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ORBITS OF MAXIMAL VECTOR SPACES

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Let V_∞ be a standard computable infinite-dimensional vector space over the field of rationals. The lattice $\mathcal{L}(V_\infty)$ of computably enumerable vector subspaces of V_∞ and its quotient lattice modulo finite dimension, $\mathcal{L}^*(V_\infty)$, have been studied extensively. At the same time, many important questions still remain open. In 1998*, R. Downey and J. Remmel posed the question of finding meaningful orbits in $\mathcal{L}^*(V_\infty)$* [4*, Question* 5.8]*. This question is important and difficult and its answer depends on significant progress in the structure theory for the lattice $\mathcal{L}^*(V_\infty)$, and also on a better understanding of its automorphisms. Here we give a necessary and sufficient condition for quasimaximal (hence maximal) vector spaces with extendable bases to be in the same orbit of $\mathcal{L}^*(V_\infty)$. More specifically, we consider two vector spaces, V_1 and V_2, which are spanned by two quasimaximal subsets of, possibly different, computable bases of V_∞. We give a necessary and sufficient condition for the principal filters determined by V_1 and V_2 in $\mathcal{L}^*(V_\infty)$ to be isomorphic. We also specify a necessary and sufficient condition for the existence of an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that Φ maps the equivalence class of V_1 to the equivalence class of V_2. Our results are expressed using m-degrees of relevant sets of vectors. This study parallels the study of orbits of quasimaximal sets in the lattice $\mathcal{E}$ of computably enumerable sets, as well as in its quotient lattice modulo finite sets, $\mathcal{E}^*$, carried out by R. Soare in* [13]*. However, our conclusions and proof machinery are quite different from Soare's. In particular, we establish that the structure of the principal filter determined by a quasimaximal vector space in $\mathcal{L}^*(V_\infty)$ is generally much more complicated than the one of a principal filter determined by a quasimaximal set*

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in $\mathcal{E}^*$. We also state that, unlike in $\mathcal{E}^*$, having isomorphic principal filters in $\mathcal{L}^*(V_\infty)$ is merely a necessary condition for the equivalence classes of two quasimaximal vector spaces to be in the same orbit of $\mathcal{L}^*(V_\infty)$.

INTRODUCTION

Computable model theory uses the tools of computability theory to explore the algorithmic content (effectiveness) of notions, results, and constructions in mathematics. Effective vector spaces and computability-theoretic complexity of their bases were first considered by Mal'tsev in [1] and Dekker in [2]. Modern study of these spaces has been introduced by Metakides and Nerode in [3]. Effective vector spaces have been further investigated in computable model theory by Ash, Dimitrov, Downey, Guhl, Guichard, Hird, Harizanov, Kalantari, Lytkina, Morozov, Nerode, Remmel, Retzlaff, Shore, Smith, Stephan, and others (see survey papers [4-6]). Many of the results about computable vector spaces can be generalized to certain effective closure systems [4]. More recently, Conidis [7] and Downey, Hirschfeldt, Kach, Lempp, Mileti, and Montalbán [8] studied effective vector spaces in the context of reverse mathematics.

We denote by V_∞ a computable $\aleph_0$-dimensional vector space over the field Q of rationals. The vectors in V_∞ are ω-sequences of elements of Q with only finitely many nonzero components. Vector addition and scalar multiplication are defined pointwise. The standard basis

$$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), \ldots$$

for V_∞ is clearly a computable set. In addition, V_∞ has a dependence algorithm; i.e., there is a uniformly effective procedure, which, when applied to a vector u and finitely many vectors $v_1, \ldots, v_n$, determines whether u is an element of the subspace spanned by $\{v_1, \ldots, v_n\}$. As usual, we write c.e. as an abbreviation for *computably enumerable*. A subspace V of V_∞ is c.e. if its domain is a c.e. subset of V_∞. Here, we identify the domain of V_∞ with the set $\omega = \{0, 1, 2, \ldots\}$ of natural numbers. It is not hard to show that every c.e. basis of V_∞ is computable.

We will now briefly review the main notions, ideas, and earlier results that inspired our investigation in this paper. The study of the space V_∞ is important because, as Metakides and Nerode showed in [3], V_∞ is canonical for exploring effective vector spaces. In [3], in particular, it was proved that every c.e. presented vector space is computably isomorphic to $\frac{V_\infty}{W}$ for some c.e. subspace W of V_∞. For $U \subseteq V_\infty$, by $\mathrm{cl}\,(U)$ (the closure of U) we denote the set of all linear combinations of the vectors in U. The c.e. subspaces of V_∞ are the closures of c.e. subsets of V_∞. More precisely, let $I_0, I_1, I_2, \ldots$ be a fixed effective enumeration of all c.e. independent subsets of V_∞. For $e \in \omega$, we let

$$V_e =_{def} \mathrm{cl}\,(I_e).$$

Then $V_0, V_1, V_2, \ldots$ is a fixed effective enumeration of all c.e. subspaces of V_∞. The c.e. subspaces of V_∞ form a lattice, which is denoted by $\mathcal{L}(V_\infty)$. For $U, V \in \mathcal{L}(V_\infty)$, we have the partial order

$$U \leq V \Leftrightarrow U \subseteq V$$

with the infimum $U \wedge V =_{def} U \cap V$ and the supremum $U \vee V =_{def} \mathrm{cl}\,(U \cup V)$.

Recall some general definitions for a lattice $(L; \leq, \wedge, \vee)$.

(i) If L has a least (greatest) element, then that element is denoted by 0 (1, resp.).

(ii) If L is a lattice with 0, then $a \in L$ is called an *atom* if

$$0 < a \wedge (\forall b \in L)\,[b < a \Rightarrow b = 0].$$

(iii) If L is a lattice with 1, then $a \in L$ is called a *coatom* if

$$a < 1 \wedge (\forall b \in L)\,[a < b \Rightarrow b = 1].$$

The lattice $\mathcal{L}(V_\infty)$ has 0 (the empty space) and 1 (the space V_∞). Its atoms are exactly one-dimensional spaces, and the coatoms are the spaces of codimension 1.

A lattice L is said to be *modular* if for every $a, b, x \in L$ we have

$$x \leq b \Rightarrow [x \vee (a \wedge b) = (x \vee a) \wedge b].$$

For example, a lattice of type 1-3-1 (or 1-∞-1) is a modular nondistributive lattice. Notice that $\mathcal{L}(V_\infty)$ is a modular nondistributive lattice, although we model its study upon the study of the distributive lattice $\mathcal{E}$ of all c.e. subsets of ω under inclusion. There are common structural results, but the differences between $\mathcal{E}$ and $\mathcal{L}(V_\infty)$ are interesting and often surprising. For example, the lattice $\mathcal{E}$ also has a least and a greatest element, atoms, and coatoms. In addition, $\mathcal{E}$ has complemented elements. These are exactly all computable subsets of ω; the subsets form a sublattice of $\mathcal{E}$, which is a Boolean algebra. However, while there are complemented elements in $\mathcal{L}(V_\infty)$, their complements may not be unique.

We will use $=^*$ to refer both to the equality of sets up to finitely many elements and to the equality of vector spaces up to finite dimension. By $\mathcal{E}^*$ we will denote the lattice $\mathcal{E}$ modulo finite sets (i.e., $\mathcal{E}^* = \mathcal{E}/_{=^*}$). Notice that $\mathcal{E}^*$ is also a distributive lattice. Similarly, $\mathcal{L}^*(V_\infty)$ is the lattice $\mathcal{L}(V_\infty)$ modulo finite dimension (i.e., $\mathcal{L}^*(V_\infty) = \mathcal{L}(V_\infty)/_{=^*}$). It is a nondistributive modular lattice. Naturally, for $E \in \mathcal{E}$ (or $V \in \mathcal{L}(V_\infty)$), we will use E^* (or V^*) to denote the equivalence class of E in $\mathcal{E}^*$ (or V in $\mathcal{L}^*(V_\infty)$). Note that both $\mathcal{E}^*$ and $\mathcal{L}^*(V_\infty)$ have a least and a greatest element, but neither quotient lattice has atoms. It turns out that both of these lattices have coatoms.

For $A \in \mathcal{E}$, we let $\mathcal{E}(A, \uparrow) = \{E \in \mathcal{E} : A \subseteq E\}$ be the principal filter of A in $\mathcal{E}$. Similarly, let $\mathcal{E}^*(A, \uparrow)$ denote the principal filter of A^* in $\mathcal{E}^*$. Recall that a c.e. set $M \subseteq \omega$ is said to be *maximal* if $\omega - M$ is infinite, and

$$(\forall E \in \mathcal{E})\,[M \subseteq E \Rightarrow |\omega - E| < \infty \vee |E - M| < \infty]\,.$$

Equivalently, a set $M \subseteq \omega$ is maximal if M is c.e. and its complement is cohesive. An infinite set of natural numbers is *cohesive* if it cannot be split into two infinite parts by a c.e. set. Friedberg [9] showed that maximal sets exist. Notice that if M is maximal, then M^* is a coatom in $\mathcal{E}^*$, and that $\mathcal{E}^*(M, \uparrow)$ is isomorphic to the Boolean algebra $\mathbf{B}_1 = \{0, 1\}$. Martin [10] established that a c.e. Turing degree is the degree of a maximal set iff it is a *high* degree. A set $B \subseteq \omega$ is *quasimaximal* iff B is the intersection of finitely many maximal sets M_i, $1 \leq i \leq n$, i.e,

$$B = \bigcap_{i=1}^{n} M_i.$$

If $M_i \neq^* M_j$ for $i \neq j$, then the number n is called the *rank* of B. It is not hard to show that in this case $\mathcal{E}^*(B, \uparrow)$ is isomorphic to the Boolean algebra $\mathbf{B}_n$ (which has 2^n elements).

Kent [11] established that $\mathcal{E}$ has $2^{\aleph_0}$ automorphisms. Lachlan (unpublished; for a proof, see [12, Chap. XV]) stated that $\mathcal{E}^*$ has $2^{\aleph_0}$ automorphisms. Every automorphism of $\mathcal{E}^*$ is induced by an automorphism of $\mathcal{E}$. A remarkable result by Soare [13] is that for any two maximal sets, M_1 and M_2, there is an automorphism Φ of $\mathcal{E}$ (or $\mathcal{E}^*$) such that $\Phi(M_1) = M_2$ (or $\Phi(M_1^*) = M_2^*$). As a consequence, Soare also proved that for any two quasimaximal sets B_1 and B_2 of the same rank, there is an automorphism Ψ of $\mathcal{E}$ such that $\Psi(B_1) = B_2$. The question arises whether there are analogs of Soare's theorem for $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$. There has been a significant progress on this question for the lattice $\mathcal{L}(V_\infty)$, and we will now give an overview of the related results.

As we already mentioned, the lattice $\mathcal{L}^*(V_\infty)$ has coatoms. The coatoms in $\mathcal{L}^*(V_\infty)$ fall in two general categories, the equivalence classes of maximal spaces with extendable bases, and the equivalence classes of maximal spaces with no extendable bases. Here, the notion of a maximal vector space is analogous to the one for maximal sets. That is, a space $V \in \mathcal{L}(V_\infty)$ is *maximal* if $\dim\left(\frac{V_\infty}{V}\right) = \infty$ and

$$(\forall W \in \mathcal{L}(V_\infty))\left[V \subseteq W \Rightarrow \left(\dim\left(\frac{V_\infty}{W}\right) < \infty \vee \dim\left(\frac{W}{V}\right) < \infty\right)\right].$$

An independent set $J \subseteq V_\infty$ is said to be *nonextendable* if $\dim\left(\frac{V_\infty}{\mathrm{cl}\,(J)}\right) = \infty$ and

$$(\forall e)[J \subseteq I_e \Rightarrow |I_e - J| < \infty].$$

A c.e. basis J of a subspace in $\mathcal{L}(V_\infty)$ is said to be *fully extendable* if there is a computable basis A of V_∞ such that $J \subseteq A$. Metakides and Nerode [3] showed that there are nonextendable independent c.e. sets of vectors, and that there are c.e. subspaces of V_∞ with no fully extendable bases. The results in our paper are about the equivalence classes in $\mathcal{L}^*(V_\infty)$ for c.e. vector spaces with fully extendable bases.

If $V \in \mathcal{L}(V_\infty)$, then by $\mathcal{L}(V,\uparrow)$ (or $\mathcal{L}^*(V,\uparrow)$) we will denote the principal filter of V in $\mathcal{L}(V_\infty)$ (or V^* in $\mathcal{L}^*(V_\infty)$). If V is a maximal space, then the structure of $\mathcal{L}(V,\uparrow)$ depends on whether a basis of V is fully extendable. However, in all the cases, V^* is a coatom in $\mathcal{L}^*(V_\infty)$ and $\mathcal{L}^*(V,\uparrow) \cong \mathbf{B}_1$. Metakides and Nerode [3] constructed a maximal space by modifying Friedberg's e-state construction of a maximal set. Shore proved that if M is a maximal subset of a computable basis of V_∞, then $\mathrm{cl}\,(M)$ is a maximal space (see [13]). If a maximal subspace of V_∞ has a c.e. basis M, which is extendable to a computable basis A of V_∞, then M must be a maximal subset of A. In this case

$$\mathcal{E}^*(M,\uparrow) \cong \mathcal{L}^*(\mathrm{cl}\,(M),\uparrow).$$

In [3] Metakides and Nerode also directly constructed a maximal space V such that no c.e. basis of V is extendable.

Kalantari and Retzlaff [14] introduced a stronger notion of maximality for vector spaces. A space $V \in \mathcal{L}(V_\infty)$ is said to be k-*thin* if $\dim \frac{V_\infty}{V} = \infty$ and

$$(\forall W \in \mathcal{L}(V_\infty)) \left[V \subseteq W \Rightarrow \left(\dim\left(\frac{W}{V}\right) < \infty \vee \dim\left(\frac{V_\infty}{W}\right) \leq k\right)\right],$$
$$(\exists U \in \mathcal{L}(V_\infty)) \left[V \subseteq U \wedge \dim\left(\frac{V_\infty}{U}\right) = k\right].$$

Clearly, every k-thin space is a maximal space with no extendable basis. Kalantari and Retzlaff [14] showed that for every $k \geq 0$, there exists a k-thin space $\mathcal{T}_k$. Hence there exists an infinite sequence of maximal spaces, $(\mathcal{T}_k)_{k\in\omega}$, such that for every automorphism Φ of $\mathcal{L}(V_\infty)$, we have

$$i \neq j \Rightarrow \Phi(\mathcal{T}_i) \neq \mathcal{T}_j.$$

The 0-thin spaces are also referred to as *supermaximal*. Equivalently, a space $V \in \mathcal{L}(V_\infty)$ is supermaximal if $\dim\left(\frac{V_\infty}{V}\right) = \infty$ and

$$(\forall W \in \mathcal{L}(V_\infty)) \left[V \subseteq W \Rightarrow \left(\dim\left(\frac{W}{V}\right) < \infty \vee W = V_\infty\right)\right].$$

Remmel [15] showed that for every c.e. Turing degree $\mathbf{d} \neq \mathbf{0}$, there exist a supermaximal space of degree $\mathbf{d}$ (and dependence degree $\mathbf{d}$). Guichard [16] proved that for every $k \geq 0$ and every c.e. Turing degree $\mathbf{d} \neq \mathbf{0}$, there are k-thin spaces U and V of degree $\mathbf{d}$ (and dependence degree $\mathbf{d}$) such that for every automorphism Φ of $\mathcal{L}(V_\infty)$, we have

$$\Phi(U) \neq V.$$

This result follows from Remmel's construction in [15] and Guichard's surprising result in [16] which says that every automorphism of $\mathcal{L}(V_\infty)$ is induced by a *computable* semilinear transformation in

V_∞. Recall the following definition. Let W_1 be a vector space over a field F_1, W_2 be a vector space over a field F_2, and $\tau : F_1 \to F_2$ be a field isomorphism; then a map $\phi : W_1 \to W_2$ is said to be *semilinear* (with respect to τ) if

$$\phi(av + bw) = \tau(a)\phi(v) + \tau(b)\phi(w).$$

Hence Guichard's result implies that there are only countably many automorphisms of $\mathcal{L}(V_\infty)$. Currently, the question about the number of automorphisms of $\mathcal{L}^*(V_\infty)$ is open. Guichard [16] showed that not every automorphism of $\mathcal{L}^*(V_\infty)$ is induced by a semilinear transformation. Ash conjectured that the automorphisms of $\mathcal{L}^*(V_\infty)$ are induced by semilinear transformations with finite-dimensional kernels and cofinite-dimensional images in V_∞ (see [16, p. 57]).

Hird [17] introduced an even stronger notion than supermaximality for vector spaces. A space $V \in \mathcal{L}(V_\infty)$ is referred to as *strongly supermaximal* if $\dim \frac{V_\infty}{V} = \infty$, and for every c.e. set of vectors $X \subseteq V_\infty - V$, there is $n \geq 0$ such that

$$(\exists a_0, \ldots, a_{n-1} \in V_\infty)[X \subseteq \mathrm{cl}\,(V \cup \{a_0, \ldots, a_{n-1}\})].$$

Hird [17] showed that strongly supermaximal spaces exist. Downey and Hird [18] established that every strongly supermaximal vector space is supermaximal, but that the converse is not true. Moreover, Downey and Hird [18] proved that every nonzero c.e. Turing degree contains two strongly supermaximal subspaces, U and V, such that for every automorphism Φ of $\mathcal{L}(V_\infty)$, we have

$$\Phi(U) \neq V.$$

In 1998, Downey and Remmel [4] posed the question of finding meaningful orbits in $\mathcal{L}^*(V_\infty)$. In our main Theorem 4.10, we give a necessary and sufficient condition for quasimaximal vector spaces with extendable bases to be in the same orbit of $\mathcal{L}^*(V_\infty)$. The condition demonstrates an intricate connection between the lattice-theoretic structure of $\mathcal{L}^*(V_\infty)$ and the degree-theoretic properties of the sets of vectors. It is stated in terms of m-degrees. For $X, Y \subseteq \omega$, we write $X \leq_m Y$ if X is many-one reducible, or m-reducible, to Y. The sets X and Y have the same *m-degree* iff $X \leq_m Y$ and $Y \leq_m X$. This is denoted by $X \equiv_m Y$ or $\deg_m(X) = \deg_m(Y)$.

Unlike for the principal filters in $\mathcal{E}^*$ determined by quasimaximal sets, there are several possibilities for the principal filters in $\mathcal{L}^*(V_\infty)$ determined by the closures of quasimaximal subsets of a computable basis. More precisely, in [19, 20], Dimitrov gave a description of all possible isomorphism types of $\mathcal{L}^*(\mathrm{cl}\,(B), \uparrow)$, where B is a quasimaximal subset of rank n in a computable basis of V_∞. It was proved that $\mathcal{L}^*(\mathrm{cl}\,(B), \uparrow)$ is isomorphic to one of the following structures:

(1) a Boolean algebra $\mathbf{B_n}$;

(2) the lattice $L(n, Q_\mathbf{a})$ of all subspaces of an n-dimensional vector space over a certain extension $Q_\mathbf{a}$ of the field Q;

(3) a finite product of lattices in items (1) and (2).

These principal filters fall into infinitely many nonisomorphic classes, even if the filters are isomorphic to the lattices of subspaces of the vector spaces of the same dimension (see [21]). Note that the Boolean algebra $\mathbf{B_n}$ in (1) can also be viewed as a product of n copies of the Boolean algebra $\mathbf{B_1}$. We call the extensions $Q_{\mathbf{a}}$ of the field Q mentioned in (2) *cohesive powers* of Q. The subscript $\mathbf{a}$ in $Q_{\mathbf{a}}$ stands for a degree and ranges over all possible m-degrees of maximal subsets of computable bases of V_∞. Various results about the structure of such fields were established in [21, 22]. These results, together with the above classification of the possible isomorphism types of $\mathcal{L}^*(\mathrm{cl}\,(B), \uparrow)$, will be used in the proof of our main theorem. We will further discuss them in Section 4.

To state our main theorem, we introduce the notion of an *m-degree type* of a quasimaximal set $E = \bigcap\limits_{i=1}^{n} M_i$ of rank n, denoted by type$\,(E)$. This notion captures the number and the m-degrees of the maximal sets M_i (see Definition 4.3). We then establish the following main

THEOREM 4.17. Let E_1 and E_2 be quasimaximal subsets of rank n in the computable bases A_1 and A_2, respectively, for V_∞. Then the following are equivalent:

(1) there is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(E_1)^*) = \mathrm{cl}\,(E_2)^*$;

(2) $\mathrm{type}_{A_1}(E_1) = \mathrm{type}_{A_2}(E_2)$.

For the special case of maximal sets, the theorem implies

COROLLARY 4.18. Let M_1 and M_2 be maximal subsets of the computable bases A_1 and A_2, respectively, for V_∞. Then the following are equivalent:

(1) there is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(M_1)^*) = \mathrm{cl}\,(M_2)^*$;

(2) $\deg_m(M_1) = \deg_m(M_2)$.

Their proofs are based on the technical results presented in Sections 2 and 3.

In Section 2, we consider an arbitrary finite collection of maximal vector spaces V_i with bases B_i, which extend to (possibly) different computable bases A_i of V_∞ (for $i \in I$). We show that, under certain assumptions, the spaces V_i have c.e. bases D_i that are extendable to a common computable basis A of V_∞. In Section 3, we prove that if V_1 and V_2 are two maximal spaces such that V_1 has an extendable c.e. basis, while no c.e. basis of V_2 is extendable, then

$$\mathcal{L}^*(V_1 \cap V_2, \uparrow) \cong \mathbf{B_2}.$$

Therefore, if the modular lattice 1-3-1 (or 1-∞-1) is a principal filter in $\mathcal{L}^*(V_\infty)$, then either all coatoms in the filter have c.e. extendable bases, or no coatom has a c.e. extendable basis.

For more detailed information about effective vector spaces and any additional computability-theoretic notions and techniques, the reader is referred to [3, 4, 6, 12, 23-25].

2. BASES OF MAXIMAL SPACES WITH A COMMON EXTENSION

Extendable c.e. bases of two or more maximal spaces may not extend to a common c.e. basis of V_∞. The following theorem gives a sufficient condition for the existence of such a common extension. Moreover, under this condition, we show that the m-degrees of the extendable bases are preserved. For more properties of m-degrees, see [23]. Recall that if A is a basis of V_∞, then, for any $x \in V_\infty$, the *support* of x with respect to A, denoted $\mathrm{supp}_A(x)$, is the set of all vectors from A, which appear with nonzero coefficients when x is written as a linear combination in the basis A.

Recall that $\lessdot$ stands for the standard lattice-theoretic *cover relation*

$$a \lessdot b \Leftrightarrow [a < b \wedge \forall c\, [a \leq c \leq b \Rightarrow (c = a \vee c = b)]]\,.$$

Definition 2.1. To simplify the notation and indexing in the statement and proof of Theorem 2.2, we will use the following notational conventions for the rest of this section only:

(1) $\bigcap X_{(n)} =_{def} \bigcap\limits_{1\leq i\leq n} X_i$ and $\bigcup X_{(n)} =_{def} \bigcup\limits_{1\leq i\leq n} X_i$;

(2) $\bigcap X_{(n-\{k\})} =_{def} \bigcap\limits_{1\leq i\leq n; i\neq k} X_i$ and $\bigcup X_{(n-\{k\})} =_{def} \bigcup\limits_{1\leq i\leq n; i\neq k} X_i$;

(3) $\sum c_{(n)} =_{def} \sum\limits_{1\leq i\leq n} c_i$ and $\sum c_{(n-\{k\})} =_{def} \sum\limits_{1\leq i\leq n; i\neq k} c_i$.

We will employ similar conventions for double subscripts as well.

THEOREM 2.2. Let V_i, $i = 1, \ldots, n$, where $n \geq 2$, be maximal subspaces of V_∞ such that each V_i has a c.e. basis B_i, which is a maximal subset of a computable basis A_i of V_∞. Assume that for every $k \in \{1, \ldots, n\}$,

$$\dim\left(\frac{\bigcap V_{(n-\{k\})}}{V_k}\right) = \infty. \tag{inf}$$

Then:

(i) there are a c.e. independent set A and a collection D_k, $1 \leq k \leq n$, of c.e. subsets of A such that

$$\mathrm{cl}\,(A) =^* V_\infty \text{ and } (\forall k \in \{1, \ldots, n\})\,[V_k =^* \mathrm{cl}\,(D_k)];$$

(ii) there are computable 1-1 functions F_k, $k = 1, \ldots, n$, such that

$$\mathrm{dom}\,(F_k) =^* A \wedge \mathrm{rng}(F_k) =^* A_k \wedge F_k(D_k) = B_k \wedge F_k(A - D_k) =^* A_k - B_k;$$

(iii) $D_k \equiv_m B_k$ for $k = 1, \ldots, n$.

Proof. (i) We will construct a c.e. set D and sets C_i, $i = 1, \ldots, n$, satisfying the following conditions:

$\mathrm{cl}\,(D) =^* \bigcap V_{(n)}$;

for every $k \in \{1, \ldots, n\}$, $D_k =_{def} D \cup \bigcup C_{(n-\{k\})}$ is a c.e. basis of V_k up to $=^*$;

$A =_{def} D \cup \bigcup C_{(n)}$ is a computable basis of V_∞ up to $=^*$.

The sets D and C_i will be enumerated in stages. Let D^s and C_i^s, $i = 1, \dots, n$, be their finite approximations by the end of stage s. Some of the elements already enumerated in C_i may at later stages be enumerated in D, and thus taken out of C_i. This will guarantee that $D \cup \bigcup_{i \in P} C_i$ will be a c.e. set for any index set $P \subseteq \{1, \dots, n\}$. Each C_i will be a difference of two c.e. sets (such a set is also called a d-c.e. set). The vectors in $D^s \cup \bigcup C^s_{(n)}$ will be linearly independent, and the construction will guarantee that $\mathrm{cl}\,(D) =^* \bigcap V_{(n)}$. We will also make sure that each C_k is an infinite subset of $\bigcap V_{(n-\{k\})}$. Therefore, for any permutation $(i_1, i_2, \dots, i_n)$ of $\{1, 2, \dots, n\}$, we have

$$D \subset_\infty D \cup C_{i_1} \subset_\infty D \cup C_{i_1} \cup C_{i_2} \subset_\infty \cdots \subset_\infty D \cup \bigcup C_{i_{(n)}}.$$

Hence in the lattice $\mathcal{L}^*(V_\infty)$, we obtain

$$\mathrm{cl}\,(D) \lessdot \mathrm{cl}\,(D \cup C_{i_1}) \lessdot \mathrm{cl}\,(D \cup C_{i_1} \cup C_{i_2}) \lessdot \cdots \lessdot \mathrm{cl}\left(D \cup \bigcup_{i_{(n)}} C_{i_{(n)}}\right).$$

Assume that we have a fixed computable enumeration of each B_i such that B_i^s is the set of elements enumerated in B_i by the end of stage s. Furthermore, suppose that at each stage s, at most one new element will be enumerated into no more than one of the sets B_i, $i = 1, \dots, n$.

CONSTRUCTION

Stage 0. Let $D^0 = \varnothing$ and $C_i^0 = \varnothing$ for $i = 1, \dots, n$.

Stage $s+1$. Before we start a sequence of substages of this stage, we put $D^{s+1} = D^s$ and $C_i^{s+1} = C_i^s$.

Substage 0. For every $i \in \{1, \dots, n\}$ and every $x \in C_i^s$, check whether $\mathrm{supp}_{A_i}(x) \subseteq B_i^s$. There is at most one such i, say, i_0, and its x is unique. For such x, let $D^{s+1,0} = D^s \cup \{x\}$ and $C_{i_0}^{s+1,0} = C_{i_0}^s - \{x\}$.

For all $i \neq i_0$, define $C_i^{s+1,0} = C_i^s$. If there is no such i, then let $D^{s+1,0} = D^s$ and $C_i^{s+1,0} = C_i^s$ for all i. (In what follows, in such cases we will say that all other sets remain unchanged.)

Substage k, $1 \le k \le n$. Look for an $x \le s+1$ such that

$x \in \bigcap \mathrm{cl}\,(B^{s+1}_{(n-\{k\})})$,

$x \notin \mathrm{cl}\,(B_k^{s+1})$,

$D^{s+1,0} \cup \bigcup C^{s+1,0}_{(n)} \cup \{x\}$ is an independent set, and

$\left(\forall y \in C_k^{s+1,0}\right) [(\mathrm{supp}_{A_k}(y) - B_k^{s+1}) \cap (\mathrm{supp}_{A_k}(x) - B_k^{s+1}) = \varnothing]$.

If such x exists, then, for the least such x, we let $C_k^{s+1} = C_k^{s+1,0} \cup \{x\}$; otherwise, let $C_k^{s+1} = C_k^{s+1,0}$. The other sets remain unchanged at this substage.

Substage $n+1$. Look for an $x \le s+1$ such that

$x \in \bigcap \mathrm{cl}\,(B^{s+1}_{(n)})$, and

$D^{s+1,0} \cup \bigcup C^{s+1}_{(n)} \cup \{x\}$ is independent.

If such x exists, then let $D^{s+1} = D^{s+1,0} \cup \{x\}$. Otherwise, let $D^{s+1} = D^{s+1,0}$.

End of Construction

We let $D_k =_{def} D \cup \bigcup C_{(n-\{k\})}$ for $k \in \{1, \ldots, n\}$ and let $A =_{def} D \cup \bigcup C_{(n)}$. $\Box$

LEMMA 2.3. For any $P \subseteq \{1, \ldots, n\}$, the set $D \cup \bigcup_{i \in P} C_i$ is c.e.

Proof. Although the sets C_i are d-c.e., when a vector x, already enumerated in C_i at some stage, is removed from C_i at substage i of a later stage, this x is enumerated into the c.e. set D. Hence $D \cup \bigcup_{i \in P} C_i$ is c.e. $\Box$

LEMMA 2.4. For every $k \in \{1, \ldots, n\}$, we have $C_k \subseteq \bigcap \mathrm{cl}\,(B_{(n-\{k\})})$.

Proof. This inclusion follows immediately from the first condition for enumerating elements into C_k at substage k of the construction. $\Box$

LEMMA 2.5. We have $\mathrm{cl}\,(D) =^* \bigcap \mathrm{cl}\,(B_{(n)})$.

Proof. Clearly, $\mathrm{cl}\,(D) \subseteq^* \bigcap \mathrm{cl}\,(B_{(n)})$. Indeed, if x is enumerated into D at substage 0, then, for some $i_0 \leq n$,

$$x \in C_{i_0}^{s+1} \subseteq \bigcap \mathrm{cl}\,(B_{(n-\{i_0\})});$$
$$\mathrm{supp}_{A_{i_0}}(x) - B_{i_0}^{s+1} = \varnothing,$$

which means that $x \in \mathrm{cl}\,(B_{i_0}^{s+1})$. If x is enumerated into D^{s+1} at substage $n+1$, then $x \in \bigcap \mathrm{cl}\,(B_{(n)})$. Thus if $x \in D$, then $x \in \bigcap \mathrm{cl}\,(B_{(n)})$.

Now suppose that $x \in \bigcap \mathrm{cl}\,(B_{(n)})$ but $x \notin \mathrm{cl}\,(D)$. Let t be the first stage such that $x < t$ and $x \in \bigcap \mathrm{cl}\,(B_{(n)}^t)$. The reason why x is not enumerated into B at substage $n+1$ of stage t is that the set $D^t \cup \bigcup C_{(n)}^t \cup \{x\}$ is not independent. Suppose that $d \in \mathrm{cl}\,(D^t)$ and $c_k \in \mathrm{cl}\,(C_k^t)$, $1 \leq k \leq n$, satisfy the condition

$$x = d + c_1 + \cdots + c_n.$$

It follows by construction that $d \in \bigcap \mathrm{cl}\,(B_{(n)})$, and for every $k \in \{1, \ldots, n\}$, $c_k \in \bigcap \mathrm{cl}\,(B_{(n-\{k\})})$. We will show that $c_k \in \mathrm{cl}\,(D)$. Clearly,

$$c_k = x - d - \sum c_{(n-\{k\})}.$$

Since $x \in \mathrm{cl}\,(B_k)$, $d \in \mathrm{cl}\,(B_k)$, and $\sum c_{(n-\{k\})} \in \mathrm{cl}\,(B_k)$, it is also true that $c_k \in \mathrm{cl}\,(B_k)$. However,

$$\left(\forall x, y \in C_k^t\right) \left[\left(\mathrm{supp}_{A_k}(y) - B_k^t\right) \cap \left(\mathrm{supp}_{A_k}(x) - B_k^t\right)\right] = \varnothing.$$

If $c_k = \sum y_{(m)}$, where $y_i \in C_k^t$ for $i = 1, \ldots, m$, then, in view of the fact that $c_k \in \mathrm{cl}\,(B_k)$, all y_i must be enumerated into B_k (and hence into D) at substage 0 of the later stages. Therefore, $c_k \in \mathrm{cl}\,(D)$ for $k \in \{1, \ldots, n\}$. Hence $x \in \mathrm{cl}\,(D)$, which contradicts our assumption. $\Box$

LEMMA 2.6. The sets D and C_i, $i = 1, \ldots, n$, are infinite and pairwise disjoint.

Proof. That $D, C_1, \ldots, C_n$ are pairwise disjoint follows from the construction where we guarantee that $D \cup \bigcup C_{(n)}$ is linearly independent. The space $\bigcap V_{(n)}$ is infinite-dimensional, and by Lemma 2.5, D is infinite.

We now fix $k \in \{1, \ldots, n\}$ and assume that C_k is finite. Let s be a stage after which no new (permanent) elements of C_k are enumerated. Since $\dim \left(\frac{\bigcap V_{(n-\{k\})}}{V_k}\right) = \infty$, we can find a sequence of vectors $x_0, x_1, \ldots$ in $\bigcap V_{(n-\{k\})}$, which are independent modulo $\mathrm{cl}\,(V_k \cup C_k)$. Obviously, for some $x \in \mathrm{cl}\,(\{x_j : j \geq 0\})$, we will have $x \notin C_k$, but

$$(\forall y \in C_k)\,[\big(\mathrm{supp}_{A_k}(y) - B_k\big) \cap \big(\mathrm{sup}_{A_k}(x) - B_k\big) = \varnothing].$$

Suppose that $s_1 > s$ is a stage such that $\mathrm{supp}_{A_k}(x) - B_k = \mathrm{supp}_{A_k}(x) - B_k^{s_1}$. No vector z with $\big(\mathrm{supp}_{A_k}(x) - B_k\big) \cap \big(\mathrm{supp}_{A_k}(z) - B_k\big) \neq \varnothing$ will be enumerated into C_k after stage s_1, since such z cannot later be removed from C_k because of its support. Consequently,

$$(\forall s \geq s_1)\,(\forall y \in C_k)\,[\big(\mathrm{supp}_{A_k}(y) - B_k^s\big) \cap \big(\mathrm{sup}_{A_k}(x) - B_k^s\big) = \varnothing],$$

and so x will be enumerated into C_k at some stage s_2 such that $s \leq s_2 \leq s_1$, and it will never be removed from C_k since $\big(\mathrm{supp}_{A_k}(x) - B_k\big) \neq \varnothing$. We are led to a contradiction with the choice of stage s. □

LEMMA 2.7. We have $\mathrm{cl}\,(A) =^* V_\infty$ and $\mathrm{cl}\,(D_k) =^* \mathrm{cl}\,(B_k)$ for $k = 1, \ldots, n$.

Proof. The $=^*$-equivalence classes of the spaces V_i are coatoms in the modular lattice $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$. Recall that if L is a modular lattice, then $[a \wedge b, a] \cong [b, a \vee b]$ for all $a, b \in L$. Let $P \subsetneq \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\} - P$. Define $a = \bigcap\limits_{i \in P} V_i$ and $b = V_j$ in the lattice $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$. Then

$$\left[\left(\bigcap_{i \in P \cup \{j\}} V_i\right),\ \bigcap_{i \in P} V_i\right] \cong \left[V_j,\ \left(\bigcap_{i \in P} V_i\right) \vee V_j\right].$$

Since V_j is a maximal space and $\dim \left(\frac{\bigcap\limits_{i \in P} V_i}{V_j}\right) = \infty$, we have $\left(\bigcap\limits_{i \in P} V_i\right) \vee V_j =^* V_\infty$. In $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$, therefore, we obtain

$$\left[\left(\bigcap_{i \in P \cup \{j\}} V_i\right),\ \bigcap_{i \in P} V_i\right] \cong [V_j,\ V_\infty].$$

Since $V_j \lessdot V_\infty$, the above lattice interval isomorphism implies that

$$\left(\bigcap_{i \in P \cup \{j\}} V_i\right) \lessdot \bigcap_{i \in P} V_i.$$

Consequently, for any sequence

$$\varnothing = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{n-1} \subsetneq P_n = \{1, \ldots, n\},$$

the chain

$$\mathrm{cl}\,(D) =^* \bigcap_{i \in P_n} V_i \lessdot \bigcap_{i \in P_{n-1}} V_i \lessdot \cdots \lessdot \bigcap_{i \in P_1} V_i \lessdot \bigcap_{i \in P_0} V_i =^* V_\infty \qquad \text{(chain 1)}$$

is a maximal chain of length n in $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$.

By construction, the set $D \cup \bigcup C_{(n)}$ is linearly independent, and by Lemma 2.6, the sets D and C_i, $i = 1, \ldots, n$, are infinite and pairwise disjoint. Therefore, for the complements $\overline{P_i} = \{1, \ldots, n\} - P_i$, we have

$$D \subset_\infty \left(D \cup \bigcup_{i \in \overline{P_{n-1}}} C_i \right) \subset_\infty \cdots \subset_\infty \left(D \cup \bigcup_{i \in \overline{P_1}} C_i \right) \subset_\infty \left(D \cup \bigcup_{i \in \overline{P_0}} C_{i_k} \right).$$

Hence

$$\mathrm{cl}\,(D) < \mathrm{cl} \left(D \cup \bigcup_{i \in \overline{P_{n-1}}} C_i \right) < \cdots < \mathrm{cl} \left(D \cup \bigcup_{i \in \overline{P_1}} C_i \right) < \mathrm{cl} \left(D \cup \bigcup_{i \in \overline{P_0}} C_{i_k} \right) \qquad \text{(chain 2)}$$

is a chain (not necessarily maximal) of length n in $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$.

Note that a modular lattice satisfies the Jordan–Dedekind chain condition (saying that any two maximal chains between two elements have the same finite length). Using this condition and the facts that $D \subseteq \bigcap V_{(n)}$ and $C_k \subseteq \bigcap V_{(n-\{k\})}$, we conclude that (chain 2) is maximal and that the sequences (chain 1) and (chain 2) are identical. In particular, if we put $P_1 = \{k\}$ we obtain

$$\mathrm{cl}\,(D_k) = \mathrm{cl} \left(D \cup \bigcup C_{(n-\{k\})} \right) = \mathrm{cl} \left(D \cup \bigcup_{i \in \overline{P_1}} C_i \right) =^* \bigcap_{i \in P_1} V_i = V_k.$$

Also,

$$\mathrm{cl}\,(A) = \mathrm{cl} \left(D \cup \bigcup C_{(n)} \right) = \mathrm{cl} \left(D \cup \bigcup_{i \in \overline{P_0}} C_i \right) =^* V_\infty.$$

This completes the proof of (i).

(ii) We will now describe only the new action needed to define functions F_k for $k = 1, \ldots, n$.

CONSTRUCTION

Stage 0. Put $F_k^0 = \varnothing$ for $k = 1, \ldots, n$.

Stage $s + 1$.

(I) Consider every substage $i \leq n$ of this stage at which a (unique) new vector x is enumerated in C_i^{s+1}. Assume also that $F_k^s(x)$ has not yet been defined for some $k \in \{1, \ldots, n\}$ such that $k \neq i$. For each such k, find the least stage $t \geq s$ for which there exists $y \in B_k^t - \mathrm{rng}(F_k^s)$. Let $F_k^{s+1}(x) = y$ for the least such y.

Suppose that at one of the substages of this stage, a new vector x is enumerated in D^{s+1}. If $F_k^s(x)$ has not yet been defined for some $k \in \{1, \ldots, n\}$, then for each such k we find the least stage $t \geq s$ for which there exists $y \in B_k^t - \mathrm{rng}(F_k^s)$. Let $F_k^{s+1}(x) = y$ for the least such y.

(II) Suppose that for some $k \leq n$, there are $x \in C_k^{s+1}$ and $a \in A_k - \mathrm{rng}(F_k^s)$ such that

$$\mathrm{supp}_{A_k}(x) - B_k^{s+1} = \{a\}. \qquad \text{(II.1)}$$

For each such $k \le n$ and for every such x, we put $F_k^{s+1}(x) = a$.

End of Construction

We will now prove that each function F_k is 1-1 and satisfies the conditions $\operatorname{dom}(F_k) =^* A$, $\operatorname{rng}(F_k) =^* A_k$, and $F_k(D_k) =^* B_k$. Let $C_k^\dagger$ be the set of all elements that have been enumerated in C_k at some stage of our construction. Note that $C_k \subseteq C_k^\dagger$, the set $C_k^\dagger$ is c.e., and

$$\big\{\big(\operatorname{supp}_{A_k}(x) - B_k\big)\big\}_{x \in C_k^\dagger}$$

is a sequence of finite sets of elements of $A_k - B_k$ satisfying the following conditions:

$$\operatorname{supp}_{A_k}(x) - B_k \neq \varnothing \text{ if } x \in C_k;$$
$$\operatorname{supp}_{A_k}(x) - B_k = \varnothing \text{ if } x \in D.$$

We claim that for all but finitely many $x \in C_k^\dagger$, either $x \in D$ or

$$x \in C_k \text{ and } \big|\big(\operatorname{supp}_{A_k}(x) - B_k\big)\big| = 1.$$

To prove this claim, suppose that

$$\big|\big(\operatorname{supp}_{A_k}(x) - B_k\big)\big| \ge 2$$

for infinitely many $x \in C_k$.

We construct a c.e. set L_k in stages as follows:

$L_k^0 = \varnothing$;

if x is enumerated in C_k at stage s of the construction, then for the least $z \in \big(\operatorname{supp}_{A_k}(x) - B_k^s\big)$ such that $z \notin L_k^s$ we let $L_k^{s+1} = L_k^{s+1} \cup \{z\}$;

if this z is enumerated into B_k at some later stage $s_1 > s$, then we check whether there is $z_1 \in \big(\operatorname{supp}_{A_k}(x) - B_k^{s_1}\big)$ such that $z_1 \notin L_k^{s_1}$. If there is such z_1, then enumerate the least such z_1 into L_k.

It is clear that

$$(\forall t \ge 0)\, \big(\forall x, y \in C_k^t\big)\, \big[\big(\operatorname{supp}_{A_k}(y) - B_k^t\big) \cap \big(\operatorname{supp}_{A_k}(x) - B_k^t\big)\big] = \varnothing.$$

Note that if $\big|\big(\operatorname{supp}_{A_k}(x) - B_k\big)\big| \ge 2$ for infinitely many $x \in C_k$, then both $L_k \cap B_k$ and $L_k \cap \overline{B_k}$ will be infinite, which contradicts the cohesiveness of B_k.

Suppose that for $x \in C_k^\dagger$, $x \in D$, but $F_k(x) = a$ was defined using part (II) of the construction at some stage s such that

$$\operatorname{supp}_{A_k}(x) - B_k^s = \{a\}$$

for some $a \in A_k$. Then, at some later stage $t > s$, the vector a will be enumerated into B_k^t, and by construction, x will be enumerated in D. Note that for every $x \in D_k$, $F_k(x)$ will be defined either via the process we have just described, or using part (I) of the construction. Therefore, if $x \in D_k$, then $F_k(x)$ is defined and $F_k(x) \in B_k$.

For almost all $x \in C_k$ $(\subseteq C_k^{\dagger})$, we will have $\left|\mathrm{supp}_{A_k}(x) - B_k\right| = 1$. In these cases $F_k(x)$ will also be defined using part (II) of the construction at some stage s so that $F_k(x) = a$ for some $a \in A_k$ such that $\mathrm{supp}_{A_k}(x) - B_k^s = \{a\}$. However, this a will not be enumerated into B_k at any later stage. We therefore conclude that $F_k(x)$ is defined for all but finitely many $x \in C_k$. Also, if $x \in C_k$ is such that $x \in \mathrm{dom}\,(F_k)$, then $F_k(x) \in A_k - B_k$. Hence $\mathrm{dom}\,(F_k) =^* A$, $F_k(D_k) \subseteq B_k$, and $F_k(A - D_k) \subseteq A_k - B_k$. By construction, a vector x is enumerated into C_k^{s+1} only if

$$\left(\forall y \in C_k^{s+1}\right) \left[\left(\mathrm{supp}_{A_k}(y) - B_k^{s+1}\right) \cap \left(\mathrm{supp}_{A_k}(x) - B_k^{s+1}\right) = \varnothing\right].$$

This means that

$$(\forall x, y \in C_k) \left[\left(\mathrm{supp}_{A_k}(y) - B_k\right) \cap \left(\mathrm{supp}_{A_k}(x) - B_k\right) = \varnothing\right],$$

and so if $x, y \in C_k$ are such that $F_k(x)$ and $F_k(y)$ are defined, then $F_k(x) \neq F_k(y)$. Also, it follows from part (I) of the construction that the function F_k is 1-1 for each $k \in \{1, \dots, n\}$. Using part (I) of the construction, we conclude that $B_k \subseteq \mathrm{rng}(F_k)$.

The set C_k is infinite and, therefore, $B_k \subset_\infty \mathrm{rng}(F_k)$. Since $\mathrm{rng}(F_k) \subseteq A_k$ is a c.e. set, and B_k is a maximal subset of A_k, we see that $\mathrm{rng}(F_k) =^* A_k$ and, consequently, $F_k(D_k) = B_k$ and $F_k(A - D_k) =^* A_k - B_k$.

(iii) The required m-equivalence follows immediately from (ii). □

Remark 2.8. Let V_i, $1 \leq i \leq n$, be as in Theorem 2.2. Since $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$ is a modular lattice in which V_i are coatoms, the condition (inf) in the statement of Theorem 2.2 implies that the collection of spaces $\left\{ \bigcap\limits_{i \in P} V_i : P \subseteq \{1, \dots, n\} \right\}$ with the lattice operations inherited from $\mathcal{L}^*(V_\infty)$ is a sublattice of $\mathcal{L}^*(\bigcap V_{(n)}, \uparrow)$, which is isomorphic to the Boolean algebra $\mathbf{B_n}$. $\Bigl($Here we assume that $V_\infty =_{def} \bigcap\limits_{i \in \varnothing} V_i$.$\Bigr)$

Remark 2.9. In part (ii) of the proof of Theorem 2.2, we showed that for almost all elements of the sequence $\left\{\left(\mathrm{supp}_{A_k}(x) - B_k\right)\right\}_{x \in C_k^{\dagger}}$, we have

$$\left|\left(\mathrm{supp}_{A_k}(x) - B_k\right)\right| \leq 1.$$

The proof of this fact is similar to the proof of Martin's theorem saying that for a maximal set M we have $\overline{\lim}(\overline{M}) = 1$ (see [25, Sec. 12.5, Thm. XIII]).

3. MAXIMAL SPACES AND MODULAR LATTICE 1-3-1

As we mentioned in the Introduction, Metakides and Nerode proved in [3] that there are spaces that have no extendable bases.

Remark 3.1. If I is a basis of a c.e. space W, which is extendable to a c.e. set J, then I must be c.e. because $I = J \cap W$.

Remark 3.2. Let I be a basis of a maximal subspace M of V_∞. If I is extendable, then I is fully extendable.

THEOREM 3.3. Suppose that M_i, $i = 1, 2, 3$, are maximal subspaces of V_∞, $M_i \neq^* M_j$ for all $i \neq j$, and for all i, j, k with $\{i, j, k\} = \{1, 2, 3\}$, we have

$$M_i \cap M_j =^* M_i \cap M_k =^* M_j \cap M_k =^* M.$$

If M_1 has an extendable basis, then the spaces M_2 and M_3 also have extendable bases.

Proof. First, we note that the assumptions of the theorem imply that the principal filter $\mathcal{L}^*(M, \uparrow)$ of the equivalence class of M contains the modular lattice 1-3-1 as its sublattice.

$$\begin{tikzcd}[arrows=dash]
 & V_\infty \arrow[dl] \arrow[d] \arrow[dr] & \\
M_1 \arrow[dr] & M_2 \arrow[d] & M_3 \arrow[dl] \\
 & M &
\end{tikzcd}$$

Diagram 1

Now, suppose that B_1 is a c.e. basis of M_1, which can be extended to a computable basis A_1 of V_∞. We will build a c.e. basis B_2 of M_2 and a d-c.e. set C_2 such that $A_2 =_{def} B_2 \cup C_2$ is a computable basis of V_∞. As before, a vector x that is enumerated into C_2 at stage s may at a later stage be removed from C_2 and enumerated into B_2. Thus both B_2 and $B_2 \cup C_2$ will be c.e. sets.

The basis B_2 and the set C_2 will be built in stages. We will use the same notation as in the construction in the proof of Theorem 2.2. If a vector x is enumerated in M_2 at stage s (i.e., in M_2^s) and the set $B_2^s \cup C_2^s \cup \{x\}$ is independent, then x is enumerated into B_2^s. We enumerate x in C_2 only if $x \in M_3$. Once such x is enumerated in C_2, it may also be enumerated in M_1. Since $M_1 \cap M_3 =^* M \subseteq^* M_2$, we assume that x will eventually appear in M_2. We will enumerate this x in B_2 and take it out of C_2. The fact that $x \in M_2^s$ for all but finitely many s will guarantee that $\mathrm{cl}\,(B_2) \subseteq^* M_2$.

As before, we will make sure that $\{(\mathrm{supp}_{A_1}(x) - B_1^s) : x \in C_2^s\}$ is a disjoint collection of nonempty sets at any stage s. This will guarantee that if some $c_2 \in \mathrm{cl}\,(C_2^s)$ is enumerated into M_1^t at some stage $t > s$, then such an enumeration occurs because $\mathrm{supp}_{A_1}(c_2) - B_1^t = \varnothing$, and so $c_2 \in M_1 \cap M_3 \subseteq^* M_2$.

CONSTRUCTION

Stage 0. Let $B_2^0 = C_2^0 = \varnothing$.

Stage $s + 1$. Put $B_2^{s+1} = B_2^s$ and $C_2^{s+1} = C_2^s$.

Substage 1. If there is an $x \in M_2^{s+1}$ such that $B_2^s \cup C_2^s \cup \{x\}$ is independent, then for the least such x we let $B_2^{s+1,1} = B_2^s \cup \{x\}$. Otherwise, let $B_2^{s+1,1} = B_2^{s+1}$. In any case put $C_2^{s+1,1} = C_2^s$.

Substage 2. If there is an $x \in M_3^s$ such that:

(1) the set $B_2^{s+1,1} \cup C_2^{s+1,1} \cup \{x\}$ is independent,

(2) $\mathrm{supp}_{A_1}(x) - B_1^s \neq \varnothing$, and

(3) $(\forall y \in C_2^{s+1,1})[(\mathrm{supp}_{A_1}(x) - B_1^s) \cap (\mathrm{supp}_{A_1}(y) - B_1^s)] = \varnothing$,

then for the least such x we let $C_2^{s+1,2} = C_2^{s+1,1} \cup \{x\}$. Otherwise, let $C_2^{s+1,2} = C_2^{s+1,1}$. In each case put $B_2^{s+1,2} = B_2^{s+1,1}$.

Substage 3. If there is an $x \in C_2^{s+1,2}$ such that $x \in \mathrm{cl}\,(B_1^s)$, then for the least such x we let $B_2^{s+1} = B_2^{s+1,2} \cup \{x\}$ and $C_2^{s+1} = C_2^{s+1,2} - \{x\}$. Otherwise, let $B_2^{s+1} = B_2^{s+1,2}$ and $C_2^{s+1} = C_2^{s+1,2}$.

End of Construction

Let $A_2 =_{def} B_2 \cup C_2$. In the lemmas below, we will prove that B_2 and A_2 are c.e. bases (up to $=^*$) for M_2 and V_∞, respectively. $\square$

LEMMA 3.4. We have $\mathrm{cl}\,(B_2) =^* M_2$.

Proof. Clearly, $\mathrm{cl}\,(B_2) \subseteq^* M_2$. Indeed, if x is enumerated in B_2 at substage 1, then $x \in M_2$. If x is enumerated in B_2^{s+1} at substage 3, then $x \in C_2^{s+1,2} \subseteq M_3^{s+1}$ and $x \in \mathrm{cl}\,(B_1^s) \subseteq M_1$. All but finitely many such x will later be enumerated in M_2 because $M_1 \cap M_3 =^* M_2$. Thus $\mathrm{cl}\,(B_2) \subseteq^* M_2$.

We will now prove that $M_2 \subseteq^* \mathrm{cl}\,(B_2)$. Suppose $B_2 = B_{2,1} \cup B_{2,2}$, where $B_{2,1} \subseteq M_2$ and $B_{2,2}$ is a finite set such that $B_{2,2} \cap M_2 = \varnothing$. We know that $M_2 \cap M_3 \subseteq^* M_1 = \mathrm{cl}\,(B_1)$. Let P be a finite set of vectors for which $M_2 \cap M_3 \subseteq \mathrm{cl}\,(B_1 \cup P)$. Assume

$$\dim\left(\frac{M_2}{\mathrm{cl}\,(B_2)}\right) = \infty.$$

Let $x_1, x_2, \ldots$ be an infinite sequence of vectors from M_2, which are independent modulo $\mathrm{cl}\,(B_2)$. For every x_i, $i \geq 1$, let s_i be the least stage such that $x_i \in M_2^{s_i}$. The vector x_i is not enumerated into $B_2^{s_i}$ at substage 1 of stage s_i+1. Hence $x_i \in \mathrm{cl}\,(B_2^{s_i} \cup C_2^{s_i})$. Suppose $x_i = b_{2,1}^i + b_{2,2}^i + c_{2,0}^i$, where $b_{2,1}^i + b_{2,2}^i \in \mathrm{cl}\,(B_2^{s_i})$, $c_{2,0}^i \in \mathrm{cl}\,(C_2^{s_i})$, and $c_{2,0}^i \neq 0$. Since $M_2 \subseteq^* \mathrm{cl}\,(B_2)$, we can assume that for every i, we have $b_{2,1}^i \in \mathrm{cl}\,(B_2^{s_i}) \cap M_2$, while $b_{2,2}^i$ is a linear combination of finitely many vectors in B_2, none of which will be enumerated in M_2. Therefore, each $b_{2,2}^i$ belongs to the finite-dimensional space $\frac{\mathrm{cl}\,(B_2)}{M_2}$. Note that we do not claim that $b_{2,1}^i$ and $b_{2,2}^i$ can be found effectively.

Using standard linear algebra, we can eliminate the vectors $b_{2,2}^i$ from almost all of the equations

$$x_i = b_{2,1}^i + b_{2,2}^i + c_{2,0}^i \text{ for } i \geq 1.$$

Thus let $y_1, y_2, \ldots$ be a sequence of vectors from M_2, which are independent modulo $\mathrm{cl}\,(B_2)$ and are such that

$y_i = b_{2,3}^i + c_{2,3}^i$ for $i \geq 1$,

each $b_{2,3}^i$ is a linear combination of some of the vectors $\{b_{2,1}^j : j \geq 1\}$, and

each $c_{2,3}^i$ is a linear combination of some of the vectors $\{c_{2,0}^j : j \geq 1\}$.

Hence $b^i_{2,3} \in \mathrm{cl}\,(B_2) \cap M_2$, and by construction, $c^i_{2,3} \in M_3$ for all $i \geq 1$. Since $y_i \in M_2$ and $b^i_{2,3} \in M_2$, we obtain $c^i_{2,3} = (y_i - b^i_{2,3}) \in M_2$, and so $c^i_{2,3} \in M_2 \cap M_3 \subseteq \mathrm{cl}\,(B_1 \cup P)$. Assume that for each $i \geq 1$, the vector $c^i_{2,3}$ in the equation

$$y_i = b^i_{2,3} + c^i_{2,3}$$

is written as a linear combination of the vectors from the set $B_1 \cup P$. Since P is a finite set, we can find a nontrivial linear combination of these equations such that the vectors from P are eliminated from almost all of them. In other words, there are nonzero vectors z, b_2, and c_2 satisfying the following conditions:

(i) $z = b_2 + c_2$,

(ii) $z \in \mathrm{cl}\,(\{y_1, y_2, \ldots\}) \subseteq M_2$,

(iii) $b_2 \in \mathrm{cl}\,(\{b^i_{2,3} : i \geq 1\})$ is such that $b_2 \in \mathrm{cl}\,(B_2) \cap M_2$, and

(iv) $c_2 \in \mathrm{cl}\,(B_1) \cap \mathrm{cl}\,(\{c^i_{2,3} : i \geq 1\})$.

Also, note that c_2 is a linear combination of vectors that have been enumerated, at different stages, into C_2. Since $\mathrm{supp}_{A_1}(z) - B_1 = \varnothing$, all these support vectors must eventually be enumerated into B_2 and removed from C_2. This implies that $c_2 \in \mathrm{cl}\,(B_2)$, and hence

$$z = (b_2 + c_2) \in \mathrm{cl}\,(B_2),$$

which contradicts the fact that $y_1, y_2, \ldots$ is a sequence of vectors that are independent modulo $\mathrm{cl}\,(B_2)$. □

The proof that C_2 is infinite is similar to the proof of Lemma 2.6. Then the space $\mathrm{cl}\,(A_2) = \mathrm{cl}\,(B_2 \cup C_2)$ is infinite-dimensional modulo the maximal space M_2 and $M_2 \subsetneq^* \mathrm{cl}\,(A_2)$. Therefore, $\mathrm{cl}\,(A_2) =^* V_\infty$.

COROLLARY 3.5. If M is a maximal space with extendable basis, and N is a maximal space with no extendable basis, then

$$\mathcal{L}^*(M \cap N, \uparrow) \cong \mathbf{B_2}.$$

4. CLASSIFICATION OF ORBITS OF QUASIMAXIMAL SPACES WITH EXTENDABLE BASES

In this section we will establish our main result. Let $U_{11}, \ldots, U_{1n}$ and $U_{21}, \ldots, U_{2n}$ be two collections of maximal spaces with extendable bases. The bases of the spaces U_{ij} may be extendable to different computable bases of V_∞. The equivalence class of each space U_{ij}, $i \in \{1, 2\}$, $j \leq n$, is a coatom in the modular lattice $\mathcal{L}^*(V_\infty)$. If a c.e. basis B of U_{ij} is extendable to a computable basis A of V_∞, then B^* is a coatom in the distributive lattice $\mathcal{E}^*_A$ of c.e. subsets of A modulo $=^*$.

The lattices $\mathcal{L}^*(V_\infty)$ and $\mathcal{E}^*_A$ contain infinite chains. However, all principal filters of $\mathcal{L}^*(V_\infty)$ (or $\mathcal{E}^*_A$), which we consider here, will be modular (or distributive) lattices in which all chains are finite. All modular (and hence distributive) lattices in which all chains are finite satisfy the Jordan–Dedekind condition. We know that rank and corank functions can be defined on posets satisfying the Jordan–Dedekind condition. Here, we will explicitly define a specific rank function for some elements of the lattices $\mathcal{L}^*(V_\infty)$ and $\mathcal{E}^*_A$, as well as of some standard lattices that we will need later.

Definition 4.1. Let L be either the modular lattice $\mathcal{L}^*(V_\infty)$, the distributive lattice $\mathcal{E}^*_A$, the lattice of all subspaces of a finite-dimensional vector space W, or the finite Boolean algebra $\mathbf{B}_n$.

For $U \in L$, the *rank* of U, denoted $\operatorname{rank}(U)$, is defined as follows:

(1) $\operatorname{rank}(U) = 0$ if U is the greatest element in L;

(2) $(\forall U, V \in L)[(\operatorname{rank}(U) = n < \omega \wedge V \lessdot U) \Rightarrow \operatorname{rank}(V) = \operatorname{rank}(U) + 1]$.

If $U \in L$ and $\operatorname{rank}(U) = n < \omega$, then we say that the principal filter of U in L, $L(U, \uparrow)$, is a *lattice of rank* n.

Remark 4.2. (1) If W is a finite-dimensional vector space and $V \subseteq W$, then $\operatorname{rank}(V) = \dim(\frac{W}{V})$.

(2) If $\mathbf{B}_n$ is the Boolean algebra of subsets of $\{1, \ldots, n\}$ and $P \subseteq \{1, \ldots, n\}$, then $\operatorname{rank}(P) = n - |P|$.

(3) For each of the above maximal spaces U_{ij}, we have $\operatorname{rank}(U^*_{ij}) = 1$ in $\mathcal{L}^*(V_\infty)$.

(4) If a c.e. basis B for U_{ij} is extendable to a computable basis A for V_∞, then $\operatorname{rank}(B^*) = 1$ in $\mathcal{E}^*_A$.

Remark 4.3. Let X_i, $i = 1, \ldots, n$, be coatoms in one of the lattices $\mathcal{L}^*(V_\infty)$ or $\mathcal{E}^*_A$. Then $X = \bigcap\limits_{1 \le i \le n} X_i$ is an *irredundant intersection* (of the X_i's) up to $=^*$ if

$$(\forall P \subsetneq \{1, \ldots, n\}) \left[\bigcap_{j \in P} X_j \neq^* X \right].$$

Suppose that $X_1 =_{def} \bigcap\limits_{1 \le i \le n} X_{1i}$ and $X_2 =_{def} \bigcap\limits_{1 \le i \le n} X_{2i}$ are irredundant intersections up to $=^*$ in $\mathcal{L}^*(V_\infty)$. Then

$$(\forall P \subsetneq \{1, \ldots, n\}) \, (\forall k \in \{1, \ldots, n\}) \left[k \notin P \Rightarrow \bigcap_{j \in P \cup \{k\}} X_{1j} \lessdot \bigcap_{j \in P} X_{1j} \right].$$

Assume $k \notin P$. Then $\bigcap\limits_{j \in P \cup \{k\}} X_{1j} \lneq \bigcap\limits_{j \in P} X_{1j}$ because X_1 is an irredundant intersection. Moreover, $\bigcap\limits_{j \in P \cup \{k\}} X_{1j} \lessdot \bigcap\limits_{j \in P} X_{1j}$ since X_{1k} is a coatom in the modular lattice $\mathcal{L}^*(V_\infty)$. Using this fact we notice that for any sequence

$$\varnothing = P_0 \subset P_1 \subset \cdots \subset P_{n-1} \subset P_n = \{1, \ldots, n\},$$

the chain

$$X_1 =^* \bigcap_{j \in P_n} X_{1j} \lessdot \bigcap_{j \in P_{n-1}} X_{1j} \lessdot \cdots \lessdot \bigcap_{j \in P_1} X_{1j} \lessdot \bigcap_{j \in P_0} X_{1j} =^* V_\infty$$

is a maximal chain of rank n in $\mathcal{L}^*(X_1, \uparrow)$. Hence $\mathrm{rank}(X_1) = \mathrm{rank}(X_2) = n$, and so both $\mathcal{L}^*(X_1, \uparrow)$ and $\mathcal{L}^*(X_2, \uparrow)$ are rank-n lattices. Assume that each X_{ij}, where $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$, has an extendable basis. By Theorem 2.2, there are computable bases A_1 and A_2 for V_∞, and also maximal c.e. sets $D_{ij} \subset A_i$ such that $\mathrm{cl}\,(D_{ij}) =^* X_{ij}$. Note that for each $i \in \{1, 2\}$, the equivalence classes of D_{ij} for $j \in \{1, \ldots, n\}$ are distinct coatoms in the distributive lattice $\mathcal{E}^*_{A_i}$, and so

$$\bigcap_{1 \leq j \leq n} D_{ij} \lessdot \bigcap_{1 \leq j \leq n-1} D_{ij} \lessdot \cdots \lessdot D_{i1} \lessdot A_i$$

is a maximal chain. Hence $\mathcal{E}^*_{A_i}\left(\bigcap_{1 \leq j \leq n} D_{ij}, \uparrow\right)$ is a rank-n lattice (in fact, the Boolean algebra $\mathbf{B}_n$).

We will next give a sufficient and necessary condition for the existence of an isomorphism between $\mathcal{L}^*(X_1, \uparrow)$ and $\mathcal{L}^*(X_2, \uparrow)$. Then, assuming that $\mathcal{L}^*(X_1, \uparrow) \cong \mathcal{L}^*(X_2, \uparrow)$, we will give a sufficient and necessary condition for the existence of an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(X_1) = X_2$. Both characterizations will use the notion of an *m-degree type* of a quasimaximal subset of a fixed set A, where A is intended to be a basis for V_∞.

Remark 4.4. Let A be a fixed computable basis of V_∞. Suppose that D_i, $i = 1, \ldots, n$, are pairwise $*$-different maximal subsets of A, which fall into s equivalence classes K_j, $j = 1, \ldots, s$, with respect to $\equiv_m$. Assume also that $K_j = \{D_{n_{j-1}+1}, \ldots, D_{n_j}\}$, where $0 = n_0 < \cdots < n_s = n$, and define $k_j = |K_j| = n_j - n_{j-1}$. Let $G = \bigcap_{1 \leq i \leq n} D_i$.

(1) The *m-degree type* of the quasimaximal set G with respect to the basis A, denoted $\mathrm{type}_A(G)$, is the pair $(\mathrm{sizes}_G; \mathrm{degrees}_G)$ of sequences where

$$\mathrm{sizes}_G = (k_1, k_2, \ldots, k_s)$$

is the sequence of the cardinalities k_j of the classes K_j, and

$$\mathrm{degrees}_G = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$$

is the sequence of the m-degrees $\mathbf{a}_j$ of the sets in the classes K_j, $j = 1, \ldots, s$.

(2) Without loss of generality, we may assume that sizes_G in $\mathrm{type}_A(G)$ is a nondecreasing sequence $k_1 \leq k_2 \leq \cdots \leq k_s$. When the basis A is clear from the context, we will simply write

$$\mathrm{type}\,(G) = (\mathrm{sizes}_G; \mathrm{degrees}_G).$$

(3) Two quasimaximal sets $G_1, G_2 \subseteq A$ have the same *m-degree type* if there is a permutation of the domain of the sequence sizes_{G_1}, which makes the sequences sizes_{G_1} and sizes_{G_2} identical. Moreover, the same permutation of the domain of the sequence $\mathrm{degrees}_{G_1}$ makes the sequences $\mathrm{degrees}_{G_1}$ and $\mathrm{degrees}_{G_2}$ identical.

Remark 4.5. The spaces X_{ij}, $i = 1, 2$ and $j = 1, \ldots, n$, introduced at the beginning of this section, may have c.e. bases C_{ij}, respectively, which are maximal subsets of different computable bases B_{ij} of V_∞. By Theorem 2.2(i), we can find computable bases A_1 and A_2 of V_∞ and the c.e. sets D_{ij} that are maximal subsets of A_i such that $\mathrm{cl}\,(D_{ij}) =^* X_{ij}$. Furthermore, by Theorem 2.2(ii), C_{ij} and D_{ij} will have the same m-degree. Therefore, the notion of $\mathrm{type}_A(G)$ for a maximal or quasimaximal subset G of an extendable basis A will be, in a certain sense, basis-invariant. This will be made precise in Lemma 4.16.

Recall that an infinite set $C \subseteq \omega$ is said to be *cohesive* if for every c.e. set W either $W \cap C$ or $\overline{W} \cap C$ is finite. By definition, if a set $M \subseteq \omega$ is maximal, then $\overline{M} = \omega - M$ is cohesive. The notion of a *cohesive power* of a computable structure F over a cohesive set C, denoted $\prod\limits_C F$, was introduced in [22] (see also [21]). The cohesive power is a structure the domain of which consists of the equivalence classes of partial computable functions $\varphi : N \to \mathrm{dom}\,(F)$, which are defined for almost all elements of C, and are equivalent if their values are equal for almost all elements of C. Operations and relations in $\prod\limits_C F$ are defined naturally. For the case where F is a field, we can prove that $\prod\limits_C F$ is also a field. In [21], we established the following results regarding comaximal (hence cohesive) powers of the field Q.

THEOREM 4.6 [21]. If M is a maximal set, then $\prod\limits_{\overline{M}} Q$ has only trivial automorphisms.

THEOREM 4.7 [21]. For any maximal sets M_1 and M_2,

$$\prod_{\overline{M_1}} Q \cong \prod_{\overline{M_2}} Q \text{ iff } M_1 \equiv_m M_2.$$

To simplify the notation, by $Q_{\mathbf{a}} =_{def} \prod\limits_{\overline{M}} Q$ we denote the cohesive power of the field Q over a comaximal set $\overline{M}$ such that $\deg_m(M) = \mathbf{a}$. The use of this notation is justified by Theorem 4.7. Let $\mathcal{L}(l, F)$ denote the lattice of subspaces of an l-dimensional vector space over the field F.

Definition 4.8. (1) Suppose that A_1 and A_2 are computable bases of V_∞, and for $i \in \{1, 2\}$, the sets $D_{i1}, \ldots, D_{in}$ are pairwise $*$-different maximal subsets of A_i.

(2) Let $E_1 =_{def} \bigcap\limits_{1 \le j \le n} D_{1j}$ and $E_2 =_{def} \bigcap\limits_{1 \le j \le n} D_{2j}$. Note that E_1 and E_2 are quasimaximal subsets of rank n in A_1 and A_2, respectively.

(3) If $\mathrm{type}_{A_1}(E_1) = (k_1, k_2, \ldots, k_s; \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$, then, for $j = 1, \ldots, s$, put

$$E_1^j =_{def} \bigcap \{D_{1i} : 1 \le i \le n \wedge \deg_m(D_{1i}) = \mathbf{a}_j\}.$$

That is, E_1^j is a quasimaximal subset of E_1, which is the intersection of all maximal subsets D_{1i} of E_1 that have m-degree $\mathbf{a}_j$.

(4) Assume that for $j = 1, \ldots, s$, we have a fixed k_j-dimensional vector space W_j over $Q_{\mathbf{a}_j}$. Let $L_j =_{def} \mathcal{L}(k_j, Q_{\mathbf{a}_j})$ be the lattice of all subspaces of W_j.

(5) Below are the diagrams that reflect the structure of the lattices $\mathcal{L}(3, Q_{\mathbf{a}})$ and $\mathcal{L}(2, Q_{\mathbf{a}})$, respectively:

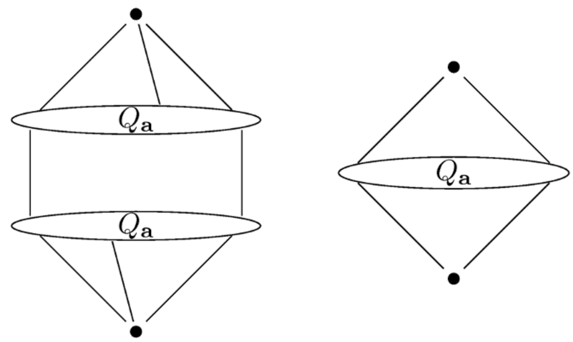

Diagram 2.1

(6) If $U_1, U_2 \in \mathfrak{L}(V_\infty)$ are maximal spaces, and we do not know the isomorphism type of the filter $\mathfrak{L}^*(U_1 \cap U_2, \uparrow)$, then the structure of this filter is reflected in the following diagram:

$$\begin{tikzcd} & V_\infty \arrow[dl, dash] \arrow[dr, dash] & \\ U_1 \arrow[dr, dash] & \bigstar & U_2 \arrow[dl, dash] \\ & U_1 \cap U_2 & \end{tikzcd}$$

Diagram 2.2

In [20], we described all possible principal filters of closures of quasimaximal subsets of a fixed computable basis of V_∞. The results, restated using the definition of an *m-degree type,* are as follows.

THEOREM 4.9 [20, Thm. 2]. If $\mathrm{type}_{A_1}(E_1) = (k_1; \mathbf{a}_1)$ (here $s = 1$), then

$$\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathfrak{L}(k_1, Q_{\mathbf{a}_1}).$$

THEOREM 4.10 [20, Thm. 3]. Suppose that

$$\mathrm{type}_{A_1}(E_1) = (k_1, k_2, \ldots, k_s; \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s).$$

Then

$$\begin{aligned} \mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) = {} & \mathfrak{L}^*\left(\bigcap_{1 \le j \le s} \mathrm{cl}\,(E_1^j), \uparrow \right) \cong \prod_{1 \le j \le s} \mathfrak{L}(\mathrm{cl}\,(E_1^j), \uparrow) \\ & \cong \prod_{1 \le j \le s} \mathfrak{L}(k_j, Q_{\mathbf{a}_j}) =_{def} \prod_{1 \le j \le s} L_j. \end{aligned}$$

Example 4.11. (a) If $\mathrm{type}_{A_1}(E_1) = (3; \mathbf{a})$, then $\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathfrak{L}(3, Q_{\mathbf{a}})$.

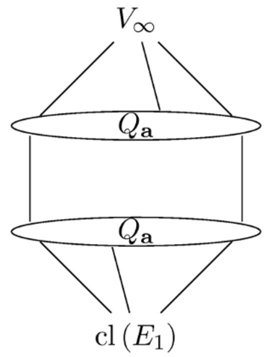

Diagram 3

(b) If $\mathrm{type}_{A_1}(E_1) = (1, 1, 1; \mathbf{a}, \mathbf{b}, \mathbf{c})$, then $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathbf{B}_1 \times \mathbf{B}_1 \times \mathbf{B}_1 \cong \mathbf{B}_3$.

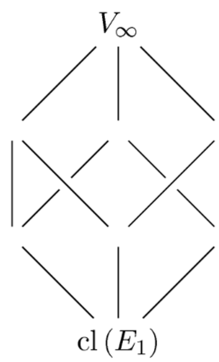

Diagram 4

(c) If $\mathrm{type}_{A_1}(E_1) = (1, 2; \mathbf{b}, \mathbf{a})$, then $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathbf{B}_1 \times \mathcal{L}(2, Q_{\mathbf{a}})$.

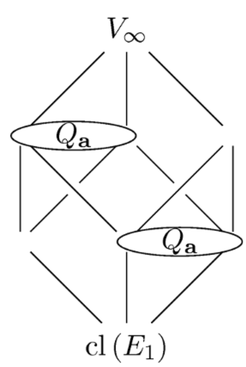

Diagram 5

In [26], we proved that a certain class of automorphisms of $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \prod_{j \le s} \mathcal{L}(\mathrm{cl}\,(E_1^j), \uparrow)$ can be extended to an automorphism of $\mathcal{L}^*\,(V_\infty)$.

THEOREM 4.12 [26, Thm. 2.1, Cor. 2.5]. Suppose that for $j \in \{1, \ldots, s\}$, there is a linear transformation ϕ_j of the vector space W_j, which induces an automorphism φ_j of $L_j =_{def} \mathcal{L}(k_j, Q_{\mathbf{a}_j})$.

Assume also that

$$\varphi =_{def} \langle \varphi_1, \ldots, \varphi_s \rangle : \prod_{j=1}^{s} L_j \to \prod_{j=1}^{s} L_j$$

is the corresponding product automorphism of $\prod_{j=1}^{s} L_j$ such that

$$\varphi((V_1, \ldots, V_s)) =_{def} (\varphi_1 (V_1), \ldots, \varphi_s (V_s)).$$

Let

$$\psi : \mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \to \prod_{1 \le j \le s} L_j$$

be the isomorphism constructed in the proof of Theorem 4.10, and let

$$\Phi_\varphi =_{def} \psi^{-1} \circ \varphi \circ \psi$$

be the induced automorphism of $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow)$. Then the automorphism Φ_φ can be extended to an automorphism Φ of $\mathcal{L}^*(V_\infty)$.

We recall the following:

Definition 4.13. Let W_1 and W_2 be vector spaces over the fields F_1 and F_2, respectively. A map $\phi : W_1 \to W_2$, together with an associated field isomorphism $\tau : F_1 \to F_2$, for which

$$(\forall v, w \in W_1)\,(\forall a, b \in F_1)\,[\phi(av + bw) = \tau(a)\phi(v) + \tau(b)\phi(w)]\,,$$

is called a *semilinear transformation*.

The fundamental theorem of projective geometry states that if the spaces W_1 and W_2 are such that $\dim(W_1) = \dim(W_2) \ge 3$, then all isomorphisms (if there are any) between the lattice of subspaces of W_1 and the lattice of subspaces of W_2 are induced by bijective semilinear transformations. The theorem also implies that the automorphisms of the lattice of the subspaces of a finite-dimensional vector space V for which $\dim(V) \ge 3$ are generated by bijective semilinear transformations of the space V. (For a good exposition of this theorem, see [27].)

By the fundamental theorem of projective geometry and Theorem 4.6, if σ is an automorphism of $L_j =_{def} \mathcal{L}(k_j, Q_{\mathbf{a}_j})$ and $k_j \ge 3$, then σ is induced by a bijective linear (not merely semilinear) transformation, since $Q_{\mathbf{a}_j}$ is rigid. Moreover, if $k_i, k_j \ge 3$ for some $i, j \le s$ with $i \ne j$, then Theorem 4.7 implies that $Q_{\mathbf{a}_i} \not\cong Q_{\mathbf{a}_j}$, even if $k_i = k_j$. Therefore, $\mathcal{L}(k_i, Q_{\mathbf{a}_i}) \not\cong \mathcal{L}(k_j, Q_{\mathbf{a}_j})$. With these observations in mind, we now discuss conditions for the existence of an isomorphism between $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow)$ and $\mathcal{L}^*(\mathrm{cl}\,(E_2), \uparrow)$.

Definition 4.14. Let $1^{s_1} 2^{s_2} (\ge 3)^{s_3}$ denote the sequence

$$\underbrace{1, \ldots, 1}_{s_1}, \underbrace{2, \ldots, 2}_{s_2}, \underbrace{k_{s_1+s_2+1}, \ldots, k_{s_1+s_2+s_3}}_{s_3},$$

where $k_i \ge 3$ for each i with $s_1 + s_2 + 1 \le i \le s_1 + s_2 + s_3$.

We give necessary and sufficient conditions for the existence of an automorphism Φ of $\mathfrak{L}^*(V_\infty)$ such that $\Phi(E_1) = E_2$. Note that such an automorphism exists only if $\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow)$ and $\mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow)$ are isomorphic. In the proposition below, we specify conditions for the existence of an isomorphism θ between $\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow)$ and $\mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow)$ in terms of $\mathrm{type}_{A_i}(E_i)$ for $i = 1, 2$.

PROPOSITION 4.15. Let $i \in \{1, 2\}$. Suppose that D_{ij}, $j = 1, \ldots, n$, are pairwise $*$-different quasimaximal subsets of a computable basis A_i of V_∞. Let $E_i = \bigcap\limits_{1 \le j \le n} D_{ij}$ and

$$\mathrm{type}_{A_i}(E_i) = (\mathrm{sizes}_{E_i}; \mathrm{degrees}_{E_i}).$$

The following statements hold:

(1) If sizes_{E_1} and sizes_{E_2} are not identical up to permutation, then $\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow)$ and $\mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow)$ are not isomorphic.

(2) If $\mathrm{sizes}_{E_1} = \mathrm{sizes}_{E_2} = 1^n$, then

$$\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathbf{B_n} \cong \mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow),$$

regardless of whether $\mathrm{degrees}_{E_1} = \mathrm{degrees}_{E_2}$.

(3) If $\mathrm{type}_{A_1}(E_1) = (2; \mathbf{a})$ and $\mathrm{type}_{A_2}(E_2) = (2; \mathbf{b})$, then

$$\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong 1\text{-}\infty\text{-}1 \cong \mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow),$$

regardless of whether $\mathbf{a} = \mathbf{b}$. Here, 1-∞-1 denotes the corresponding modular lattice.

(4) If $\mathrm{sizes}_{E_1} = \mathrm{sizes}_{E_2} = 1^{s_1} 2^{s_2}$, then

$$\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow),$$

regardless of whether $\mathrm{degrees}_{E_1} = \mathrm{degrees}_{E_2}$.

(5) If $p \ge 3$, $\mathrm{type}_{A_1}(E_1) = (p; \mathbf{a})$, and $\mathrm{type}_{A_2}(E_2) = (p; \mathbf{b})$, then

$$\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow)$$

iff $\mathbf{a} = \mathbf{b}$.

(6) If $\mathrm{sizes}_{E_1} = \mathrm{sizes}_{E_2} = 1^{s_1} 2^{s_2} (\ge 3)^{s_3}$, then

$$\mathfrak{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathfrak{L}^*(\mathrm{cl}\,(E_2), \uparrow)$$

iff the sequences

$$\mathrm{degrees}_{E_1}(s_1 + s_2 + 1), \ldots, \mathrm{degrees}_{E_1}(s_1 + s_2 + s_3)$$

and

$$\mathrm{degrees}_{E_2}(s_1 + s_2 + 1), \ldots, \mathrm{degrees}_{E_2}(s_1 + s_2 + s_3)$$

are identical up to the same permutation that also makes the sequences

$$\mathrm{sizes}_{E_1}(s_1 + s_2 + 1), \ldots, \mathrm{sizes}_{E_1}(s_1 + s_2 + s_3)$$

and

$$\text{sizes}_{E_2}(s_1+s_2+1),\ldots,\text{sizes}_{E_2}(s_1+s_2+s_3)$$

identical.

Proof. (1) Follows from Theorem 4.10.

(2) Follows from Theorem 4.7.

(3) By Theorem 4.7, we have $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow) \cong \mathcal{L}(2,Q_{\mathbf{a}})$ and $\mathcal{L}^*(\text{cl}\,(E_2),\uparrow) \cong \mathcal{L}(2,Q_{\mathbf{b}})$. Both $Q_{\mathbf{a}}$ and $Q_{\mathbf{b}}$ are countable. Let $\sigma : Q_{\mathbf{a}} \to Q_{\mathbf{b}}$ be a bijection for which $\sigma(0_{Q_{\mathbf{a}}}) = 0_{Q_{\mathbf{b}}}$.

Suppose that $\mathcal{L}(2,Q_{\mathbf{a}})$ and $\mathcal{L}(2,Q_{\mathbf{b}})$ are the lattices of all subspaces of two-dimensional vector spaces W_1 and W_2, respectively. Let $\{w_{11},w_{12}\}$ be a basis of W_1 and $\{w_{21},w_{22}\}$ be one of W_2. The map $\theta : \mathcal{L}(2,Q_{\mathbf{a}}) \to \mathcal{L}(2,Q_{\mathbf{b}})$ such that

$\theta(W_1) = W_2$,

$\theta(\text{cl}\,(w_{11}+aw_{12})) = \text{cl}\,(w_{21}+\sigma(a)w_{22})$, and

$\theta(0_{W_1}) = 0_{W_2}$,

is an isomorphism.

(4) Follows from parts (2) and (3) of this theorem and Theorem 4.10.

(5) Follows from the fundamental theorem of projective geometry and Theorems 4.7 and 4.9.

(6) Follows from parts (4) and (5) of this theorem and Theorem 4.10. □

Assume that $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)$ and $\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$ are isomorphic via an isomorphism θ. Our next goal is to find additional conditions which will guarantee that the isomorphism θ between $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)$ and $\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$ can be extended to an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(\text{cl}\,(E_1)^*) = \text{cl}\,(E_2)^*$ (which we will write as $\Phi(\text{cl}\,(E_1)) = \text{cl}\,(E_2)$ if it is clear from the context that Φ is an automorphism of $\mathcal{L}^*(V_\infty)$). The construction of such Φ will depend on whether $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)$ and $\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$ have common elements other than V_∞^*. In the lemma below we give conditions for the existence of a nontrivial intersection of $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)$ and $\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$. This lemma will be used in the proof of the main Theorem 4.17.

LEMMA 4.16. Suppose that $E_1 = \bigcap_{1\le j\le n_1} D_{1j}$ and $E_2 = \bigcap_{1\le j\le n_2} D_{2j}$, where for each $i \in \{1,2\}$, the sets D_{ij} for $j \in \{1,\ldots,n_i\}$ are pairwise $*$-different maximal subsets of a computable basis A_i. Assume that $\text{type}_{A_i}(E_i) = (n_i;\mathbf{a}_i)$ for $i = 1,2$.

(1) If $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)\cap\mathcal{L}^*(\text{cl}\,(E_2),\uparrow) \neq \{V_\infty^*\}$, then $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)\cap\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$ contains a coatom in $\mathcal{L}^*(V_\infty)$.

(2) If $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)\cap\mathcal{L}^*(\text{cl}\,(E_2),\uparrow) \neq \{V_\infty^*\}$, then $\mathbf{a}_1 = \mathbf{a}_2$.

(3) All coatoms of $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)$ have fully extendable bases. Every fully extendable basis of any such coatom is of m-degree $\mathbf{a}_1$.

Proof. (1) Suppose that $V \neq^* V_\infty$ is such that $V \in \mathcal{L}^*(\text{cl}\,(E_1),\uparrow)\cap\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$. Assume that $\text{rank}(V) = n$. Then $0 < n \le \min(n_1,n_2)$. Any maximal chain $V \subset \cdots \subset V_\infty$ in $\mathcal{L}^*(V_\infty)$ will contain a coatom that is an element of $\mathcal{L}^*(\text{cl}\,(E_1),\uparrow)\cap\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$.

(2) Suppose that $W \in \mathcal{L}^*(\text{cl}\,(E_1),\uparrow)\cap\mathcal{L}^*(\text{cl}\,(E_2),\uparrow)$ is a coatom in $\mathcal{L}^*(V_\infty)$.

Case 1. Let $n_1 = n_2 = 1$. In this event $W =^* \mathrm{cl}(E_1) =^* \mathrm{cl}(E_2)$. Note that $D_{11} = E_1$ and let D_{12} and D_{13} be other maximal subsets of A_1 of m-degree $\mathbf{a}_1$ such that

$$\mathcal{L}^*\left(\bigcap_{1\le i\le 3} \mathrm{cl}\ (D_{1i}), \uparrow\right) \cong \mathcal{L}(3, Q_{\mathbf{a}_1}).$$

On the other hand, $W =^* \mathrm{cl}\ (E_2)$ and $\deg_m(E_2) = \mathbf{a}_2$. We apply Theorem 2.2 to find a new computable basis A of V_∞, and also sets $D'_{21}, D_{22}, D_{23} \subset_{\max} A$ having m-degrees $\mathbf{a}_2, \mathbf{a}_1, \mathbf{a}_1$, respectively, and satisfying $W =^* \mathrm{cl}\ (D'_{21})$, $\mathrm{cl}\ (D_{12}) =^* \mathrm{cl}\ (D_{22})$, and $\mathrm{cl}\ (D_{13}) =^* \mathrm{cl}\ (D_{23})$. By Theorem 4.10,

$$\mathcal{L}^*(\mathrm{cl}\ \left(D'_{21}\right) \cap \mathrm{cl}\ (D_{22}) \cap \mathrm{cl}\ (D_{23}), \uparrow) \cong \begin{cases} \mathcal{L}(2, Q_{\mathbf{a}_1}) \times \mathbf{B}_1 & \text{if } \mathbf{a}_1 \neq \mathbf{a}_2; \\ \mathcal{L}(3, Q_{\mathbf{a}_1}) & \text{if } \mathbf{a}_1 = \mathbf{a}_2. \end{cases}$$

Since $\bigcap_{i\le 3} \mathrm{cl}\ (D_{1i}) =^* \mathrm{cl}\ (D'_{21}) \cap \mathrm{cl}\ (D_{22}) \cap \mathrm{cl}\ (D_{23})$, we must have $\mathbf{a}_1 = \mathbf{a}_2$.

Case 2. Let $\max(n_1, n_2) > 1$. Assume that $n_1 \geq 2$. The space W is a coatom in $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathcal{L}(n_1, Q_{\mathbf{a}_1})$. It is possible that either $W =^* \mathrm{cl}\,(D_{11})$ or $W =^* \mathrm{cl}\,(D_{12})$, but we cannot have both. Without loss of generality, we may assume that $W \neq^* \mathrm{cl}\,(D_{11})$. Note that $W \cap \mathrm{cl}\,(D_{11})$ has rank 2 in $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cong \mathcal{L}(n_1, Q_{\mathbf{a}_1})$. This implies that

$$\mathcal{L}^*(W \cap \mathrm{cl}\,(D_{11}), \uparrow) \cong \mathcal{L}(2, Q_{\mathbf{a}_1}).$$

Therefore, the lattice 1-3-1 is embeddable into $\mathcal{L}^*(W \cap \mathrm{cl}\,(D_{11}), \uparrow)$.

$$\begin{tikzcd}
& V_\infty \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathrm{cl}\,(D_{11}) \arrow[dr, dash] & W \arrow[d, dash] & {} \arrow[dl, dash] \\
& W \cap \mathrm{cl}\,(D_{11}) &
\end{tikzcd}$$

Diagram 6

In this embedding, W and $\mathrm{cl}\,(D_{11})$ are two coatoms of the lattice 1-3-1, and $W \cap \mathrm{cl}\,(D_{11})$ is the smallest element. Since $\mathrm{cl}\,(D_{11})$ is a maximal space with an extendable basis, it follows by Theorem 3.1 that the space W is also maximal with an extendable basis.

By Theorem 2.2, there is a computable basis A for V_∞, and there are sets $D'_{11}, D'_{12} \subset_{\max} A$ for which $\mathrm{cl}\,(D'_{11}) =^* \mathrm{cl}\,(D_{11})$ and $\mathrm{cl}\,(D'_{12}) = W$. Since $\mathcal{L}^*(W \cap \mathrm{cl}\,(D_{11}), \uparrow) \ncong \mathbf{B}_2$, we can apply Theorem 4.10 to obtain $\deg_m(D'_{11}) = \deg_m(D'_{12})$. By Theorem 2.2(ii), we have

$$\mathbf{a}_1 = \deg_m(D_{11}) = \deg_m(D'_{11}) = \deg_m(D'_{12}).$$

If $n_2 \geq 2$, then we can similarly prove that W has an extendable basis of m-degree $\mathbf{a}_2$. If $n_2 = 1$, then $W =^* \mathrm{cl}\,(E_2)$, and so W has an extendable basis of degree $\mathbf{a}_2$. In either case $\mathbf{a}_1 = \mathbf{a}_2$ by virtue of Case 1.

(3) Let $E_2 = E_1$ and W be a coatom in $\mathcal{L}^*(\mathrm{cl}\,(E_1), \uparrow) \cap \mathcal{L}^*(\mathrm{cl}\,(E_2), \uparrow)$. We then follow the proof of part (2) to find an extendable basis of W of m-degree $\mathbf{a}_1$. □

THEOREM 4.17. Let E_1 and E_2 be quasimaximal subsets of the computable bases A_1 and A_2, respectively. Then there is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that

$$\Phi(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2) \text{ iff } \mathrm{type}_{A_1}(E_1) = \mathrm{type}_{A_2}(E_2).$$

Proof. We will consider several cases for $\mathrm{type}_{A_1}(E_1)$ and $\mathrm{type}_{A_2}(E_2)$. The proofs of the if and the only if directions will have several cases.

Case 1 ($\Rightarrow$). Suppose that Φ is an automorphism of $\mathcal{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$. Assume that $\mathrm{type}_{A_1}(E_1) = (2; \mathbf{a})$ and $\mathrm{type}_{A_2}(E_2) = (2; \mathbf{b})$. We will prove that $\mathbf{a} = \mathbf{b}$. Let D_{13} be a maximal subset of A_1 with $\deg_m(D_{13}) = \mathbf{a}$ and $D_{13} \neq^* D_{1i}$ for $i = 1, 2$. Then

$$\begin{aligned} \mathcal{L}(3, Q_{\mathbf{a}}) &\cong \mathcal{L}^*(\mathrm{cl}\,(E_1 \cap D_{13}), \uparrow) \\ &\cong \mathcal{L}^*(\Phi(\mathrm{cl}\,(E_1)) \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow) \\ &\cong \mathcal{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow). \end{aligned}$$

Therefore, $\mathcal{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow)$ is a rank-3 lattice:

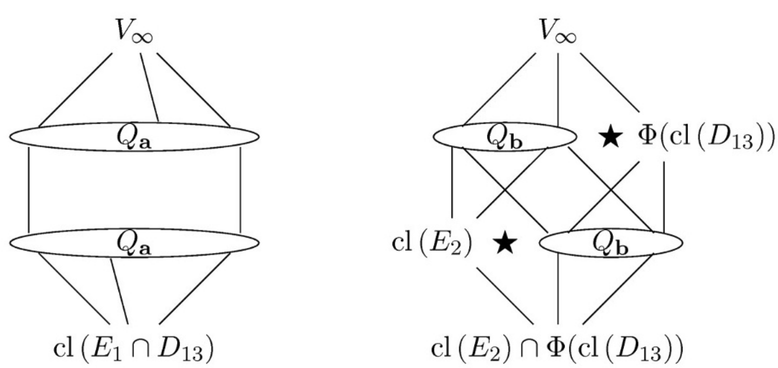

Diagram 7.1

We have the following subcases for the maximal space $\Phi(\mathrm{cl}\,(D_{13}))$.

Case 1.1 ($\Rightarrow$). Assume that $\Phi(\mathrm{cl}\,(D_{13}))$ is the equivalence class of a maximal subspace of V_∞ with no extendable basis. We know that $\mathcal{L}^*(\mathrm{cl}\,(E_2), \uparrow) \cong \mathcal{L}(2, Q_{\mathbf{b}})$. By Theorem 3.3, every coatom V in $\mathcal{L}^*(\mathrm{cl}\,(E_2), \uparrow)$ has an extendable basis. Again, by Theorem 3.3, for every coatom V in $\mathcal{L}^*(\mathrm{cl}\,(E_2), \uparrow)$, we have

$$\mathcal{L}^*(V \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow) \cong \mathbf{B_2}.$$

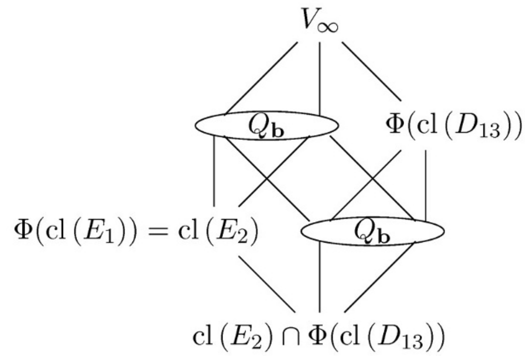

Diagram 7.2

Hence

$$\mathfrak{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow) \cong \mathbf{B_1} \times \mathfrak{L}(2, Q_{\mathbf{b}}) \ncong \mathfrak{L}(3, Q_{\mathbf{a}}),$$

and so this case is impossible.

Case 1.2 ($\Rightarrow$). Assume that $\Phi(\mathrm{cl}\,(D_{13}))$ is the equivalence class of a maximal subspace of V_∞, which has a basis extendable to a computable basis A_3 of V_∞. Since $\mathfrak{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(D_{13}), \uparrow)$ is a rank-3 lattice, we can apply Theorem 2.2. Without loss of generality, we may assume that D'_{23} is a maximal subset of A_2 such that $\mathrm{cl}\,(D'_{23}) =^* \Phi(\mathrm{cl}\,(D_{13}))$.

If $\deg_m(D'_{23}) \neq \deg_m(D_{21}) = \deg_m(D_{22}) = \mathbf{b}$, then, by Theorem 4.10,

$$\mathfrak{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(D_{13}), \uparrow) \cong \mathbf{B_1} \times \mathfrak{L}(2, Q_{\mathbf{b}}) \ncong \mathfrak{L}(3, Q_{\mathbf{a}}).$$

Thus $\deg_m(D'_{23}) = \deg_m(D_{21}) = \deg_m(D_{22})$, and again by Theorem 4.10,

$$\mathfrak{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(D_{13}), \uparrow) \cong \mathfrak{L}(3, Q_{\mathbf{b}}).$$

We already know that

$$\mathfrak{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(D_{13}), \uparrow) \cong \mathfrak{L}(3, Q_{\mathbf{a}}).$$

By the fundamental theorem of projective geometry, $\mathfrak{L}(3, Q_{\mathbf{a}}) \cong \mathfrak{L}(3, Q_{\mathbf{b}})$ iff $Q_{\mathbf{a}} \cong Q_{\mathbf{b}}$. In view of Theorem 4.7, $Q_{\mathbf{a}} \cong Q_{\mathbf{b}}$ iff $\mathbf{a} = \mathbf{b}$. Therefore, $\mathbf{a} = \mathbf{b}$.

Case 1 ($\Leftarrow$). Suppose $\mathrm{type}_{A_1}(E_1) = (2; \mathbf{a}) = \mathrm{type}_{A_2}(E_2)$. We will prove that there is an automorphism Φ of $\mathfrak{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$. Again, for $j \in \{1, 2\}$, let D_{j3} be a maximal subset of A_j having m-degree $\mathbf{a}$ and satisfying $D_{j3} \neq^* D_{ji}$ for $i = 1, 2$. Assume $V = \mathrm{cl}\,(E_1 \cap D_{13}) \cap \mathrm{cl}\,(E_2 \cap D_{23})$. Let r be the rank of $\mathfrak{L}^*(V, \uparrow)$. Notice that $3 \leq r \leq 6$.

If $r = 3$, then

$$\begin{aligned}\mathfrak{L}^*(V, \uparrow) &= \mathfrak{L}^*(\mathrm{cl}\,(D_{11}) \cap \mathrm{cl}\,(D_{12}) \cap \mathrm{cl}\,(D_{13})) \\ &= \mathfrak{L}^*(\mathrm{cl}\,(D_{21}) \cap \mathrm{cl}\,(D_{22}) \cap \mathrm{cl}\,(D_{23})) \cong \mathfrak{L}(3, Q_{\mathbf{a}}).\end{aligned}$$

Let δ be an isomorphism that maps $\mathcal{L}^*(V,\uparrow)$ to the lattice of subspaces of a fixed 3-dimensional space X over $Q_{\mathbf{a}}$. For $k=1,2$, we let

$v_{k1} \in X$ be a basis vector for $\delta(\mathrm{cl}\,(D_{k2})) \cap \delta(\mathrm{cl}\,(D_{k3}))$,

$v_{k2} \in X$ be a basis vector for $\delta(\mathrm{cl}\,(D_{k1})) \cap \delta(\mathrm{cl}\,(D_{k3}))$, and

$v_{k3} \in X$ be a basis vector for $\delta(\mathrm{cl}\,(D_{k1})) \cap \delta(\mathrm{cl}\,(D_{k2}))$.

Note that both $\{v_{11}, v_{12}, v_{13}\}$ and $\{v_{21}, v_{22}, v_{23}\}$ are bases for X. Let σ_0 be a linear map on X such that $\sigma_0(v_{1i}) = v_{2i}$ (for $i=1,2,3$), and let $\overline{\sigma_0}$ be the automorphism of the lattice of subspaces of X that is induced by the linear map σ_0. Define $\sigma =_{def} \delta^{-1} \circ \overline{\sigma_0} \circ \delta$. Notice that σ is an automorphism of $\mathcal{L}^*(V,\uparrow)$, which is induced by the linear transformation σ_0 and is such that $\sigma(\mathrm{cl}\,(D_{1i})) = \mathrm{cl}\,(D_{2i})$ for $i \in \{1,2,3\}$. By virtue of Theorem 4.12, σ can be extended to an automorphism Φ of $\mathcal{L}^*(V_\infty)$.

If $4 \le r \le 6$, then the equivalence class of V in $\mathcal{L}^*(V,\uparrow)$ is an irredundant intersection of r of the (six) coatoms $\mathrm{cl}\,(D_{ij})$, where $i=1,2,3$ and $j=1,2$. There is no loss of generality in assuming that

$$V =^* \mathrm{cl}\,(D_{11}) \cap \mathrm{cl}\,(D_{12}) \cap \mathrm{cl}\,(D_{13}) \cap \mathrm{cl}\,(D_{21}) \cap \cdots \cap \mathrm{cl}\,(D_{2,(r-3)}).$$

By Theorem 2.2, we can suppose that D_{1i}, $1 \le i \le 3$, and D_{2i}, $1 \le i \le r-3$, are all maximal subsets of the same computable basis A of V_∞, of which each has m-degree $\mathbf{a}$. Then

$$\mathcal{L}^*(V,\uparrow) \cong \mathcal{L}(r, Q_{\mathbf{a}}).$$

Every $\mathrm{cl}\,(D_{ij})$ is a coatom in $\mathcal{L}^*(V,\uparrow)$ and the equivalence classes of both $\mathrm{cl}\left(\bigcap_{1\le i\le 3} D_{1i}\right)$ and $\mathrm{cl}\left(\bigcap_{1\le i\le 3} D_{2i}\right)$ have rank 3 (hence corank $r-3$) in $\mathcal{L}^*(V,\uparrow)$. Thus we can find an automorphism σ of $\mathcal{L}^*(V,\uparrow)$, which is induced by a linear transformation (of an r-dimensional space over the rigid field $Q_{\mathbf{a}}$) and satisfies $\sigma(\mathrm{cl}\,(D_{1i})) = \mathrm{cl}\,(D_{2i})$ for $1 \le i \le 3$. In view of Theorem 4.12, σ can be extended to an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$.

Case 2 ($\Rightarrow$). Suppose Φ is an automorphism of $\mathcal{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$. Assume $\mathrm{type}_{A_1}(E_1) = (1;\mathbf{a})$ and $\mathrm{type}_{A_2}(E_2) = (1;\mathbf{b})$. We will prove that $\mathbf{a}=\mathbf{b}$. Let D_{1j}, $j=2,3$, be maximal subsets of A_1 of m-degree $\mathbf{a}$ such that D_{1j}, $j=1,2,3$, are pairwise distinct up to $=^*$.

Then

$$\begin{aligned}
\mathcal{L}(3,Q_{\mathbf{a}}) &\cong \mathcal{L}^*(\mathrm{cl}\,(E_1) \cap \mathrm{cl}\,(D_{12}) \cap \mathrm{cl}\,(D_{13}), \uparrow) \\
&\cong \mathcal{L}^*(\Phi(\mathrm{cl}\,(E_1)) \cap \Phi(\mathrm{cl}\,(D_{12})) \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow) \\
&= \mathcal{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(\mathrm{cl}\,(D_{12})) \cap \Phi(\mathrm{cl}\,(D_{13})), \uparrow).
\end{aligned}$$

We will consider the following subcases.

Case 2.1 ($\Rightarrow$). Suppose that both $\Phi(\mathrm{cl}\,(D_{12}))$ and $\Phi(\mathrm{cl}\,(D_{13}))$ are equivalence classes of maximal subspaces of V_∞ with no extendable bases. By Theorem 3.3, $\mathcal{L}^*(\mathrm{cl}\,(E_2) \cap \Phi(\mathrm{cl}\,(D_{1j})), \uparrow) \cong \mathbf{B_2}$ for $j=2,3$. This implies that

$$\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(\mathrm{cl}\,(D_{12}))\cap\Phi(\mathrm{cl}\,(D_{13})),\uparrow)\not\cong\mathcal{L}(3,Q_{\mathbf{a}}),$$

and so this case is impossible.

Case 2.2 ($\Rightarrow$). Assume that exactly one of $\Phi(\mathrm{cl}\,(D_{12}))$ and $\Phi(\mathrm{cl}\,(D_{13}))$ has an extendable c.e. basis. There is no loss of generality in letting it be $\Phi(\mathrm{cl}\,(D_{12}))$. By virtue of Theorem 3.3, $\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(\mathrm{cl}\,(D_{13})),\uparrow)\cong\mathbf{B_2}$. This implies that

$$\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(\mathrm{cl}\,(D_{12}))\cap\Phi(\mathrm{cl}\,(D_{13})),\uparrow)\not\cong\mathcal{L}(3,Q_{\mathbf{a}}),$$

and so this case is also impossible.

Case 2.3 ($\Rightarrow$). Suppose that both $\Phi(\mathrm{cl}\,(D_{12}))$ and $\Phi(\mathrm{cl}\,(D_{13}))$ have extendable bases. Since

$$\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(\mathrm{cl}\,(D_{12}))\cap\Phi(\mathrm{cl}\,(D_{13})),\uparrow)\cong\mathcal{L}(3,Q_{\mathbf{a}})$$

is a rank-3 lattice, by Theorem 2.2, we can assume the following:

(i) the sets D_{22} and D_{23} are such that $\Phi(\mathrm{cl}\,(D_{12}))=^*\mathrm{cl}\,(D_{22})$ and $\Phi(\mathrm{cl}\,(D_{13}))=^*\mathrm{cl}\,(D_{23})$;

(ii) D_{2i}, $i=1,2,3$, are maximal subsets of the same basis A_2 for V_∞.

If $(\deg_m(D_{22})=\mathbf{b}\wedge\deg_m(D_{23})\neq\mathbf{b})\vee(\deg_m(D_{22})\neq\mathbf{b}\wedge\deg_m(D_{23})=\mathbf{b})$ then, by Theorem 4.10,

$$\begin{aligned}\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(D_{12})\cap\Phi(D_{13}),\uparrow)&=\mathcal{L}^*(\mathrm{cl}\,(D_{21})\cap\mathrm{cl}\,(D_{22})\cap\mathrm{cl}\,(D_{23}),\uparrow)\\&\cong\mathbf{B_1}\times\mathcal{L}(2,Q_{\mathbf{b}})\not\cong\mathcal{L}(3,Q_{\mathbf{a}}).\end{aligned}$$

If $(\deg_m(D_{22})=\mathbf{c}\wedge\deg_m(D_{23})=\mathbf{c}\wedge\mathbf{c}\neq\mathbf{b})$ then, by Theorem 4.10,

$$\begin{aligned}\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(D_{12})\cap\Phi(D_{13}),\uparrow)&=\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\mathrm{cl}\,(D_{22})\cap\mathrm{cl}\,(D_{23}),\uparrow)\\&\cong\mathbf{B_1}\times\mathcal{L}(2,Q_{\mathbf{c}})\not\cong\mathcal{L}(3,Q_{\mathbf{a}}).\end{aligned}$$

If $(\deg_m(D_{22})=\mathbf{c}\wedge\deg_m(D_{23})=\mathbf{d}\wedge\mathbf{c}\neq\mathbf{b}\wedge\mathbf{d}\neq\mathbf{b})$ then, by Theorem 4.10,

$$\begin{aligned}\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(D_{12})\cap\Phi(D_{13}),\uparrow)&=\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\mathrm{cl}\,(D_{22})\cap\mathrm{cl}\,(D_{23}),\uparrow)\\&\cong\mathbf{B_3}\not\cong\mathcal{L}(3,Q_{\mathbf{a}}).\end{aligned}$$

Therefore, $\deg_m(D_{21})=\deg_m(D_{22})=\deg_m(D_{23})=\mathbf{b}$ and

$$\begin{aligned}\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(D_{12})\cap\Phi(D_{13}),\uparrow)&=\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\mathrm{cl}\,(D_{22})\cap\mathrm{cl}\,(D_{23}),\uparrow)\\&\cong\mathcal{L}(3,Q_{\mathbf{b}}).\end{aligned}$$

We already know that

$$\mathcal{L}^*(\mathrm{cl}\,(E_2)\cap\Phi(D_{12})\cap\Phi(D_{13}),\uparrow)\cong\mathcal{L}(3,Q_{\mathbf{a}}).$$

By the fundamental theorem of projective geometry, $\mathfrak{L}(3, Q_{\mathbf{a}}) \cong \mathfrak{L}(3, Q_{\mathbf{b}})$ iff $Q_{\mathbf{a}} \cong Q_{\mathbf{b}}$. In view of Theorem 4.7, $Q_{\mathbf{a}} \cong Q_{\mathbf{b}}$ iff $\mathbf{a} = \mathbf{b}$. Hence $\mathbf{a} = \mathbf{b}$.

Case 2 ($\Leftarrow$). The proof is similar to the proof for Case 1 ($\Leftarrow$) above.

Case 3 ($\Rightarrow$). Suppose that Φ is an automorphism of $\mathfrak{L}^*(V_\infty)$ such that $\Phi(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$. Then $\mathfrak{L}^*(E_1, \uparrow) \cong \mathfrak{L}^*(E_2, \uparrow)$, and by Proposition 4.15(1), we have $\mathrm{sizes}_{E_1} = \mathrm{sizes}_{E_2}$. Assume that

$$\mathrm{type}_{A_1}(E_1) = (1^{s_1}2^{s_2}(\geq 3)^{s_3}; \mathbf{a}_1, \ldots, \mathbf{a}_{s_1+s_2+s_3}),$$
$$\mathrm{type}_{A_2}(E_2) = (1^{s_1}2^{s_2}(\geq 3); \mathbf{b}_1, \ldots, \mathbf{b}_{s_1+s_2+s_3}).$$

By Case 2 ($\Rightarrow$), the sequences $(\mathbf{a}_1, \ldots, \mathbf{a}_{s_1})$ and $(\mathbf{b}_1, \ldots, \mathbf{b}_{s_1})$ will be identical up to the permutation naturally induced by the map Φ. By Case 1 ($\Rightarrow$), the sequences $(\mathbf{a}_{s_1+1}, \ldots, \mathbf{a}_{s_1+s_2})$ and $(\mathbf{b}_{s_1+1}, \ldots, \mathbf{b}_{s_1+s_2})$, too, will be identical (up to the permutation naturally induced by the map Φ). By Proposition 4.15(6), the sequences

$$(\mathbf{a}_{s_1+s_2+1}, \ldots, \mathbf{a}_{s_1+s_2+s_3}) \text{ and } (\mathbf{b}_{s_1+s_2+1}, \ldots, \mathbf{b}_{s_1+s_2+s_3})$$

will also be identical (up to the permutation naturally induced by the map Φ). Therefore, $\mathrm{type}_{A_1}(E_1) = \mathrm{type}_{A_2}(E_2)$.

Case 3 ($\Leftarrow$). Suppose that

$$\mathrm{type}_{A_1}(E_1) = \mathrm{type}_{A_2}(E_2) = (1^{s_1}2^{s_2}(\geq 3)^{s_3}; \mathbf{a}_1, \ldots, \mathbf{a}_{s_1+s_2+s_3}).$$

Let $s =_{def} s_1 + s_2 + s_3$. Assume that $E_1 = \bigcap_{i=1}^{n} D_{1i}$ and $E_2 = \bigcap_{i=1}^{n} D_{2i}$, where D_{1i} is a maximal subset of A_1 and D_{2i} is one of A_2, for $i \in \{1, \ldots, n\}$. Suppose that the collections $\{D_{1i}\}_{i=1}^{n}$ and $\{D_{2i}\}_{i=1}^{n}$ each is partitioned into s equivalence classes according to the m-degrees of its members. Let the jth equivalence class have k_j members for $j \leq s$. Therefore,

$$\mathrm{type}_{A_1}(E_1) = \mathrm{type}_{A_2}(E_2) = (k_1, \ldots, k_s; \mathbf{a}_1, \ldots, \mathbf{a}_s),$$

where $k_i = 1$ for $i \leq s_1$, $k_i = 2$ for $s_1 + 1 \leq i \leq s_1 + s_2$, $k_i \geq 3$ for $s_1 + s_2 + 1 \leq i \leq s$, $\sum_{i=1}^{s} k_i = n$, and the m-degrees $\mathbf{a}_1, \ldots, \mathbf{a}_s$ are pairwise distinct.

Suppose that $V = \mathrm{cl}\,(E_1) \cap \mathrm{cl}\,(E_2)$ and the rank of $\mathfrak{L}^*(V, \uparrow)$ is r. Note that $n \leq r \leq 2n$. Without loss of generality, we may assume that

$$V =^* \begin{cases} \mathrm{cl}\,(D_{11}) \cap \cdots \cap \mathrm{cl}\,(D_{1n}) & \text{if } r = n, \\ \bigcap\limits_{1 \leq i \leq n} \mathrm{cl}\,(D_{1i}) \cap \bigcap\limits_{1 \leq j \leq r-n} \mathrm{cl}\,(D_{2j}) & \text{if } r > n \end{cases}$$

is an irredundant intersection of the above r coatoms of $\mathfrak{L}^*(V_\infty)$. By Theorem 2.2, we assume that $D_{11}, \ldots, D_{1n}, D_{21}, \ldots, D_{2,(r-n)}$ (or $D_{11}, \ldots, D_{1n}$ if $r = n$) are subsets of the same computable basis A for V_∞. Define

$$E = \begin{cases} D_{11} \cap \cdots \cap D_{1n} & \text{if } r = n; \\ \bigcap_{1 \le i \le n} D_{1i} \cap \bigcap_{1 \le j \le r-n} D_{2j} & \text{if } r > n. \end{cases}$$

For each coatom $\mathrm{cl}\,(D_{2j})$, where $r - n < j \le n$, let U_j be a minimal subset of $\{D_{11}, \ldots, D_{1n}, D_{21}, \ldots, D_{2,(r-n)}\}$ such that $\mathrm{cl}\,(D_{2j})$ is a coatom in $\mathcal{L}^*\left(\bigcap_{D \in U_j} \mathrm{cl}\,(D), \uparrow\right)$. Suppose $|U_j| = k$. Note that, by Theorems 4.9 and 4.10, we have

$$\mathcal{L}^*\left(\bigcap_{D \in U_j} \mathrm{cl}\,(D), \uparrow\right) \cong \begin{cases} \mathcal{L}(k, Q_{\mathbf{a}}) & \text{if } (\forall C, D \in U_j)\,[\deg_m(C) = \deg_m(D) = \mathbf{a}]; \\ \prod_i \mathcal{L}(k_i, Q_{\mathbf{a}_i}) & \text{otherwise.} \end{cases}$$

Every coatom in any product lattice of type $\prod_i \mathcal{L}(k_i, Q_{\mathbf{a}_i})$ is the union of the coatoms of $\mathcal{L}(k_i, Q_{\mathbf{a}_i})$, where each $\mathcal{L}(k_i, Q_{\mathbf{a}_i})$ is viewed as a principal filter in $\mathcal{L}^*(V_\infty)$ in the context of Theorem 4.10. Therefore, $\mathrm{cl}\,(D_{2j})$ is a coatom in exactly one of the lattices $\mathcal{L}(k_i, Q_{\mathbf{a}_i})$. Since the set $U_j \subseteq \{D_{11}, \ldots, D_{1n}, D_{21}, \ldots, D_{2,(r-n)}\}$ is minimal such that $\mathrm{cl}\,(D_{2j})$ is a coatom in $\mathcal{L}^*\left(\bigcap_{D \in U_j} \mathrm{cl}\,(D), \uparrow\right)$, there is a unique m-degree $\mathbf{a} \in \{\mathbf{a}_1, \ldots, \mathbf{a}_s\}$ for which

$$\mathcal{L}^*\left(\bigcap_{D \in U_j} \mathrm{cl}\,(D), \uparrow\right) \cong \mathcal{L}(k, Q_{\mathbf{a}}).$$

By virtue of Lemma 4.16(3), we may conclude that

$$(\forall j \in \{r - n + 1, \ldots, n\})\,(\forall C \in U_j)\,[\deg_m(D_{2j}) = \deg_m(C)].$$

Parts (2) and (3) of Lemma 4.16 allow us to uniquely determine the m-degree of *any* extendable basis of a maximal space. For each $i \in \{1, \ldots, s\}$, we can now define

$$\begin{aligned} U^{(1)}_{\mathbf{a}_i} &\subseteq \{D_{11}, \ldots, D_{1n}\}, \\ U^{(2)}_{\mathbf{a}_i} &\subseteq \{D_{21}, \ldots, D_{2n}\}, \\ U^{(3)}_{\mathbf{a}_i} &\subseteq \{D_{11}, \ldots, D_{1n}, D_{21}, \ldots, D_{2,(r-n)}\} \text{ if } r > n, \\ U^{(3)}_{\mathbf{a}_i} &\subseteq \{D_{11}, \ldots, D_{1n}\} \text{ if } r = n \end{aligned}$$

to be maximal collections for which

$$(\forall C \in U^{(l)}_{\mathbf{a}_i})[\deg_m(C) = \mathbf{a}_i], \text{ where } l = 1, 2, 3.$$

Then, for any $i \in \{1, \ldots, s\}$ and any $j = 1, 2$, the following hold:
$\left|U^{(1)}_{\mathbf{a}_i}\right| = \left|U^{(2)}_{\mathbf{a}_i}\right| = k_i$;
$(\forall D \in U^{(j)}_{\mathbf{a}_i})\left[\mathrm{cl}\,(D) \text{ is a coatom in } \mathcal{L}^*\left(\bigcap_{D \in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right)\right]$.

Suppose $\left|U^{(3)}_{\mathbf{a}_i}\right| = m_i$. Then

$$\text{type}_A(E) = (m_1, \ldots, m_s; \mathbf{a}_1, \ldots, \mathbf{a}_s),$$

where $k_i \leq m_i \leq 2k_i$ for every $i \in \{1, \ldots, s\}$ and $r = \sum\limits_{i=1}^{s} m_i$. Furthermore, by Theorem 4.10, we have

$$\mathcal{L}^*(\text{cl}\,(E), \uparrow) \cong \prod_{i=1}^{s} \mathcal{L}^*\left(\bigcap_{D \in U^{(3)}_{\mathbf{a}_i}} \text{cl}\,(D), \uparrow\right) \cong \prod_{i=1}^{s} \mathcal{L}(m_i, Q_{\mathbf{a}_i}),$$

and also for $j = 1, 2$,

$$\mathcal{L}^*(\text{cl}\,(E_j), \uparrow) \cong \prod_{i=1}^{s} \mathcal{L}^*\left(\bigcap_{D \in U^{(j)}_{\mathbf{a}_i}} \text{cl}\,(D), \uparrow\right) \cong \prod_{i=1}^{s} \mathcal{L}(k_i, Q_{\mathbf{a}_i}).$$

We obtain the following diagram:

$$\begin{tikzcd}
\mathcal{L}^*\left(\bigcap\limits_{D \in U^{(1)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[r, "\cong"] \arrow[d, "\text{principal}"', "\text{filter}"] & \mathcal{L}(k_i, Q_{\mathbf{a}_i}) \arrow[r, hook] \arrow[d] & \prod\limits_{i=1}^{s} \mathcal{L}(k_i, Q_{\mathbf{a}_i}) \arrow[r, "\cong"] \arrow[d] & \mathcal{L}^*(cl(E_1), \uparrow) \arrow[d, "\text{principal}"', "\text{filter}"] \\
\mathcal{L}^*\left(\bigcap\limits_{D \in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[r, "\cong"] & \mathcal{L}(m_i, Q_{\mathbf{a}_i}) \arrow[r, hook] & \prod\limits_{i=1}^{s} \mathcal{L}(m_i, Q_{\mathbf{a}_i}) \arrow[r, "\cong"] & \mathcal{L}^*(cl(E), \uparrow) \\
\mathcal{L}^*\left(\bigcap\limits_{D \in U^{(2)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[r, "\cong"] \arrow[u, "\text{principal}", "\text{filter}"'] & \mathcal{L}(k_i, Q_{\mathbf{a}_i}) \arrow[r, hook] \arrow[u] & \prod\limits_{i=1}^{s} \mathcal{L}(k_i, Q_{\mathbf{a}_i}) \arrow[r, "\cong"] \arrow[u] & \mathcal{L}^*(cl(E_2), \uparrow) \arrow[u, "\text{principal}", "\text{filter}"']
\end{tikzcd}$$

Both $\mathcal{L}^*\left(\bigcap\limits_{D \in U^{(1)}_{\mathbf{a}_i}} \text{cl}\,(D), \uparrow\right)$ and $\mathcal{L}^*\left(\bigcap\limits_{D \in U^{(2)}_{\mathbf{a}_i}} \text{cl}\,(D), \uparrow\right)$ are principal filters and rank-k_i sublattices of the lattice

$$\mathcal{L}^*\left(\bigcap_{D \in U^{(3)}_{\mathbf{a}_i}} \text{cl}\,(D), \uparrow\right),$$

which in turn is isomorphic to $\mathcal{L}(m_i, Q_{\mathbf{a}_i})$. Suppose that $L_i = \mathcal{L}(m_i, Q_{\mathbf{a}_i})$ for $i \in \{1, \ldots, s\}$ is the lattice of all subspaces of a fixed m_i-dimensional vector space W_i, and that the map

$$\sigma_i : \mathcal{L}^*\left(\bigcap_{D \in U^{(3)}_{\mathbf{a}_i}} \text{cl}\,(D), \uparrow\right) \to L_i$$

is an isomorphism. Then $\sigma_i\left(\bigcap\limits_{D \in U^{(1)}_{\mathbf{a}_i}} \text{cl}\,(D)\right)$ and $\sigma_i\left(\bigcap\limits_{D \in U^{(2)}_{\mathbf{a}_i}} \text{cl}\,(D)\right)$ are both elements of L_i, which are $(m_i - k_i)$-dimensional subspaces of W_i. Suppose that each ϕ_i is a linear transformation of W_i

such that

$$\phi_i\left(\sigma_i\left(\bigcap_{D\in U^{(1)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right)\right) = \sigma_i\left(\bigcap_{D\in U^{(2)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right).$$

Assume that ϕ_i induces an automorphism φ_i of L_i for which

$$\varphi_i\left(\sigma_i\left(\bigcap_{D\in U^{(1)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right)\right) = \sigma_i\left(\bigcap_{D\in U^{(2)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right).$$

Let $F_i = \sigma_i^{-1} \circ \varphi_i \circ \sigma_i$. We have the following diagram:

$$\begin{array}{ccc}
\mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right) & \xrightarrow{\sigma_i} & L_i \\
\Big\downarrow F_i & & \varphi_i \Big\downarrow \text{ induced by } \phi_i \\
\mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right) & \xleftarrow{\sigma_i^{-1}} & L_i.
\end{array}$$

We will now construct an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi\,(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$. Example 4.19 below gives us an idea of how to build a map for the case where $s = 2$. In general, note, each map F_i is an automorphism of the filter $\mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right)$ for which

$$F_i\left(\bigcap_{D\in U^{(1)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right) = \bigcap_{D\in U^{(2)}_{\mathbf{a}_i}} \mathrm{cl}\,(D).$$

Hence the product map

$$\bigotimes_{i=1}^{s} F =_{def} \langle F_1, \ldots, F_s\rangle : \prod_{i=1}^{s} \mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right) \to \prod_{i=1}^{s} \mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right),$$

which is defined by

$$\langle F_1, \ldots, F_s\rangle\,((V_1, \ldots, V_s)) =_{def} (F_1\,(V_1)\,, \ldots, F_s\,(V_s))$$

for any $(V_1, \ldots, V_s) \in \prod_{i=1}^{s} \mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right)$, naturally gives rise to an automorphism of

$$\mathcal{L}^*(\mathrm{cl}\,(E), \uparrow) \cong \prod_{i=1}^{s} \mathcal{L}^*\left(\bigcap_{D\in U^{(3)}_{\mathbf{a}_i}} \mathrm{cl}\,(D), \uparrow\right).$$

For simplicity, we will also denote this automorphism by $\bigotimes_{i=1}^{s} F$. Then

$$\bigotimes_{i=1}^{s} F\left(\bigcap_{D\in U^{(1)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right) = \bigcap_{D\in U^{(2)}_{\mathbf{a}_i}} \mathrm{cl}\,(D) \text{ for every } i = 1,\dots,s$$

and

$$\bigotimes_{i=1}^{s} F\left(\bigcap_{i=1}^{s}\bigcap_{D\in U^{(1)}_{\mathbf{a}_i}} \mathrm{cl}\,(D)\right) = \bigcap_{i=1}^{s}\bigcap_{D\in U^{(2)}_{\mathbf{a}_i}} \mathrm{cl}\,(D) \text{ or, equivalently,}$$

$$\bigotimes_{i=1}^{s} F(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2).$$

The map $\bigotimes_{i=1}^{s} F$ is an automorphism of $\mathfrak{L}^*(\mathrm{cl}\,(E),\uparrow)$, which is generated by the linear maps ϕ_i, where $i \in \{1,\dots,s\}$. By virtue of Theorem 4.12, $\bigotimes_{i=1}^{s} F$ can be extended to an automorphism Φ of $\mathfrak{L}^*(V_\infty)$. Hence $\Phi\,(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2)$. □

COROLLARY 4.18. Let M_1 and M_2 be maximal subsets of the computable bases A_1 and A_2, respectively, for V_∞. Then there is an automorphism Φ of $\mathfrak{L}^*(V_\infty)$ such that

$$\Phi(\mathrm{cl}\,(M_1)) = \mathrm{cl}\,(M_2) \text{ iff } \deg_m(M_1) = \deg_m(M_2).$$

Example 4.19. In this example and the corresponding Diagram 8, we give an idea of how to build an automorphism $\bigotimes_{i=1}^{2} F =_{def} \langle F_1, F_2\rangle$ of $\mathfrak{L}^*(\mathrm{cl}\,(E),\uparrow)$ for the case where $s = 2$. Let $x_j =_{def} \bigcap_{D\in U^{(j)}_{\mathbf{a}_1}} \mathrm{cl}\,(D)$ and $y_j =_{def} \bigcap_{D\in U^{(j)}_{\mathbf{a}_2}} \mathrm{cl}\,(D)$, where $j = 1,2,3$. Then F_1 and F_2 are automorphisms of $\mathfrak{L}^*(x_3,\uparrow)$ and $\mathfrak{L}^*(y_3,\uparrow)$, respectively, with $F_1(x_1) = x_2$ and $F_2(y_1) = y_2$. Since

$$\mathfrak{L}^*(\mathrm{cl}\,(E),\uparrow) \cong \mathfrak{L}^*(x_3,\uparrow) \otimes \mathfrak{L}^*(y_3,\uparrow),$$

$\bigotimes_{i=1}^{2} F =_{def} \langle F_1, F_2\rangle$ is an automorphism of $\mathfrak{L}^*(\mathrm{cl}\,(E),\uparrow)$, for which

$$\langle F_1, F_2\rangle\,((x_1, y_1)) = (x_2, y_2).$$

Note that $\mathfrak{L}^*(\mathrm{cl}\,(E_i),\uparrow) \cong \mathfrak{L}^*(x_i,\uparrow) \otimes \mathfrak{L}^*(y_i,\uparrow)$ for $i = 1,2$. Moreover, (x_i, y_i) corresponds to the smallest element in the principal filter in $\mathfrak{L}^*(\mathrm{cl}\,(E_i),\uparrow)$ for $i = 1,2$, and this element is the equivalence class of $\mathrm{cl}\,(E_i)$. Therefore,

$$\langle F_1, F_2\rangle\,(\mathrm{cl}\,(E_1)) = \mathrm{cl}\,(E_2).$$

Look at the following diagram (to improve readability, we do not draw all the lines):

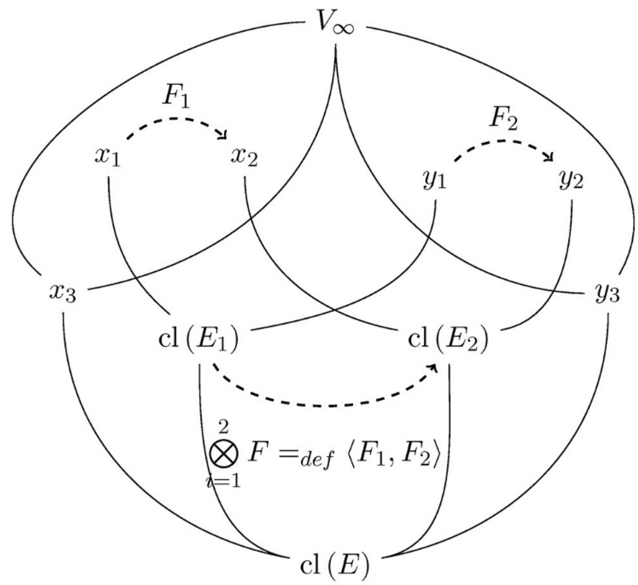

Diagram 8

Example 4.20. The diagram below summarizes our construction of the map Φ:

$$\begin{tikzcd}
\bigcap\limits_{D\in U^{(1)}_{\mathbf{a}_i}} cl(D) \arrow[r, "F_i", "\text{induced by } \phi_i"'] \arrow[d, "\text{element}"', "\text{of}"] & \bigcap\limits_{D\in U^{(2)}_{\mathbf{a}_i}} cl(D) \arrow[d, "\text{element}"', "\text{of}"] \\
\mathcal{L}^*\left(\bigcap\limits_{D\in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[r, "F_i", "\text{induced by } \phi_i"'] \arrow[d, "\text{principal filter}"', "\text{in}"] & \mathcal{L}^*\left(\bigcap\limits_{D\in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[d, "\text{principal filter}"', "\text{in}"] \\
\mathcal{L}^*\left(\bigcap\limits_{i=1}^{s} \bigcap\limits_{D\in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[r, equal] \arrow[d, "\cong"'] & \mathcal{L}^*\left(\bigcap\limits_{i=1}^{s} \bigcap\limits_{D\in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[d, "\cong"'] \\
\prod\limits_{i=1}^{s} \mathcal{L}^*\left(\bigcap\limits_{D\in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[r, "\langle F_1,\ldots,F_s\rangle"] \arrow[d, "\cong"'] & \prod\limits_{i=1}^{s} \mathcal{L}^*\left(\bigcap\limits_{D\in U^{(3)}_{\mathbf{a}_i}} cl(D), \uparrow\right) \arrow[d, "\cong"'] \\
\mathcal{L}^*(cl(E), \uparrow) \arrow[r, "\Phi"] \arrow[d, "\text{principal}"', "\text{filter}"] & \mathcal{L}^*(cl(E), \uparrow) \arrow[d, "\text{principal}"', "\text{filter}"] \\
\mathcal{L}^*(V_\infty) \arrow[r, "\Phi"] & \mathcal{L}^*(V_\infty)
\end{tikzcd}$$

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