightarrow{c_j}))$. Hence $u \notin X_{s-1}$. We set

$$u_{e,k} =_{def} u.$$

For every $e' \leq j$, let $d^s_{e'} =_{def} d^{s-1}_{e'}$, and for every $e' > j$, let $d^s_{e'}$ be undefined. Let $f_s =_{def} f_{s-1}$ and $\Psi^s =_{def} \Psi^{s-1}$.

End of the construction.

Lemma 3.2. *No Q-requirement is injured by an action of a lower priority N-requirement.*

Proof. Let us consider N_j for some $j = \langle e, k\rangle$, and $Q_{e'}$ for some $e' \leq j$. Let s be a stage at which the requirement N_j is being satisfied. We will use the same notation as in (v) of the construction. We assume that $f_{s-1}(d^{s-1}_{e'})$ is free over

$$f_{s-1}(0, \dots, e')\hat{\ }(0, \dots, e')\hat{\ }f_{s-1}(\overrightarrow{d^{s-1}_{<e'}})\hat{\ }f_{s-1}(\overrightarrow{u^{s-1}}).$$

We want to show that $f_s(d^s_{e'})$ remains free over

$$f_s(0, \dots, e')\hat{\ }(0, \dots, e')\hat{\ }f_s(\overrightarrow{d^s_{<e'}})\hat{\ }f_s(\overrightarrow{u^s}).$$

Since $d^s_{e'} = d^{s-1}_{e'}$, $f_s(d^s_{e'}) = f_{s-1}(d^{s-1}_{e'})$, $f_s(0, \dots, e') = f_{s-1}(0, \dots, e')$, $\overrightarrow{d^s_{<e'}} = \overrightarrow{d^{s-1}_{<e'}}$, $f_s(\overrightarrow{d^s_{<e'}}) = f_{s-1}(\overrightarrow{d^{s-1}_{<e'}})$, $\overrightarrow{u^s} = \overrightarrow{u^{s-1}}\hat{\ }u_{e,k}$, $f_s(\overrightarrow{u^{s-1}}) = f_{s-1}(\overrightarrow{u^{s-1}})$, and $f_s(u_{e,k}) = f_{s-1}(u_{e,k})$, it is sufficient to conclude that $f_{s-1}(d^{s-1}_{e'})$ is free over

$$f_{s-1}(0, \dots, e')\hat{\ }(0, \dots, e')\hat{\ }f_{s-1}(\overrightarrow{d^{s-1}_{<e'}})\hat{\ }f_{s-1}(\overrightarrow{u^{s-1}})\hat{\ }f_{s-1}(u_{e,k}).$$

First, we assume that $f_{s-1}(u_{e,k}) \in bd(f_{s-1}(\overrightarrow{u^{s-1}}))$. Then the required conclusion follows from condition (4) of the theorem if we set

$$\overrightarrow{c} = f_{s-1}(0, \dots, e')\hat{\ }(0, \dots, e')\hat{\ }f_{s-1}(\overrightarrow{d^{s-1}_{<e'}})\hat{\ }f_{s-1}(\overrightarrow{u^{s-1}}),$$

$$a = f_{s-1}(d^{s-1}_{e'}) \text{ and } v = f_{s-1}(u_{e,k}).$$

That is, $a \in fr(\overrightarrow{c})$ by our first assumption, and $v \in bd(\overrightarrow{c})$ since, by our second assumption, v is bounded over a subsequence of $\overrightarrow{c}$.

Now, we assume that $f_{s-1}(u_{e,k}) \in fr(f_{s-1}(\overrightarrow{u^{s-1}}))$. Then the required conclusion follows from condition (5) of the theorem if we set

$$\overrightarrow{c} = f_{s-1}(0, \dots, e')\hat{\ }(0, \dots, e')\hat{\ }f_{s-1}(\overrightarrow{d^{s-1}_{<e'}}),$$

$$\overrightarrow{u} = f_{s-1}(\overrightarrow{u^{s-1}}),$$

$$a = f_{s-1}(d_{e'}^{s-1}) \text{ and } v = f_{s-1}(u_{e,k}).$$

That is, $a \in fr(\overrightarrow{c}\,\hat{}\,\overrightarrow{u})$ by our first assumption, $v \in fr(\overrightarrow{c}\,\hat{}\,a)$ by a condition in (v) of the construction, and $v \in fr(\overrightarrow{u})$ by our second assumption. ∎

It follows from the construction that a Q-requirement can be injured at most once by every higher priority N-requirement. Since P^1-requirements, P^2-requirements and Q-requirements may be injured only by higher priority Q-requirements, it is easy to show that each requirement is attacked and injured only finitely often. Thus, all P-requirements are met. Hence there is an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ such that $f = \lim\limits_{s\to\infty} f_s$. The construction is effective because of the condition (3) of the theorem. Therefore, $\mathcal{B}$ is a computable structure. Let $X = \bigcup\limits_{s\geq 0} X_s$. Clearly, X is c.e. It follows by construction that $X = f^{-1}(R)$. Since every Q-requirement is satisfied, X is noncomputable.

Lemma 3.3. *Every N-requirement is satisfied.*

Proof. We fix $e \in \omega$. If $u_{e,k}$ is defined for every $k \in \omega$, then every $N_{\langle e,k\rangle}$ is satisfied. Therefore, let us assume that k is the least number such that $u_{e,k}$ is not defined. Let s_0 be a stage by which all requirements of higher priority than $N_{\langle e,k\rangle}$ have been attacked for the last time. Hence, at s_0, the sequence of numbers in B coming from the higher priority P-requirements and Q-requirements has reached its final value $\overrightarrow{b}$. Then for $s > s_0$, for every $u \in W_{e,s} - X_s$ such that $u \in dom(f_{s-1})$, we have that $f_{s-1}(u) \in bd(f(\overrightarrow{b})\,\hat{}\,f(u_{e,<k}))$. It then follows from condition (2) of the theorem that $W_e - X$ is contained in a subset of $\overline{X}$ which is definable by a computable Σ_1^Γ formula. Hence $N_{\langle e,k\rangle}$ is satisfied for every k. ∎

Finally, we will prove that X is effectively nowhere simple on $\mathcal{B}$. Let $e \in \omega$. If for some k, $u_{e,k}$ is not defined, then $W_e - X$ is contained in some $F \subseteq \overline{X}$ such that F is definable by a computable Σ_1^Γ formula. We now assume that $u_{e,k}$ is defined for every k. We set $W_e' = \{u_{e,0}, u_{e,1}, u_{e,2}, \dots\}$. It is enough to show that W_e' is not contained in any subset of $\overline{X}$ which is definable by a computable Σ_1^Γ formula. Let us assume otherwise. That is, let W_e' be contained in some subset of $\overline{X}$ which is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{d}$. Then $f(u_{e,k}) \in bd(f(\overrightarrow{d}))$ for every k. This is not possible since, by (v) of the construction, for every k such that $\langle e,k\rangle \geq \max(rng(\overrightarrow{d}))$, we have that $f(u_{e,k}) \in fr(f(\overrightarrow{d}))$. ∎

Theorem 3.1 can be modified, by using coding and permitting, as described in Theorem 2.5 in [7], to obtain an effectively nowhere simple set in $Im_{\mathcal{A}}(R)$ of an arbitrary c.e. degree.

Theorem 3.4. *Let conditions (1)-(5) from Theorem 3.1 hold for a unary computable relation R on the domain of a computable structure $\mathcal{A}$. Then, for every c.e. degree* $\mathbf{c}$, *there is a computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that the image X of R in B is of degree* $\mathbf{c}$ *and effectively nowhere simple on $\mathcal{B}$.*

Proof. We just sketch the main idea. Let $C \subseteq \omega$ be an arbitrary noncomputable c.e. set such that at every stage s, C receives at most one new element c, and $c \leq s$. We replace the Q-requirements of Theorem 3.1 by the requirement $X \equiv_T C$. At every stage s of the construction, we have a sequence of movable markers on the elements in $\overline{X}_s$, where $\overline{X}_s = \{d_0^s < d_1^s < d_2^s < \dots\}$. More precisely, we effectively define a partial binary function γ with the intention that if $c \in C_s - C_{s-1}$, then $\gamma(c, s-1) \downarrow$ and $d_{\gamma(c,s-1)}^{s-1} \in X_s$. ∎

Example 3.1. *Let $\mathcal{A}$ be $(\omega, =)$ and let $R \subseteq A$ be a computable infinite co-infinite set. It is shown in* [3] *that $\mathcal{A}$ and R satisfy conditions (1)-(4). The condition (5) is also satisfied because for $\overrightarrow{c} \in A^{<\omega}$ and $a \in \overline{R}$, we have*

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \notin rng(\overrightarrow{c}).$$

Hence, we obtain Shore's and Miller-Remmel's theorem that every c.e. Turing degree contains an effectively nowhere simple set.

Example 3.2. *Let $\mathcal{A}$ be a computable linear order of type η and let $R \subseteq A$ be a computable dense co-dense set. It is shown in* [3] *that $\mathcal{A}$ and R satisfy conditions (1)-(4). The condition (5) is also satisfied because for $\overrightarrow{c} \in A^{<\omega}$ and $a \in \overline{R}$, we have*

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \notin rng(\overrightarrow{c}).$$

Thus, we have the following corollary.

Corollary 3.5. *For every c.e. set C, there is a computable linear order of type η with a dense co-dense effectively nowhere simple set X such that $X \equiv_T C$.*

Example 3.3. *Let $\mathcal{B}$ be a computable linear order of type $\omega + \omega^*$ and let X be the ω-part of B. Then it is easy to see that X can not be nowhere simple unless it is computable. That is, if X is c.e. and if W is an infinite c.e. set such that $W \subseteq \overline{X}$, then X is computable. On the other hand, Ash, Knight and Remmel* [3] *have shown that for any* *noncomputable c.e. set C, there is a computable linear order of type $\omega + \omega^*$ with the ω-part X such that X is a simple set and $X \equiv_T C$. Indeed, if $\mathcal{A} = (\omega, \prec)$ is a computable linear order of type $\omega + \omega^*$ with the computable ω-part R, it is shown in* [3] *that conditions (1)-(4) are satisfied. Condition (5) is not satisfied because for $\overrightarrow{c} \in A^{<\omega}$ and $a \in \overline{R}$, we have*

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \prec c_0,$$

where $ran(\overrightarrow{c}) \cap \overline{R} = \{c_0 \prec \ldots \prec c_l\}$.

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