Kleene's O, Harrison orderings, and Turing degree spectra

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- We consider *countable* structures for *computable* languages.

- In measuring complexity, we identify a structure $\mathcal{A}$ with its *atomic diagram* $D(\mathcal{A})$.

- *Intrinsic*: bounds on complexity of some aspect of a structure are *preserved under isomorphisms*.

- Let $\mathcal{A}$ be a *computable* structure, and let R be an additional relation on $\mathcal{A}$.

(i) R is *intrinsically* $\mathcal{P}$ *on* $\mathcal{A}$ if in all *computable* isomorphic copies of $\mathcal{A}$, the image of R is $\mathcal{P}$.

(ii) R is *relatively intrinsically* $\mathcal{P}$ *on* $\mathcal{A}$ if in *all* isomorphic copies $\mathcal{B}$ of $\mathcal{A}$, the image of R is $\mathcal{P}$ relative to $\mathcal{B}$.

- Results on *definability versus* intrinsic computability-theoretic complexity of relations.

- Results on *intrinsically c.e.* and *intrinsically* $\mathbf{\Sigma}^0_\alpha$ relations.

- Equivalent syntactic conditions for a relation to be *intrinsically* $\mathbf{\Pi}^1_1$ on a structure.

- Examples of computable structures $\mathcal{A}$ with intrinsically $\mathbf{\Pi}^1_1$ relations R (Harrison orderings with the maximal well-ordered initial segments).

- Turing degree spectra of these relations, and connections with $\mathbf{\Pi}^1_1$ *paths* through *Kleene's* O.

Kleene's system of notations for computable ordinals

Consists of a set O of notations, with a partial ordering $<_O$:

- 0 gets notation 1,
- a is a notation for $\alpha \Rightarrow 2^a$ is a notation for $\alpha + 1$; $a <_O 2^a$,
- α is a limit ordinal

  φ_e a total function with values forming a sequence of notations for an increasing sequence of ordinals with limit α

  $\Rightarrow 3 \cdot 5^e$ is a notation for α;

  $$(\forall n)[\varphi_e(n) <_O 3 \cdot 5^e].$$

$\alpha = |a|$ is the ordinal with notation a

$$pred(a) =_{def} \{b \in O : b <_O a\}$$

- $(pred(a), <_O)$ well-ordering of type $|a|$
- $pred(a)$ c.e., uniformly in a
- O is Π^1_1 complete
- a Π^1_1 subset of O is Δ^1_1 iff

  it is contained in $\{b \in O : |b| < \alpha\}$ for some computable ordinal α

Paths through O

A path P through O is *regular* if for $a \in P$, $pred(a)$ is *computable*, uniformly in a.

- (Feferman and Spector)

  There are no $\mathbf{\Sigma}_1^1$ paths through O. There are $\mathbf{\Pi}_1^1$ paths through O.

- (Jockusch)

  (i) Every $\mathbf{\Pi}_1^1$ path through O is Turing equivalent to a regular path.

  (ii) (Parikh) There is a $\mathbf{\Pi}_1^1$ path P through O such that $K \leq_m P$, where K is the halting set.

- (Friedman)

  O is computable in some $\mathbf{\Pi}_1^1$ path through O.

(Goncharov, Harizanov, Knight, Shore)

- There exist two Turing incomparable Π^1_1 paths through O.

- For any computable ordinal α, there is a pair of Π^1_1 paths that are incomparable with respect to $\equiv_{\Delta^0_\alpha}$.

- (Steel, Harrington) There is a Π^1_1 path P through O such that no noncomputable hyperarithmetical set is computable in P.

- There exist two Π^1_1 paths through O whose Turing degrees form a minimal pair.

- There is no Π^1_1 path of minimal Turing degree.

Classification of computable (infinitary) formulas

- A computable Σ_0 (Π_0) formula is a finitary open formula.

- A computable Σ_α formula is a *c.e. disjunction* of formulas
$$\exists \overline{u}\, \psi(\overline{x}, \overline{u}),$$
where ψ is computable Π_β for some $\beta < \alpha$.

- A computable Π_α formula is a *c.e. conjunction* of formulas
$$\forall \overline{u}\, \psi(\overline{x}, \overline{u}),$$
where ψ is computable Σ_β for some $\beta < \alpha$.

- (Ash) The relation defined in a countable structure $\mathcal{A}$ by a computable Σ_α (Π_α) formula is Σ^0_α (Π^0_α) relative to the open diagram of $\mathcal{A}$.

Kresel-Barwise compactness theorem

Let Γ be a Π^1_1 set of computable (infinitary) sentences.

- If every Δ^1_1 subset has a model, then Γ has a model.

- If every Δ^1_1 subset has a *computable* model, then Γ has a *computable* model.

- Let $\mathcal{A}$ be a hyperarithmetical structure. Let $\vec{a}, \vec{b}$ be tuples of the same length, satisfying the same computable (infinitary) formulas. Then then there is an *automorphism* of $\mathcal{A}$ mapping $\vec{a}$ to $\vec{b}$.

(Ash and Nerode)

(i) The relation R is *formally c.e.* on $\mathcal{A}$ if it is definable by a computable $\mathbf{\Sigma}_1$ formula with finitely many parameters.

(ii) Under some effectiveness condition (enough to have the existential diagram of $(\mathcal{A}, R)$ computable), R is *intrinsically c.e.* on (computable) $\mathcal{A}$ iff R is *formally c.e.* on $\mathcal{A}$.

(Ash-Knight-Manasse-Slaman, Chisholm)

R is relatively intrinsically c.e. on $\mathcal{A}$ iff it is formally c.e. on $\mathcal{A}$ (*no additional effectiveness* needed).

(Manasse) There is an intrinsically c.e., but *not relatively* intrinsically c.e. relation.

(Barker) Under some effectiveness conditions, R is *intrinsically* $\mathbf{\Sigma}^0_\alpha$ on $\mathcal{A}$ iff R is *formally* $\mathbf{\Sigma}_\alpha$ on $\mathcal{A}$.

(Goncharov, Harizanov, Knight and McCoy)

A relation S on a structure $\mathcal{B}$ is *immune relative to* $\mathcal{B}$ if S is infinite and S contains no infinite subset that is c.e. relative to $\mathcal{B}$.

Let R be an infinite co-infinite relation on computable $\mathcal{A}$. The following are equivalent:

(i) For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms f from $\mathcal{A}$ onto $\mathcal{B}$, $f(R)$ is not immune relative to $\mathcal{B}$.

(ii) There are an infinite set D and a finite tuple $\vec{c}$ such that $D \subseteq R$ and D is definable in $\mathcal{A}$ by a computable $\mathbf{\Sigma_1}$ formula $\psi(x, \vec{c})$.

Similar result holds for *relatively simple relations*.

The set of Turing degrees of images of R in computable copies of $\mathcal{A}$ is called the *degree spectrum of* R, $DgSp(R)$.

(Harizanov) Under some effectiveness condition, if R is *intrinsically c.e.* and *not intrinsically computable*, then $DgSp(R)$ includes *all c.e. degrees*.

(Note that $\overline{R}$ is not definable in (A, R) by a computable $\mathbf{\Sigma}_1$ formula in which the symbol R occurs only positively.)

Degrees coarser than Turing degrees:

$$X \leq_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow X \leq_T Y \oplus \mathbf{\Delta}^0_\alpha$$

$$X \equiv_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow (X \leq_{\mathbf{\Delta}^0_\alpha} Y \wedge Y \leq_{\mathbf{\Delta}^0_\alpha} X)$$

($\equiv_{\mathbf{\Delta}^0_1}$ is $\equiv_T$)

(Ash and Knight) Under some effectiveness conditions, if R is *not intrinsically* $\mathbf{\Delta}^0_\alpha$ on computable $\mathcal{A}$, then for every $\mathbf{\Sigma}^0_\alpha$ set C, there is an isomorphism f from $\mathcal{A}$ onto a computable structure such that $f(R) \equiv_{\mathbf{\Delta}^0_\alpha} C$.

(Not possible to replace these by Turing degrees.)

(Harizanov) Under some effectiveness condition (enough to have the existential diagram of $(\mathcal{A}, R)$ computable), if R is *not intrinsically computable*, then $DgSp(R)$ includes *all c.e. degrees*.

Note that at least one of R, $\neg R$ is not definable in A by a computable $\mathbf{\Sigma_1}$ formula with parameters.

(Moses, Hirschfeldt) A *computable* relation R on a *computable linear ordering* is either definable by a *quantifier-free* formula with parameters (hence R is intrinsically computable), or $DgSp(R)$ is infinite.

(Downey, Goncharov and Hirschfeldt)

A *computable* relation on a *computable Boolean algebra* is either definable by a *quantifier-free* formula with parameters, or $DgSp(R)$ is infinite.

(Hirschfeldt, Khoussainov, Shore and Slinko) Let T be any of the following theories

symmetric irreflexive directed graphs,

lattices,

rings,

integral domains of arbitrary characteristic,

commutative semigroups,

2-step nilpotent groups.

Then for every countable graph $\mathcal{G}$ and relation U on its domain, there is a structure $\mathcal{A} \models T$ and a relation R on $\mathcal{A}$ such that $DgSp(U) = DgSp(R)$.

(Kueker) The following are equivalent for a relation R on a (countable) structure $\mathcal{A}$:

(i) R has fewer than $2^{\aleph_0}$ different images under automorphisms of $\mathcal{A}$;

(ii) R is definable in $\mathcal{A}$ by an $L_{\omega_1\omega}$ formula with finitely many parameters.

(Harizanov, Ash-Cholak-Knight)

For a computable relation R on computable $\mathcal{A}$, if $DgSp(R)$ contains *every* Δ^0_3 *Turing degree*, obtained via an automorphism f of the same Turing degree as $f(R)$, then $DgSp(R)$ contains *every Turing degree*.

Intrinsically $\mathbf{\Delta}_1^1$ relations

(Soskov, revisited)

Suppose that $\mathcal{A}$ is computable, R is $\mathbf{\Delta}_1^1$ and invariant under automorphisms of $\mathcal{A}$. Then R is definable in $\mathcal{A}$ by a computable (infinitary) formula (without parameters).

For a computable $\mathcal{A}$, and R, the following are equivalent:

(i) R is intrinsically $\mathbf{\Delta}_1^1$ on $\mathcal{A}$,

(ii) R is relatively intrinsically $\mathbf{\Delta}_1^1$ on $\mathcal{A}$,

(iii) R is definable in $\mathcal{A}$ by a computable formula with finitely many parameters.

R is intrinsically $\mathbf{\Delta}_1^1$ on $\mathcal{A}$

$\Rightarrow R$ has countably many automorphic images

$\Rightarrow (\exists \vec{c})$ [R invariant under automorphisms of $(A, \vec{c})$]

$\Rightarrow R$ definable by a computable formula $\psi(x,\vec{c})$.

Intrinsically Π^1_1 relations

A relation R on $\mathcal{A}$ is *formally* Π^1_1 *on* $\mathcal{A}$ if it is definable in $\mathcal{A}$ by a Π^1_1 disjunction of computable (infinitary) formulae with finitely many parameters.

(Soskov) For a computable structure $\mathcal{A}$ and a relation R on $\mathcal{A}$, the following are equivalent:

(i) R is *relatively intrinsically* Π^1_1 on $\mathcal{A}$,

(ii) R is *formally* Π^1_1 on $\mathcal{A}$.

(Goncharov, Harizanov, Knight, Shore) For a computable structure $\mathcal{A}$ and a relation R, the following are equivalent:

(i) R is *intrinsically* Π^1_1 on $\mathcal{A}$,

(ii) R is *relatively intrinsically* Π^1_1 on $\mathcal{A}$.

(Goncharov, Harizanov, Knight, Shore) Assume R is a Π^1_1 relation on a computable structure $\mathcal{A}$, which is *invariant* under automorphisms of $\mathcal{A}$. Then R is formally Π^1_1.

- For each $a \in R$ (assume R unary), the *orbit* $O_1(a)$ is a Σ^1_1 subset of R, with index computed from a.

- By *Kleene's Separation Theorem*, there is a Δ^1_1 set $D_1(a)$ separating $O_1(a)$ and $\neg R$, with index computed from a.

- For each $a \in R$, we can find a Δ^1_1 index for

$$D(a) =_{def} \bigcup_{n \geq 1} D_n(a)$$

such that

$$a \in D(a) \subseteq R$$

$D(a)$ is invariant under automorphisms of $\mathcal{A}$

A *Harrison ordering* is a *computable* ordering of type

$$\omega_1^{CK}(1+\eta),$$

where η is the order type of the rationals.

For a Harrison ordering $\mathcal{A}$, let $R^{\mathcal{A}}$ be the initial segment of type ω_1^{CK}.

$R^{\mathcal{A}}$ is *intrinsically* Π^1_1 since it is defined by the disjunction of computable infinitary formulae saying that the interval to the left of x has order type α, for computable ordinals α.

(Harrison) For any *computable tree* $T \subseteq \omega^{<\omega}$, if T has paths but *no hyperarithmetic paths*, then the *Kleene-Brouwer ordering* on T is a computable ordering of type $\omega_1^{CK}(1+\eta)+\alpha$, for some computable ordinal α.

(Goncharov, Harizanov, Knight, Shore)

For an ordering $(R,<)$ of type ω_1^{CK}, the following are equivalent:

(i) There is a Harrison ordering $\mathcal{A}$ whose $R^{\mathcal{A}}$ is $(R,<)$.

(ii) $(R,<)$ is Π^1_1 and for $a \in R$, $pred(a)$ (the set of elements to the left of a) is computable, uniformly in a.

(iii) $(R,<)$ is the maximal well-ordered initial segment in a computable linear ordering $\mathcal{B}$ that has initial segments of type α for every computable ordinal α.

(iv) (iii) $+$ $\mathcal{B}$ has no infinite hyperarithmetical decreasing sequences.

(Goncharov, Harizanov, Knight, Shore)

The following sets are equal:

(i) The set of Turing degrees of maximal well-ordered initial segments of Harrison orderings.

(ii) The set of Turing degrees of maximal well-ordered initial segments of computable orderings, whose order types are not computable ordinals.

(iii) The set of Turing degrees of Π^1_1 paths through O.

(iii) The set of Turing degrees of *left-most* paths of computable subtrees of $\omega^{<\omega}$ in which there is a path but not a hyperarithmetical one.

(iv) The set of Turing degrees of *left-most* paths of computable subtrees of $\omega^{<\omega}$ in which there is a path and th left-most one is not hyperarithmetical.

If P is a Π^1_1 path through O, then there is a Harrison ordering $\mathcal{A}$ with maximal well-ordered initial segment R such that $R \equiv_T P$.

There is a regular path R through O such that $P \equiv_T R$.

There is a Harrison ordering $\mathcal{A}$ whose maximal well-ordered initial segment is R, ordered by $<_O$.

If $\mathcal{A}$ is a computable linear ordering with the maximal well-ordered initial segment L. If L is not hyperarithmetical, then it has order type ω_1^{CK} and there is a Π^1_1 path P through O such that $L \equiv_T P$.

Replace $\mathcal{A}$ by another Harrison ordering $\mathcal{A}^*$: each $a \in A$ replaced by a copy of ω with elements $\langle a, n \rangle$ ordered lexicographically. Let the maximal well-ordered initial segment of $\mathcal{A}^*$be L^*. Hence, $L^* \equiv_T L$.

Define (using the Recursion Theorem) a computable function f, mapping L^* isomorphically onto an initial segment of O. Moreover, for $x = \langle a, 0 \rangle$, $x < f(x)$.

$f[L^*]= P$ is a path through O, and $L^* \equiv_T P$.

Recall that for a linear ordering $\mathcal{A}$,

the *interval algebra* $I(\mathcal{A})$ is

Boolean algebra generated

by the half-open intervals $[a, b)$, $[a, \infty)$, with endpoints in $\mathcal{A}$.

$I(\omega)$: atoms are intervals $[n, n+1)$

$I(\eta)$: atomless Boolean algebra

b is an *atomic* element $\Leftrightarrow$

$(\forall x \leq b)[x \neq 0 \Rightarrow (\exists y \leq x)(y \text{ is an atom})]$

Congruence relations $\sim_a$ on a Boolean algebra B

$x \sim_0 y \Leftrightarrow x = y$

$x \sim_1 y \Leftrightarrow x, y$ differ in B by finitely many atoms

$x \sim_{\alpha+1} y \Leftrightarrow x/\sim_\alpha, y/\sim_\alpha$ differ in $B/\sim_\alpha$ by finitely many atoms

α limit, $\quad x \sim_\alpha y \Leftrightarrow (\exists \beta < \alpha)[x \sim_\beta y]$

0-*atom* = atom

x is an α-*atom* iff x cannot be expressed as a finite join of β-atoms for $\beta < \alpha$, but for every y, either $x \cap y$ or $x \cap \overline{y}$ is a finite join of β-atoms

B is *superatomic* $\quad \Leftrightarrow (\exists \alpha)[B/\sim_\alpha$ consists of only one element]

A countable superatomic Boolean algebra is isomorphic to $I(\alpha)$ for some countable α.

A *Harrison Boolean algebra* is a *computable* Boolean algebra $\mathcal{B}$ of the form $I(\omega_1^{CK}(1+\eta))$.

For a Harrison Boolean algebra $\mathcal{B}$, let $R^{\mathcal{B}}$ be the set of its *superatomic* elements.

$R^{\mathcal{B}}$ is intrinsically Π^1_1 since it is defined by the disjunction of computable infinitary formulas, saying that x is a finite join of α-atoms, for computable α.

(Goncharov, Harizanov, Knight, Shore)

For Harrison Boolean algebras, the Turing degrees of superatomic parts are the same as the Turing degrees of Π^1_1 paths through O.

Abelian p-group $\mathcal{G}$

$$x \in G - \{0\} \Rightarrow (\exists n)[o(x) = p^n]$$

Descending chain of subgroups

$$G = G_0 \supsetneq G_1 \supsetneq \ldots \supsetneq G_\alpha \supsetneq G_{\alpha+1} \supsetneq \ldots \supsetneq G_{\lambda(G)}$$

$$G_{\alpha+1} = pG_\alpha$$

$$\alpha \text{ limit}, \qquad G_\alpha = \bigcap_{\beta<\alpha} G_\beta$$

$G_\alpha - G_{\alpha+1}$ = elements of *height* α

P = subgroup of G elements of order p (plus 0)

$u_\alpha(G) = \dim(G_\alpha \cap P / G_{\alpha+1} \cap P)$, as a vector space over Z_p

length of G = $\lambda(G)$ = $\mu\lambda[G_\lambda = G_{\lambda+1}]$

$G_{\lambda(G)}$ *divisible* part (unique)

Ulm sequence $(u_\alpha(G))_{\alpha<\lambda(G)}$

- If G is a direct sum of cyclic groups, then $u_n(G)$ is the number of cyclic summands of order p^n

- $dim(\ G_{\lambda(G)}\ ) \ =$ maximum number of algebraically independent elements

- G *reduced* $\Leftrightarrow \ G_{\lambda(G)} = \{0\}$

  $\Leftrightarrow$ every nonzero element has a height

- $G \ = \ G_{\lambda(G)} \oplus G_{reduced}$

- isomorphism type of $G_{reduced}$ determined by the Ulm sequence

Recall that a countable Abelian p-group $\mathcal{G}$ is determined, up to isomorphism, by its Ulm sequence $(u_\alpha(\mathcal{G}))_{\alpha<\lambda(\mathcal{G})}$, and the dimension of the divisible part.

A *Harrison group* is a computable Abelian p-group $\mathcal{G}$ with length $\lambda(\mathcal{G}) = \omega_1^{CK}$, and Ulm invariants $u_{\mathcal{G}}(\alpha) = \infty$, for all computable α, and with infinite dimensional divisible part.

For a Harrison group $\mathcal{G}$, let $R^{\mathcal{G}}$ be the set of elements that have computable ordinal height, the complement of the divisible part.

$R^{\mathcal{G}}$ is intrinsically Π^1_1 since it is defined by the disjunction of computable infinitary formulas saying that x has height α, for computable α.

(Goncharov, Harizanov, Knight, Shore)

For Harrison groups, the Turing degrees of the height-possessing parts are the same as the Turing degrees of Π^1_1 paths through O.