Spring 2024: Department of Mathematics
Math 6720 Topics in Logic (CRN 98403)

Computable Algebra

MW 2:20–3:35pm

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Description. We will cover several topics of current research interest in *computable algebra*, where we use methods of computability theory to investigate algorithmic nature of notions and constructions in algebraic structures. While some constructions are algorithmic, or can be replaced by algorithmic ones yielding the same results, others are intrinsically non-algorithmic. For example, while the standard model of arithmetic, the natural numbers with addition and multiplication, is computable, Tennenbaum showed that there is no computable model of arithmetic that is *nonstandard* (one that contains infinite numbers).

In the last few decades there has been increasing interest in computable algebra. However, this investigation goes back to van der Waerden who in his 1930 book *Modern Algebra* defined an *explicitly given field* as one the elements of which are uniquely represented by distinguishable symbols with which we can perform the field operations of addition and multiplication (and hence of subtraction and division) algorithmically. In his pioneering paper, van der Waerden essentially proved that an explicit field does not necessarily have an algorithm for *splitting polynomials* into their irreducible factors. This was before the Turing machine was invented and the precise mathematical theory of algorithms emerged. Van der Waerden's definition eventually led to the notion of a *computable structure*, one of the main notions in computable algebra. An example of a positive result concerning fields is Rabin's celebrated theorem that every computable field can be embedded into a computable algebraic closure.

Other examples of famous negative results include Dehn's problems for finitely presented groups. A group is finitely presented if it has finitely many generators and relations they must satisfy. The group elements are words on the generators and their inverses, and the operation is writing. In 1911, Dehn formulated three fundamental decision problems, one of which was the *word problem* – whether two words represent the same element. It was established independently by Novikov and Boone in the 1950s that the word problem is undecidable in general. It turns out that a finitely presented group has a computable isomorphic copy if and only if it has a decidable word problem. Around 1945, Tarski asked whether free groups (ones without relations) with different finite numbers of generators greater than one, satisfy the same first-order quantifier sentences. Sela, in a series of six papers in the early 2000's, gave a positive answer. However, we can use infinitary sentences to distinguish different free groups, and computable infinitary sentences to characterize different computable free groups.

Specific topics will include algorithmic properties of equivalence structures, groups, orders, fields, and vector spaces, computable isomorphism problems, and Turing degrees of algebraic structures and their isomorphism types. Additional topics will be chosen based on students' interests.

Course Material. All course material will be provided in class.

Required background. MATH 2971 or an equivalent, or CS 2312 or CS 3313 or an equivalent. Familiarity with the notion of an algorithm and algebraic structure.

Math 6720 can be taken for credit repeatedly. Advanced undergraduate students may also take this course for credit.

Grading. Based on take-home assignments and individual projects, and their presentations.