Book Review

Sergei S. Goncharov. **Countable Boolean Algebras and Decidability.**
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Boolean algebras are mathematical structures important in many branches of mathematics and computer science. Thus, there are several approaches to Boolean algebras. In *Countable Boolean Algebras and Decidability*, Goncharov examines Boolean algebras from the algebraic, model-theoretic and computable model-theoretic perspectives. The model-theoretic part focuses on countable Boolean algebras, and leads naturally to a study of effectiveness in Boolean algebras. The computable model-theoretic approach follows, for the most part, the Russian school of constructive mathematics, which is closely connected with the theory of enumerations. This book is an extension of the author's previous book *Countable Boolean Algebras*, ("Nauka," 1988), available only in Russian.

The book is self-contained and includes a substantial survey of necessary results from classical and computable model theory. It is divided into three chapters. Each chapter begins at the beginning, with a sizable review section. In some sense, each chapter is independent: the author introduces notions when they are needed, and repeats them often, so a knowledgable reader can read any chapter of the book without reading the previous ones. Almost every section ends with exercises. The reader should not skip over the exercises, because they often contain important results. In a few cases, results in the exercises are used in the text without explicit reference. The book ends with a relatively small index, but an extensive bibliography.

Chapter 1 starts with a survey of elementary logic, which introduces first-order languages, models, and partially ordered sets. It then defines a Boolean algebra as an *algebraic system* (a *model*) with one unary and two binary operations. Boolean algebras are closely related to Boolean lattices and Boolean rings. The rest of Chapter 1 develops the relationships between Boolean algebras and other mathematical structures. It first reviews the basic algebraic concepts of quotient algebras, canonical homomorphisms, and ideals in the case of Boolean algebras. A sequence of iterated Frechét ideals leads to the ordinal type of a Boolean algebra. The important notion of an atom is introduced and the classification of Boolean algebras as atomless, atomic, and superatomic is discussed. Countable superatomic Boolean algebras are exactly those with at most countably many ultrafilters. A later section, in Chapter 2, examines how the notions of atomic and superatomic Lindenbaum algebras relate to prime and countably saturated models. Chapter 1 also presents the standard topological analysis of Boolean algebras by establishing the set-theoretic version of Stone's representation theorem and by defining the Stone topology on the ultrafilters of a Boolean algebra. A linear order with a least element generates a Boolean algebra, in such a way that the ultrafilters on Boolean algebra uniquely correspond to the non-empty initial segments of the linear order. Furthermore, every countable Boolean algebra is generated by a binary tree. In particular, the full binary tree generates atomless Boolean algebras. The final section of the chapter explores the so-called Ershov algebras, a generalization of Boolean algebras. A Boolean algebra can be viewed as an Ershov algebra with a greatest element, and an Ershov algebra can be viewed as an ideal of a Boolean algebra. Vaught's isomorphism theorem for Boolean algebras is extended to the case of Ershov algebras. The isomorphism types of countable Ershov algebras are completely characterized.

Chapter 2 begins with a survey of notions and results of basic model theory of countable models. (The term enumerable set of formulae is used for a computably enumerable set of formulae.) While some proofs are somply outlined, others, such as the characterization of model complete theories, are given completely. The omitting types theorem is proved by using a topological approach. An *elementary characteristic*, which is a triple of invariants, is associated with every Boolean algebra by using a sequence of ideals, which the author calls Ershov-Tarski ideals. This characteristic completely determines the theory of a Boolean algebra. Hence it follows that the theory of any Boolean algebra is decidable. A later section establishes that every so-called admissible triple of numbers from $\omega + 1$ can be realized as the elementary characteristic of a Boolean algebra. The chapter continues by characterizing a countably saturated Boolean algebra, and a countably homogeneous Boolean algebra, and by establishing the isomorphism of certain countable Boolean algebras. It is easy to see that the theory of Boolean algebras (in the usual language of three operation symbols) is not model complete. But the theory of Boolean algebras in a language expanded by infinitely many suitable predicate symbols (such as $Atom_n$, $Atomless_n$, and so forth) is model complete. Moreover, it is shown that in certain situations it is enough to consider only fragments of the whole extension.

Chapter 3, the largest chapter, investigates computable aspects of Boolean algebras. It starts with a survey of computability and computable model theory. This section introduces the fundamental notions of a *constructive* and a *strongly constructive* model. As the author points out, these notions correspond in the West to the notions of a *computable* and a *decidable* model, respectively. An infinite model is computable (decidable) if its domain is ω and its atomic (elementary) diagram is a decidable theory. A model has a constructivization (or a strong constructivization) if and only if it is isomorphic to a computable (or decidable, respectively) model. A constructive (strongly constructive) model is a model with any countable domain, together with a constructivization (strong constructivization). This section has exercises which include interesting facts about algorithmic properties of Boolean algebras. The chapter further investigates the relationship of constructive Boolean algebras with computable generating linear orders, as well as with computably enumerable generating trees. The chapter then returns to the general computable model theory and proves fundamental results involving realizing and omitting types of decidable theories in decidable models. It gives a complete proof of a characterization of a homogeneous model with a strong constructivization. From this theorem, similar characterizations of prime and saturated models with strong constructivizations follow. In particular, a countably saturated Boolean algebra is isomorphic to a decidable one. Another rather technical section investigates when a given Boolean algebra with various algorithmic properties is isomorphic to a decidable one. The *algorithmic dimension* of a computable model is the number of computable isomorphism types. Two computable models are of the same computable isomorphism type if they are computably isomorphic. A model is *autostable* (*computably categorical*) if its algorithmic dimension is 1. A series of beautiful results involving the algorithmic dimension of Boolean algebras is given, involving characterization of autostable computable Boolean algebras as those with finitely many atoms, and determining that the algorithmic dimension of a Boolean algebra can only be $0, 1$, or ∞. Algorithmic properties of subalgebras and quotient algebras of computable Boolean algebras are studied, and it is shown that the theory of the lattice of computably enumerable subalgebras of a computable Boolean algebra is undecidable. The chapter ends with results on automorphism groups of countable Boolean algebras and the information that they provide about the models.

This is a Russian book with a strong Western influence. Sometimes, a single section presents a mix of Western and Russian results, often following the organization of the original proofs. The book is well translated. It contains a rich collection of results on Boolean algebras and is suitable for those interested in Boolean algebras, model theory, or computable model theory.