Left ordered and discretely ordered groups

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Joint with Rhemtulla and Rolfsen

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Definition

A group G is left-ordered means that it has a total order $\leq$ such that for all $g, x, y \in G$ with $x \leq y$, then $gx \leq gy$.

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A group G is bi-ordered means that it has a total order $\leq$ such that for all $g, x, y \in G$ with $x \leq y$, then $gx \leq gy$ and $xg \leq yg$.

Remark

Often the term "ordered" is used for "bi-ordered".

Motivation

Theorem (Well Known)

If G is left orderable, then $\mathbb{C}G$ satisfies the zero divisor conjecture: if $0 \neq \alpha, \beta \in \mathbb{C}G$, then $\alpha\beta \neq 0$.

Theorem (Linnell 91)

Let $\ell^2(G)$ denote the Hilbert space with basis the elements of G. If G is left orderable, $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in \ell^2(G)$, then $\alpha\beta \neq 0$.

Applications to L^2-cohomology of left-ordered groups.

Conjecture (Atiyah conjecture)

Let $\mathcal{U}(G)$ denote the ring of unbounded operators affiliated to the group von Neumann algebra of G. If G is torsion-free, then there exists a division ring D such that $\mathbb{C}G \subseteq D \subseteq \mathcal{U}(G)$.

Conjecture

If G is left orderable, then $\mathbb{C}G$ can be embedded in a division ring.

Definition

Let $(G, \leq)$ be a left-ordered group. The order $\leq$ is *discrete* if there exists a least positive $g \in G$. Otherwise the order $\leq$ is said to be dense.

Remark

A least positive element in a left-ordered group is necessarily unique.

Remark

Let $(G, \leq)$ be a left-ordered group with $\leq$ dense. Then

- If $h, k \in G$ and $h < k$, then there exists $x \in G$ with $h < x < k$.
- If $G \neq 1$ is countable, then G is order isomorphic to $\mathbb{Q}$.

Example

$\mathbb{Q}$ is left orderable, but has *no* discrete left order.

Definition

Let $(G, \leq)$ be a left-ordered group and let $H \leq G$. Then H is a *convex* subgroup if whenever $h, k \in H$ and $x \in G$ with $h < x < k$, then $x \in H$.

Remark

Let $(G, \leq)$ be a discrete left-ordered group and let $g \in G$ be its least element. Then $\langle g \rangle$ is a convex subgroup of G.
Thus if $\langle g \rangle \lhd G$, then $G/\langle g \rangle$ is left-orderable.

Results with Rhemtulla and Rolfsen: bi-orders

Theorem

Let G be a group with a discrete bi-order. If z is the minimal positive element, then $z \in \mathrm{Z}(G)$ (the center of G).
Also $G/\langle z\rangle$ is bi-orderable (because $\langle z\rangle$ is convex).

Corollary

Thus nonabelian free groups have no discrete bi-order.

Theorem

Let G be a bi-orderable group and let $1 \neq z \in \mathrm{Z}(G)$. If $\mathrm{Z}(G)/\langle z\rangle$ is torsion free (i.e. z is isolated in $\mathrm{Z}(G)$), then G has a discrete bi-order.

Example

If G is a bi-orderable group, then $\mathbb{Z} \times G$ has a discrete bi-order (with $(1, 1)$ as minimal positive element).

Results with Rhemtulla and Rolfsen: left-orders

Theorem

Nontrivial finitely generated residually torsion-free nilpotent groups have discrete left orders.

Remark

These left orders can be chosen to be of "Conrad type" (lexicographic).

Examples

Example

The following groups have discrete left orders.

- Nontrivial free groups (not necessarily finitely generated).
- Surface groups, except projective plane.
- Pure Braid groups.
- Nontrivial right-angled Coxeter groups.

Proof.

These groups are all residually torsion-free nilpotent, except the Klein bottle. Result follows if we assume the group is finitely generated and is not the Klein bottle. □

Remark

With the exception of the torus, the surface groups above have trivial center, so do not have discrete bi-orders.

Further examples

Remark

Examples so far have all been locally indicable: G is locally indicable means if $1 \neq H \leq G$ and H is finitely generated, then H/H' if infinite (H' is the commutator subgroup).

Example

Artin Braid groups B_n ($n \geq 2$) have discrete left orders. Some of these left orders are dense when restricted to B_n' (Clay-Rolfsen).
B_n' is finitely generated and perfect for $n \geq 5$ (so B_n is not locally indicable for $n \geq 5$).

The next example involve groups which Bergman used to exhibit left-orderable groups that are not locally indicable.

Example

Let H be a finitely generated subgroup of $\mathrm{SL}_2(\mathbb{R})$ with $-I \in H$ (where I is the identity matrix), and suppose H contains a diagonal matrix other than $\pm I$. Then there is a group G with infinite cyclic center Z, $G/Z \cong H$, and G has a discrete left order. Furthermore in many cases $G = G'$.

Conjecture

Conjecture

A group with infinitely many left orders has a dense order.

Space of left orders

Definition

Let G be a left-orderable group and let $\mathcal{O}_G$ denote the set of left orders on G. For $g \in G$, set $U_g = \{< \in \mathcal{O}(G) \mid 1 < g\}$. Then the space of left orders on G is $\mathcal{O}(G)$ with topology given by the subbase $\{U_g \mid 1 \neq g \in G\}$.

Can use this to prove

Theorem (Linnell, Navas)

$|\mathcal{O}(G)|$ *is either finite or uncountable.*

Definition (Derived subset)

Let X be a Hausdorff topological space. Then X' is the subset obtained from X by removing all isolated points.

Remark

- X' is the set of limit points of X
- X' is a closed subset of X
- X' may still contain isolated points

Definition (Derived series)

Let X be a Hausdorff topological space. For each ordinal α, define $X^{(\alpha)}$ by transfinite induction.

- $X^{(0)} = X$.
- $X^{(\alpha+1)} = (X^{(\alpha)})'$.
- $X^{(\alpha)} = \bigcap_{\lambda<\alpha} X^{(\lambda)}$ if α is a limit ordinal.

Use

Proposition

Let X be a nonempty countable compact Hausdorff space. Then there exists an ordinal α such that $X^{(\alpha)}$ is finite and nonempty.