Spaces of Orders and Their Turing Degree Spectra*

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Abstract

We investigate computability theoretic and topological properties of spaces of orders on computable orderable groups. A left order on a group G is a linear order of the domain of G, which is left-invariant under the group operation. Right orders and bi-orders are defined similarly. In particular, we study groups for which the spaces of left orders are homeomorphic to the Cantor set, and their Turing degree spectra contain certain upper cones of degrees. Our approach unifies and extends Sikora's investigation of orders on groups in topology [28] and Solomon's investigation of these orders in computable algebra [31]. Furthermore, we establish that a computable free group F_n of rank $n > 1$ has a bi-order in every Turing degree.

1 Introduction and Preliminaries

Orderable groups have recently become a popular topic of study among topologists (see, for example, [26], [10], [28], [9], [2], [8], [24], etc.). It follows from the results in [3] that important geometric properties of 3-manifolds can be related to the spaces of left orders of their fundamental groups. In this paper, we apply the techniques of computability theory to further analyze the spaces of left orders and bi-orders on computable groups. Recall that a countable group $(G, \cdot)$ is *computable* if its domain G is a computable set and its group-theoretic operation $\cdot$ is computable. For any infinite computable group we may assume, without loss of generality, that its domain is ω. For more on computable structures see [17], [1], and [15]. A group G is *left-orderable* (*partially left-orderable*, respectively) if there is an order (a partial order,

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respectively) $\leq$ of the domain G such that $\leq$ is left-invariant under the group operation; that is, for every $x,\, y,\, z \in G$,

$$x \leq y \Rightarrow zx \leq zy.$$

Similarly, a group G is *right-orderable* (*partially right-orderable*, respectively) if there is an order $\leq$ (partial order, respectively) of elements of G, which is right-invariant under the group operation. Every left order $\leq$ on a group G induces the associated right order $\leq_r$ on G defined by

$$g \leq_r h \Leftrightarrow h^{-1} \leq g^{-1}.$$

A group G is called *bi-orderable,* or simply *orderable,* if there is a *bi-order* (*order*) $\leq$ on G; that is, for all $x, y, z \in G$,

$$x \leq y \Rightarrow (zx \leq zy \wedge xz \leq yz).$$

Analogously, we define a *partial order* on G.

Let $e \in G$ be the identity element. A partial left order $\leq$ on a group G is determined by and often identified with its *positive partial cone*:

$$P = \{a \in G : e \leq a\}.$$

Similarly, the *negative partial cone is*

$$P^{-1} = \{a \in G : a^{-1} \in P\} = \{a \in G : a \leq e\}.$$

One can easily verify that P is a *subsemigroup* of G (i.e., $PP \subseteq P$), which is *pure* (i.e., $P \cap P^{-1} = \{e\}$). Such a subsemigroup $P \subseteq G$ defines a left order on G if and only if P is *total* (i.e., $P \cup P^{-1} = G$). Moreover, P defines a bi-order on G if, in addition, P is a *normal* subsemigroup (i.e., $g^{-1}Pg \subseteq P$ for every $g \in G$). Let

$$LO(G) = \{P \subseteq G \mid P \text{ is a total and pure subsemigroup of } G\}.$$

We call the set $LO(G)$ the *space of left orders* of G. Analogously, we define the space of bi-orders on G,

$$BiO(G) = \{P \subseteq G \mid P \text{ is a total, pure and normal subsemigroup of } G\}.$$

Recall that for a given finite subset $A = \{g_1, g_2, ..., g_n\} \subset G \backslash \{e\}$, the subsemigroup $S(A)$ of G generated by A is

$$S(A) = sgr(A) = \{a_1 a_2 ... a_k |\; k \in \omega;\, a_1, ..., a_k \in A\}.$$

For convenience we assume that $S(\emptyset) = \emptyset$. One can verify that the subset $S(A) \cup \{e\}$ is a positive partial cone if and only if $e \notin S(A)$.

Not all partial left orders on G can be extended to (total) left orders. As it was shown in [6], a partial left order determined by P can be extended to a left order on G if and only if for every finite set $\{g_1, g_2, ..., g_n\} \subset G \backslash \{e\}$ there is a corresponding sequence of integers $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where $\epsilon_i \in \{1, -1\}$, such that

$$e \notin sgr((P - \{e\}) \cup \{g_1^{\epsilon_1}, g_2^{\epsilon_2}, ..., g_n^{\epsilon_n}\}).$$

A group G with the property that every partial left order (partial bi-order, respectively) on G can be extended to a left order (bi-order, respectively) is called *fully left-orderable* (*fully orderable,* respectively). Necessary and sufficient conditions for a group to be fully left-orderable were established by Todorinov [33]. Groups from many important classes, such as torsion-free abelian groups or, more generally, torsion-free nilpotent groups, are fully left-orderable [27]. On the other hand, free groups of ranks > 1 are bi-orderable, but not fully left-orderable [21]. For more on orders on groups see [20], [23], [13], and [7].

As usual, we use $\leq_T$ for Turing reducibility and $\equiv_T$ for Turing equivalence of sets. The Turing degree of X is denoted by $\deg(X)$, the *nth* Turing jump of the empty set by $\emptyset^{(n)}$, and $\mathbf{0}^{(n)} = \deg(\emptyset^{(n)})$. In particular, $\mathbf{0}'$ denotes the Turing degree of the halting set $\emptyset'$. The Turing degrees form an upper semilattice for which the partial order $\leq$ and the join $\vee$ are defined as follows. For $X, Y \subseteq \omega$ with Turing degrees $\mathbf{x}$ and $\mathbf{y}$, we have

$$\mathbf{x} \leq \mathbf{y} \Longleftrightarrow X \leq_T Y,$$

and

$$\mathbf{x} \vee \mathbf{y} = \deg(X \oplus Y),$$

where $X \oplus Y = \{2x \mid x \in X\} \cup \{2x+1 \mid x \in Y\}$. We denote by $\mathcal{D}$ the set of all Turing degrees. A finite set of natural numbers can be coded (indexed) using the *canonical indexing* as follows. Let $D_0 =_{def} \emptyset$. For $m > 0$, define $D_m = \{d_0, \ldots, d_{k-1}\}$, where $m = 2^{d_0} + \ldots + 2^{d_{k-1}}$ and $d_0 < \ldots < d_{k-1}$. A sequence $\{p_i\}_{i\in\omega}$ of finite sets is called a *strong array* ($\mathbf{x}$-*computable strong array,* respectively) if there is a unary computable ($\mathbf{x}$-computable, respectively) function ν such that for every $i \in \omega$, $p_i = D_{\nu(i)}$. For more on computability theoretic notions see [30].

For a computable torsion-free abelian group of rank 1, the space of bi-orders has exactly two elements, both of which are computable. In [31], Solomon showed that a computable, torsion-free abelian group G of a finite rank greater than 1 has an order in every Turing degree. Therefore, such G has $2^{\aleph_0}$ bi-orders and, as it was shown by Sikora [28], a topology on $LO(G)$ can be defined in a natural way. The space $LO(G)$ with this topology is compact, metrizable, and totally disconnected topological space, as it was established in [28] and [8]. Furthermore, this $LO(G)$ is homeomorphic to the Cantor set [28].

Navas-Flores [24] has recently established that for a free group F_n of rank $n > 1$, $LO(F_n)$ is homeomorphic to the Cantor set. Hence, his result confirms Sikora's Conjecture 2.2 in [28]. However, it still remains unknown whether the closed subspace $BiO(F_n)$ of $LO(F_n)$ is homeomorphic to the Cantor set.

These results of Solomon, Sikora, and Navas-Flores sparked our interest in the relationship between topological and computability theoretic results for orderable groups. Key results in Section 2 are Theorem 2, Proposition 8 and Corollary 9, which relate topological and computability theoretic properties of $LO(G)$. In Section 3 we provide a geometric insight for orders on torsion-free abelian groups. In particular, we show that $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set. This result complements the computability theoretic result obtained by Solomon in [31] for computable torsion-free abelian groups of infinite rank. Similarly, our result in Section 3 about the Turing degree spectrum of $BiO(F_n)$ (hence of $LO(F_n)$) complements Navas-Flores' result on the homeomorphism of $LO(F_n)$ to the Cantor set.

2 Uncountable spaces of orders on computable groups

In this section, we analyze algorithmic and topological properties of the spaces of left orders. For a computable group G, we define the notion of a Turing degree spectrum of $LO(G)$. (For another notion of a degree spectrum of a relation on a computable structure see [16], [18].)

Definition 1 *The* Turing degree spectrum of LO(G) *for a computable group G, DgSp(LO(G)), is the set of the Turing degrees of all left orders on G:*

$$DgSp(LO(G)) = \{\deg(P) \mid P \in LO(G)\}.$$

Similarly, for right orders and bi-orders on G, we define $DgSp(RO(G))$ and $DgSp(BiO(G))$. The following result gives a natural general sufficient condition for $DgSp(LO(G))$ to contain certain Turing degrees depending on the upper cone determined by a given degree $\mathbf{d}$.

Theorem 2 *Let G be a computable group, and $\mathbf{d}$ be a Turing degree. Assume that there is a $\mathbf{d}$-computable strong array $\mathbb{P} = \{p_i\}_{i\in\omega}$ of finite subsets of $G\backslash\{e\}$ such that for all elements $p \in \mathbb{P}$, we have*:

(**i**) $e \notin sgr(p)$;

(**ii**) $(\exists q, r \in \mathbb{P})\ (\exists a \in G\backslash\{e\})\ [q \supseteq p \wedge r \supseteq p \wedge a \in q \wedge a^{-1} \in r]$;

(**iii**) $(\forall a \in G\backslash\{e\})\ (\exists q \in \mathbb{P})\ [q \supseteq p \wedge (a \in q \vee a^{-1} \in q)]$.

Then for every Turing degree $\mathbf{x} \geq \mathbf{d}$, there exists $\mathbf{z} \in DgSp(LO(G))$ such that $\mathbf{x} = \mathbf{z} \vee \mathbf{d}$.

Proof. Assume that $D \subseteq \omega$ is an infinite set of degree $\mathbf{d}$. Let $\mathbf{x} \geq \mathbf{d}$, and let $X \subset \omega$ be such that $\deg(X) = \mathbf{x}$. We use the finite extension argument to construct a subsequence $\{p_{i_s}\}_{s\in\omega}$ of finite sets of generators for partial left orders $\{P_{i_s}\}_{s\in\omega}$ on G, where $P_{i_s} = sgr(p_{i_s}) \cup \{e\}$, such that $Q = \bigcup_{s\in\omega} P_{i_s}$ defines a left order on G with $\deg(Q) \vee \mathbf{d} = \mathbf{x}$.

Construction

Stage $s = 0$. Set $p_{i_0} = p_0$.

Stage $s + 1 = 2k + 1$. At the end of stage s we have p_{i_s}. Using oracle D, find the first pair of sets r_1, $r_2 \in \mathbb{P}$ such that

$$r_1 \supseteq p_{i_s} \wedge r_2 \supseteq p_{i_s},$$

and for some $a \in G\backslash\{e\}$, we have

$$a \in r_1 \wedge a^{-1} \in r_2.$$

Hence, clearly, $sgr(r_1) \neq sgr(r_2)$. Choose the least such a and define $p_{i_{s+1}}$, using oracle X, by

$$p_{i_{s+1}} = \begin{cases} r_1 & \text{if } k \in X, \\ r_2 & \text{if } k \notin X. \end{cases}$$

Stage $s + 1 = 2k + 2$. Let a be the least element in G such that

$$a \notin p_{i_s} \wedge a^{-1} \notin p_{i_s}.$$

Define $p_{i_{s+1}}$ to be the first set $q \in \mathbb{P}$ such that $q \supseteq p_{i_s}$ and $a \in q \vee a^{-1} \in q$.

End of the construction.

Let

$$Q =_{def} \bigcup_{s\in\omega} P_{i_s}.$$

Clearly, Q is a left order on G.

Lemma 3 $Q \leq_T X$

Proof. Since the construction is computable in D and X, and $D \leq_T X$, the sequence $\{p_{i_s}\}_{s\in\omega}$ is X-computable. Let $a \in G\backslash\{e\}$. To decide whether $a \in Q$ or $a^{-1} \in Q$, find the least s such that $a \in p_{i_s}$ or $a^{-1} \in p_{i_s}$. ■

Lemma 4 $X \leq_T Q \oplus D$

Proof. By induction, we show that the sequence $\{p_{i_s}\}_{s\in\omega}$ is $(Q \oplus D)$-computable and $X \leq_T Q \oplus D$. Given p_{i_s}, $s = 2k$, D-computably find the corresponding r_1, r_2, and then the corresponding a. Since

$$a \in p_{i_{s+1}} \text{ if } k \in X,$$

and

$$a^{-1} \in p_{i_{s+1}} \text{ if } k \notin X,$$

use $(Q \oplus D)$-oracle to determine whether $k \in X$. Since

$$p_{i_{s+1}} = r_1 \text{ if } k \in X,$$

and $p_{i_{s+1}} = r_2$ if $k \notin X$,use a $(Q \oplus D)$-oracle to determine $p_{i_{s+1}}$. Now, given $p_{i_{s+1}}$, find $p_{i_{s+2}}$ computably in D. ■

Let $\mathbf{z} = \deg(Q)$. Since $X \leq_T Q \oplus D$ and $Q \leq_T X$ and $D \leq_T X$, we have $X \equiv_T Q \oplus D$, which implies that $\mathbf{x} = \mathbf{z} \vee \mathbf{d}$. ■

As a corollary of Theorem 2 when $\mathbf{d} = \mathbf{0}$, we obtain the following result, which gives a sufficient condition for the Turing degree spectrum of left orders on G to contain all Turing degrees.

Corollary 5 *Let G be a computable group. Assume that there is a strong array $\mathbb{P} = \{p_i\}_{i\in\omega}$ of finite subsets of $G\backslash\{e\}$, which satisfies conditions* (**i**) − (**iii**) *of* Theorem 2. *Then $DgSp(LO(G)) = \mathcal{D}$.*[1]

Recall that $\mathcal{U} \subset \mathcal{D}$ is *closed upward* if for all $\mathbf{x} \in \mathcal{D}$ we have

$$(\mathbf{z} \in \mathcal{U} \wedge \mathbf{x} \geq \mathbf{z}) \Rightarrow \mathbf{x} \in \mathcal{U}.$$

Corollary 6 *Assume that the conditions of Theorem 2 are satisfied for a computable group G and a Turing degree* $\mathbf{d}$. *If, in addition, the set $DgSp(LO(G))$ is closed upward, then $DgSp(LO(G))$ contains the upper cone of Turing degrees above* $\mathbf{d}$*; that is,*

$$\{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\} \subseteq DgSp(LO(G)).$$

Note that in Theorem 2 one does not require that $sgr(p) \neq sgr(q)$ for $p, q \in \mathbb{P}$ with $p \neq q$. We also observe that the family $\mathbb{P}$ does not have to include all finite subsets of $G\backslash\{e\}$ such that the partial left orders $sgr(p) \cup \{e\}$ on G can be extended to total left orders. In particular, one does not have to assume that G is fully left-orderable. This motivates the next definition.

Definition 7 *A family $\mathbb{P} = \{p_s\}_{s\in\omega}$ of finite subsets of $G\backslash\{e\}$ is called* complete *if $\mathbb{P}$ consists of all finite subsets p such that each subsemigroup $sgr(p) \cup \{e\}$ is a partial left order that can be extended to a total left order on G.*

[1] Chubb showed in [5] that the same conditions guarantee that there are left orders in all truth table degrees.

Recall the definition of the topology on $LO(G)$ introduced in [28]. Let $\mathcal{S}=\{S_g\}_{g\in G}$ be a family of subsets of $LO(G)$, where

$$S_g = \{P \in LO(G) \mid g \in P\}.$$

The topology $\tau_{\mathcal{S}}$ on $LO(G)$ is defined by taking $\mathcal{S}$ as its subbasis. Recall that a topological space is zero-dimensional if it is a T_1-space with a clopen (closed and open) basis. It follows directly from the definition of the topology $\tau_{\mathcal{S}}$ that the space $LO(G)$ is zero-dimensional.

Theorem 8 *Let G be a countable group. There exists a complete family $\mathbb{P} = \{p_s\}_{s\in\omega}$ satisfying conditions* (**i**) – (**iii**) *of Theorem 2 if and only if $LO(G)$ with topology $\tau_{\mathcal{S}}$ is homeomorphic to the Cantor set.*

Proof. Since $LO(G)$ is zero-dimensional with weight[2] $\aleph_0$, it has a clopen subbasis of cardinality $\aleph_0$. By Vedenissoff's theorem in [34], we conclude that $LO(G)$ can be embedded into the Cantor's cube $\{0,1\}^{\aleph_0}$ as its closed subspace (see [8]). Hence $LO(G)$ is compact and metrizable. We show that if there is a complete family $\mathbb{P}$ of finite sets satisfying conditions (**i**) – (**iii**) of Theorem 2, then $LO(G)$ has no isolated points. The topological space $(LO(G), \tau_{\mathcal{S}})$ has no isolated points iff for all finite subsets $p = \{g_1, g_2, ..., g_k\} \subset G\backslash\{e\}$, $k \geq 1$, the intersection $\bigcap\limits_{j=1}^{k} S_{g_j}$ is either empty or infinite.

Suppose $\bigcap\limits_{j=1}^{k} S_{g_j} \neq \emptyset$ and let $P \in \bigcap\limits_{j=1}^{k} S_{g_j}$. Then $sgr(p) \subseteq P$ for $p = \{g_1, g_2, ..., g_k\}$ (since $\mathbb{P}$ is complete), and now, using property (**ii**) of the family $\mathbb{P}$, we find q, $r \in \mathbb{P}$ and the group element $a \neq e$ such that

$$p \subset q \wedge p \subset r \text{ and } a \in q \wedge a^{-1} \in r.$$

Obviously, for all $Q \in LO(G)$, if $sgr(q) \subseteq Q$, then $sgr(r) \nsubseteq Q$. The existence of extensions Q and R for $q, r \in \mathbb{P}$, respectively, is guaranteed by the property (**i**) of the family $\mathbb{P}$. Therefore, $\bigcap\limits_{j=1}^{k} S_{g_j}$ must be infinite. Since every compact, metrizable, totally disconnected, perfect topological space is homeomorphic to the Cantor space, $LO(G)$ is homeomorphic to the Cantor space.

Now, suppose that $LO(G)$ is homeomorphic to the Cantor space. We define a family $\mathbb{P}$ of finite subsets of $G\backslash\{e\}$ as follows. Let $p \subset G\backslash\{e\}$ be a finite subset. Then

$$p \in \mathbb{P} \Leftrightarrow \bigcap_{g\in p} S_g \neq \emptyset.$$

Clearly, the family $\mathbb{P}$ defined above satisfies condition (**i**) of Theorem 2.

Moreover, for every $p \in \mathbb{P}$ we have $\bigcap\limits_{g\in p} S_g \neq \emptyset$ and the space $LO(G)$ has no isolated points, so $\bigcap\limits_{g\in p} S_g$ contains infinitely many cones P, for which $sgr(p) \subseteq P$. Therefore, there are $a \neq e$, $q = p \cup \{a\}$, and $r = p \cup \{a^{-1}\}$ such that

$$(\bigcap_{g\in q} S_g \neq \emptyset) \wedge (\bigcap_{g\in r} S_g \neq \emptyset).$$

Consequently, $\mathbb{P}$ satisfies condition (**ii**).

Suppose that $p \in \mathbb{P}$, $a \neq e$, and $P \in \bigcap\limits_{g\in p} S_g$ $(sgr(p) \subseteq P)$. Since P is total, $a \in P \vee a^{-1} \in P$. Hence, we take either $q = p \cup \{a\}$ or $q = p \cup \{a^{-1}\}$ (depending on whether $a \in P$ or $a^{-1} \in P$) to see that $\mathbb{P}$ satisfies condition (**iii**) of Theorem 2. Clearly, from the definition of $\mathbb{P}$ it follows that $\mathbb{P}$ is complete. ■

[2] The *weight* of a topological space X is the minimal cardinality κ of a basis for the topology on X. For technical convenience the weight is defined to be $\aleph_0$ when the minimal basis is finite.

Note that $LO(G)$ being homeomorphic to the Cantor set does not imply that $DgSp(LO(G)) = \mathcal{D}$. This is because the family $\mathbb{P}$ constructed in the proof of Theorem 8 requires that we check whether $\bigcap_{g \in p} S_g \neq \emptyset$ (which may not be decidable).

Let $A \subset G \backslash \{e\}$ be a finite set, and consider the set of all left orders on G extending A:

$$SO(A) = \{P \in LO(G) \mid A \subset P\} = \bigcap_{g \in A} S_g.$$

The following corollary gives general conditions that will be applied to specific spaces of orders in the next section.

Corollary 9 *Let G be a computable fully left-orderable group and let $\mathbf{d}$ be a Turing degree. If $\mathbf{d} \neq \mathbf{0}$, assume that $DgSp(LO(G))$ is closed upward. Assume that for all finite subsets $A \subset G \backslash \{e\}$ the following two conditions hold.*

(1) *If $SO(A) \neq \emptyset$, then $SO(A)$ contains more that one element.*

(2) *The problem $e \in sgr(A)$ is $\mathbf{d}$-decidable.*

Then $DgSp(LO(G)) \supseteq \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\}$ and $LO(G)$ with topology $\tau_{\mathcal{S}}$ is homeomorphic to the Cantor set.

Proof. Consider the following family $\mathbb{P}$ of finite subsets of G:

$$\mathbb{P} = \{p \subset G \backslash \{e\} \mid p \text{ is finite } \wedge e \notin sgr(p)\}.$$

Since the problem $e \in sgr(p)$ is $\mathbf{d}$-decidable, it follows that $\mathbb{P}$ can be written as a $\mathbf{d}$-computable strong array. Since G is fully left-orderable, we have

$$p \in \mathbb{P} \Leftrightarrow \bigcap_{g \in p} S_g \neq \emptyset.$$

Now, because $LO(G)$ is a closed subspace of 2^ω, condition (1) implies that $LO(G)$ is homeomorphic to 2^ω. Since $\mathbb{P}$ is complete, we conclude from Theorem 8 that $\mathbb{P}$ satisfies conditions (**i**) – (**iii**) of Theorem 2. By assumption, $DgSp(LO(G))$ is closed upward, so by Corollary 6 we obtain that $DgSp(LO(G)) \supseteq \{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\}$. ■

3 Applications to abelian groups and free groups

We first give a unified approach of Sikora's results in topology and Solomon results in logic, which were obtain completely independently. We exploit a geometric interpretation of the orders. We note that every finite rank[3], computable, torsion-free, abelian group G has a finite basis. Clearly, since G is abelian, $LO(G) = BiO(G)$. Moreover, G is fully orderable and there is a Turing degree preserving bijection between $BiO(G)$ and $BiO(D)$, where D denotes the divisible closure of G. We first consider the case of $\mathbb{Z}^n$ and its divisible closure $\mathbb{Q}^n$. Regarding $\mathbb{Q}^n$ as a subspace of the Euclidean n-dimensional space $\mathbb{R}^n$, we see that the closure of every order P on $\mathbb{Q}^n$ is a half-space H_n [28]. Thus, every cone P on $\mathbb{Q}^n$ is determined by the choice of a hyperplane $H^{n-1} = \partial(P)$ which is the topological boundary of P in

[3]The *rank* of a torsion-free abelian group is defined to be $\dim_{\mathbb{Q}}(D)$, where D is the divisible closure of G.

$\mathbb{R}^n$. This hyperplane splits $\mathbb{R}^n$ into two closed half-spaces, H_n^+ and H_n^-, such that $H_n^+ \cup H_n^- = \mathbb{R}^n$ and $H_n^+ \cap H_n^- = H^{n-1}$.

Moreover, either $P \subset H_n^+$ or $P \subset H_n^-$, and in the case when $P \cap H^{n-1} = \{0\}$, the order P is uniquely determined by the hyperplane H^{n-1} and either one of the two half-spaces H_n^+ or H_n^-. Otherwise, if $P \cap H^{n-1} \neq \{0\}$ then $Q = P \cap H^{n-1}$ determines an order on $\mathbb{Q}^n \cap H^{n-1}$, which again could be determined either by the subspace H^{n-2} and the half-space H_{n-1}^+ of $\mathbb{Q}^n \cap H^{n-1}$ or $P \cap H^{n-2} \neq \{0\}$, and we could repeat this process. Since the dimension of $\mathbb{Q}^n$ is finite, every $P \in LO(\mathbb{Q}^n)$ is completely determined by the choice of a collection of subspaces and half-subspaces of $\mathbb{R}^n$.

For $A = \{g_1, g_2, ..., g_j\} \subset \mathbb{Z}^n \backslash \{0\}$, let

$$P_{\mathbb{Q}}(A) = \{x \in \mathbb{Q}^n \mid x = \sum_{i=1}^{j} x_i g_i,\ x_i \geq 0,\ \sum_{i=1}^{j} x_i^2 \neq 0,\ x_i \in \mathbb{Q},\ i = 1, 2, \ldots, j\}.$$

We will call $P_{\mathbb{Q}}(A)$ the "rational closure" of $sgr(A) \subset \mathbb{Z}^n$. Clearly, for any finite set $A \subset \mathbb{Z}^n \backslash \{0\}$ we have

$$0 \in sgr(A) \Leftrightarrow 0 \in P_{\mathbb{Q}}(A).$$

Proposition 10 *Let $n \geq 2$. The problem whether*

$$0 \in sgr(A),$$

where $A \subset \mathbb{Z}^n \backslash \{0\}$ is finite, is decidable. In addition, if $SO(A) \neq \emptyset$, then $SO(A)$ is infinite.

Proof. Let $A \subset \mathbb{Z}^n \backslash \{0\}$, and consider the rational closure $P_{\mathbb{Q}}(A)$ of $sgr(A)$. Since $\mathbb{Q}^n$ is fully orderable, the condition

$$0 \notin P_{\mathbb{Q}}(A)$$

is equivalent to

$$(\exists P \in LO(\mathbb{Q}^n))[P_{\mathbb{Q}}(A) \subset P].$$

Using the topological description of orders on $\mathbb{Q}^n$, this is equivalent to the existence of an affine hyperplane $H \subset \mathbb{Q}^n$ ($0 \notin H$) such that the *convex hull* of A,

$$Conv(A) = \{x \in \mathbb{Q}^n \mid x = \sum_{i=1}^{k} t_i g_i,\ \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0 \text{ and } t_i \in \mathbb{Q}, \text{ for } i = 1, 2, ..., k\},$$

can be separated from 0 by an affine hyperplane H given by:

$$H = \{x \in \mathbb{Q}^n \mid \langle v, x \rangle = z_0\},$$

where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbb{R}^n$, $v \in \mathbb{Q}^n$ is a normal to H, and $z_0 \in \mathbb{Q} \backslash \{0\}$. We also define

$$H^- = \{x \in \mathbb{Q}^n \mid \langle v, x \rangle \leq z_0\} \text{ and } H^+ = \{x \in \mathbb{Q}^n \mid \langle v, x \rangle > z_0\},$$

the two half-spaces into which H splits $\mathbb{Q}^n$. We observe that $0 \notin Conv(A)$ if and only if for all $g \in A$ we have $g \in H^-$ and $0 \in H^+$. Therefore, our problem reduces to the following linear programming problem:

Find

$$z_0 \in \mathbb{Q} \text{ and } v \in \mathbb{Q}^n$$

such that

$$\begin{cases} \langle v,g\rangle < z_0 & \text{for every } g \in A, \\ z_0 < 0. \end{cases}$$

It is well-known that solutions to this problem can be determined algorithmically (using the simplex method). Hence, we can determine whether for a given finite set $A \subset \mathbb{Z}^n \backslash \{0\}$ of generators, we have $0 \in P_{\mathbb{Q}}(A)$. Thus, it follows that we can algorithmically determine whether $0 \in sgr(A)$.

Suppose $SO(A) \neq \emptyset$. Then there are $z_0 \in \mathbb{Q}$ and $v \in \mathbb{Q}^n$ such that $Conv(A) \subset H^-$. Let

$$H^{n-1} = \{\mathbf{x} \in \mathbb{Q}^n \mid \langle v,x\rangle = 0\}.$$

Since $H^{n-1} \cap (\mathbb{Z}^n \backslash \{0\}) \neq \emptyset$, there is $g \in H^{n-1} \cap (\mathbb{Z}^n \backslash \{0\})$. For any such element g, we have

$$0 \notin Conv(A_g) \wedge 0 \notin Conv(A_{g^{-1}}),$$

where $A_g = A \cup \{g\}$ and $A_{g^{-1}} = A \cup \{g^{-1}\}$.

Otherwise, there exist $\lambda_1, \lambda_2 \in \mathbb{Q} \cap [0,1]$ and $x, y \in Conv(A)$ such that

$$\lambda_1 x + (1-\lambda_1)g = 0 \text{ or } \lambda_2 y + (1-\lambda_2)g^{-1} = 0.$$

Hence

$$0 = \langle v, \lambda_1 x + (1-\lambda_1)g\rangle = \lambda_1 \langle v, x\rangle \leq \lambda_1 z_0$$

or

$$0 = \left\langle v, \lambda_2 y + (1-\lambda_2)g^{-1}\right\rangle = \lambda_2 \langle v, y\rangle \leq \lambda_2 z_0.$$

Since $z_0 < 0$, it follows that $\lambda_1 = 0$ or $\lambda_2 = 0$. In either case, we obtain that $g = 0$, which contradicts the choice of $g \in H^{n-1} \cap (\mathbb{Z}^n \backslash \{0\})$.

Since $\mathbb{Q}^n$ is fully orderable, both $P_{\mathbb{Q}}(A_g)$ and $P_{\mathbb{Q}}(A_{g^{-1}})$ extend to cones Q and R on $\mathbb{Q}^n$, respectively. Obviously, $Q \neq R$ so $SO(A)$ has at least two, hence infinitely many elements. ■

As a corollary of Proposition 10 and Corollary 9 we obtain the following results of Solomon [31] and Sikora [28].

Corollary 11 *For $n \geq 2$, $DgSp(LO(\mathbb{Z}^n)) = \mathcal{D}$ and $LO(\mathbb{Z}^n)$ with topology τ_S is homeomorphic to the Cantor set.*

Let $\mathbb{Z}^\omega$ denote the direct sum of ω copies of $\mathbb{Z}$; that is, $\mathbb{Z}^\omega = \bigoplus_{i\in\omega} \mathbb{Z}$. Hence $\mathbb{Z}^\omega$ has a computable basis. As mentioned before, $\mathbb{Z}^\omega$ is fully orderable. Moreover, as in the case of $\mathbb{Z}^n$, both conditions (**i**) and (**ii**) of Corollary 9 are satisfied for $\mathbf{d} = \mathbf{0}$. Hence we have the following results.

Corollary 12 *The space $LO(\mathbb{Z}^\omega)$ with the topology τ_S is homeomorphic to the Cantor set, and $DgSp(LO(\mathbb{Z}^\omega)) = \mathcal{D}$.*

Similarly, we can apply Corollary 9 for $\mathbf{d} = \mathbf{0}'$ to obtain Solomon's result in [31] that for a computable, torsion-free, abelian group G of infinite rank, we have

$$\{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{0}'\} \subseteq DgSp(LO(G)).$$

It also follows from Solomon's proof that if G has a $\mathbf{d}$-computable basis, then

$$\{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\} \subseteq DgSp(LO(G)).$$

Metakides and Nerode [22] showed that for any c.e. Turing degree $\mathbf{d}$, there is a computable vector subspace V of $\mathbb{Q}^\omega$ such that the dependence degree of V is $\mathbf{d}$. The *dependence degree* of V is $\deg(D(V))$, where

$$D(V) =_{def} \bigcup_{k\geq 1}\{\langle v_0,\ldots,v_{k-1}\rangle : v_0,\ldots,v_{k-1} \text{ are dependent over } V\}.$$

Hence V and the corresponding additive abelian group $G_{\mathbf{d}}$ have a $\mathbf{d}$-computable basis.

Solomon [32] established that for every orderable computable group G, there is a computable binary branching tree $\mathcal{T}$ and a Turing degree preserving bijection from $BiO(G)$ to the set of all infinite paths of $\mathcal{T}$. Hence, by the Low Basis Theorem of Jockusch and Soare [19], $\mathcal{T}$ has a low infinite path. Hence $BiO(G)$ contains an order of low Turing degree. Recall that a set X and its Turing degree $\mathbf{x}$ are *low* if $\mathbf{x}' = \mathbf{0}'$. Thus, every computable, torsion-free, abelian group has an order of low Turing degree. On the other hand, Downey and Kurtz [11] constructed a computable, abelian, torsion-free group H with no computable orders, hence $DgSp(LO(H)) \neq \mathcal{D}$. In fact, the group H is isomorphic to $\mathbb{Z}^\omega$.

We will now study complexity of orders on computable free groups F_n of finite ranks $n > 1$. Note that such a group is computably categorical, that is, for every computable isomorphic copy there is a computable isomorphism. A convenient tool in dealing with the lower central series of groups is the free differential calculus developed by Fox in [12]. Recall that for a group G, a derivation of its group ring $\mathbb{Z}[G]$ is a $\mathbb{Z}$-linear map

$$D : \mathbb{Z}[G] \to \mathbb{Z}[G],$$

which satisfies the condition

$$(\forall g, h \in G)[D(gh) = D(g) + gD(h)].$$

The existence and uniqueness of a derivation on F_n, $n > 1$, is a consequence of a result by Fox in [12]. Let $\varepsilon : \mathbb{Z}[F_n] \to \mathbb{Z}$ denote the augmentation homomorphism, defined by: $\varepsilon(f) = \sum_{i=1}^{k}\alpha_i$ for every $f = \sum_{i=1}^{k}\alpha_i w_i \in \mathbb{Z}[F_n]$, where $\alpha_i \in \mathbb{Z}$, $w_i \in F_n$.

For a free semigroup S generated by the set $\{x_1, x_2, ..., x_n\}$, let

$$D_a^0(f) = \varepsilon\left(\frac{\partial^k f}{\partial a_1 \partial a_2 ... \partial a_k}\right),$$

where $f \in \mathbb{Z}[F_n]$, $a = a_1a_2...a_k \in S$, and

$$\frac{\partial x_k}{\partial x_i} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Fox showed that an element $w \in F_n$ belongs to $\gamma_k(F_n)$ if and only if $D_a^0(w) = 0$ for every $a = a_1a_2...a_j$ of length less than k (see Corollary 3.6 in [4]). Therefore, the predicate "$w \in \gamma_k(F_n)$" is decidable.

Recall some basic group theoretic notation and definitions. As usual, for $a, b \in G$, the commutator of a and b is $[a, b] = a^{-1}b^{-1}ab$. For subgroups $H, K \leq G$, by $[H, K]$ we denote their commutator, that is, the subgroup of G generated by $\{[a, b] \mid a \in H, b \in K\}$. Let $\gamma_0(G) = G$, and for $i \geq 0$, let $\gamma_{i+1}(G) = [\gamma_i(G), G]$. The sequence of subgroups

$$\gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots \geq \gamma_i(G) \geq \cdots$$

is the lower central series of G.

It is known that $\bigcap\limits_{i=1}^{\infty}\gamma_i(F_n)=\{e\}$ (see [12]). Recall that the *Möbius function* μ is a number theoretic function defined for every $n\in\omega$, with values in $\{-1,0,1\}$ depending on the factorization of n into prime factors:

$$\mu(n)=\begin{cases} 1 & \text{if } n \text{ is square-free with an even number of distinct prime factors,}\\ -1 & \text{if } n \text{ is square-free with an odd number of distinct prime factors,}\\ 0 & \text{if } n \text{ is not square-free.}\end{cases}$$

Using free differential calculus [4], we can show that

$$\gamma_i(F_n)/\gamma_{i+1}(F_n)\cong\mathbb{Z}^{n_i},$$

where

$$n_i=\frac{1}{i}\sum_{d|i}\mu(\frac{i}{d})n^d,\ i=1,2,\ldots.$$

To simplify the notation, denote simply by γ_k the kth term $\gamma_k(F_n)$ of the lower central series of F_n. Every group γ_i is computable, and there is an isomorphism from γ_i/γ_{i+1} onto $\mathbb{Z}^{n_i}$, which is computable uniformly in i by the algorithm for constructing a basis of γ_i/γ_{i+1} given in [14].

Example. If $n=2$ and $i=1$, then γ_1/γ_2 has a basis given by the set $H_1=\{x_1,x_2\}$, where $F_2=\langle x_1,x_2\ |-\rangle$. In the case when $i=2$, γ_2/γ_3 has a basis given by $H_2=\{[x_1,x_2]\}$, and for $i=3$ we have a basis $H_3=\{[[x_1,x_2],x_1],\ [[x_1,x_2],x_2]\}$, etc.

Thus, in general, we denote the basis of γ_i/γ_{i+1} by H_i, and call it the *Hall basis*. Note that $|H_i|=n_i$.

Lemma 13 *There is an algorithm which for a given $g\in\gamma_i\backslash\gamma_{i+1}$ finds the projection of g onto γ_i/γ_{i+1}, uniformly in i.*

Proof. For $i\geq 1$, let $H_i=\{b_1,b_2,...,b_{n_i}\}$ be the Hall basis of γ_i/γ_{i+1}. We show that there is an algorithm, which for a given $g\in\gamma_i\backslash\gamma_{i+1}$, computes $(\alpha_1,\alpha_2,...,\alpha_{n_i})\in\mathbb{Z}^{n_i}$ such that

$$g\equiv\prod_{l=1}^{n_i}b_l^{\alpha_l}\ \mathrm{mod}\,\gamma_{i+1}.$$

Let $\varphi:\omega\to\mathbb{Z}^{n_i}$ be an effective enumeration of elements of $\mathbb{Z}^{n_i}$. Since H_i is a basis of γ_i/γ_{i+1}, there is a unique $(\alpha_1,\alpha_2,...,\alpha_{n_i})\in\mathbb{Z}^{n_i}$ such that $g\equiv\prod\limits_{j=1}^{n_i}b_j^{\alpha_i}\ \mathrm{mod}\,\gamma_{i+1}$. The algorithm starts by listing the elements of $\mathbb{Z}^{n_i}$: $\varphi(0),\varphi(1),\ldots$ For every element $\varphi(j)=(a_1^j,a_2^j,...,a_{n_i}^j)\in\mathbb{Z}^{n_i}$, we test whether $g\equiv\prod\limits_{l=1}^{n_i}b_l^{a_l^j}\ \mathrm{mod}\,\gamma_{i+1}$ by checking whether $g^{-1}\prod\limits_{l=1}^{n_i}b_l^{a_l^j}\in\gamma_{i+1}$. Since the predicate "$w\in\gamma_{i+1}(F_n)$" is decidable, the procedure must terminate after finitely many steps, when we find a unique $j\in\omega$ such that $\varphi(j)=(\alpha_1,\alpha_2,...,\alpha_{n_i})$. ■

Exploiting an idea introduced in [25], we construct orders on F_n using orders on quotients of the successive terms of the lower central series of F_n. We also observe that different choices of orders on quotients of the lower central series of F_n yield different bi-orders on F_n. To prove the existence of an embedding of the Cantor set into the space of bi-orders for free groups we will apply the following version of a theorem by Šimbireva [29] and Neumann [25].

Theorem 14 (*Šimbireva, Neumann*) *Let $\gamma_1(G) \geq \gamma_2(G) \geq ... \geq \gamma_n(G) \geq ...$ be the lower central series of a group G. Assume that $\bigcap_{i=1}^{\infty} \gamma_i(G) = \{e\}$. Then any order on $\gamma_i(G)/\gamma_{i+1}(G)$ induces a bi-order on G.*

We can now use Corollary 12 to produce on F_n the orders of given Turing degrees.

Theorem 15 *For a free group F_n of rank $n > 1$, $DgSp(BiO(F_n)) = \mathcal{D}$.*

Proof. We construct a family $\mathbb{P} = \{p_i\}_{i\in\omega}$ of finite subsets of F_n, which satisfies the assumptions of Corollary 5 as follows. We start by fixing an effective enumeration φ of all finite subsets of $F_n\backslash\{e\}$. Let $A = \{w_1, w_2, ..., w_j\} \subset F_n\backslash\{e\}$. We show that there is an algorithm that gives a sufficient condition for $sgr(A)$ to extend to a bi-order on F_n. Let

$$\{k_1, k_2, ..., k_l\} = \{k \in \mathbb{Z}_+ \mid (\exists w \in A)[w \in \gamma_k \wedge w \notin \gamma_{k+1}]\}.$$

It follows from [4] that such a set can be effectively found. We assume that $k_1 < k_2 < ... < k_l$ and define

$$s_{k_i} = \{w \in A \mid w \in \gamma_{k_i} \wedge w \notin \gamma_{k_i+1}\},\ 1 \leq i \leq l.$$

Let

$$\overline{s_{k_i}} = \{\pi_{k_i}(w) \mid w \in s_{k_i}\},\ 1 \leq i \leq l,$$

where $\pi_{k_i} : F_n \to \gamma_{k_i}/\gamma_{k_i+1}$ is a computable projection.

Let $d(i) = \frac{1}{k_i}\sum_{d|k_i}\mu(\frac{k_i}{d})n^d$. Since $\overline{s_{k_i}} \subset \mathbb{Z}^{d(i)} \cong \gamma_{k_i}/\gamma_{k_i+1}$, by applying Proposition 10, we can decide whether the subsemigroup $sgr(\overline{s_{k_i}})$ defines a partial order on $\mathbb{Z}^{d(i)}$. In the case when all $sgr(\overline{s_{k_i}})$ define partial orders on $\mathbb{Z}^{d(i)}$, $1 \leq i \leq l$, we define

$$Q(A) = sgr(\bigcup_{m=1}^{l}\overline{s_{k_m}}) \cup \{e\},$$

a partial order on $\mathbb{Z}^\omega = \bigoplus_{i\in\omega}\mathbb{Z}^{n_i}$. Since $\mathbb{Z}^\omega$ is fully orderable, $Q(A)$ can be extended to a cone Q on $\mathbb{Z}^\omega$, and it thus induces cones Q_i on subgroups $\mathbb{Z}^{n_i}$ of $\mathbb{Z}^\omega$. It follows from Theorem 14 that given orders Q_i on $\mathbb{Z}^{n_i}$ for all $i = 1, 2, ...$, we can define an order on F_n by taking

$$P = \{w \in F_n \mid \pi_i(w) \in Q_i^+ \text{ for some } i \in \mathbb{Z}_+\},$$

where $\pi_i(w)$ denotes the image of the projection of w onto γ_i/γ_{i+1}. Therefore, if $Q(A)$ defines a partial order on $\mathbb{Z}^\omega$, then $sgr(A)$ extends to an order on F_n. In such a case, we can decide whether $e \in sgr(A)$. This implies that we have a sufficient condition that allows us to decide whether a given set $A \subset F_n$ induces an order on F_n.

Now, let

$$O(A) = \bigcup_{i=1}^{l}\overline{s_{k_i}},$$

where $\overline{s_{k_i}}$'s are defined as above. To construct a family $\mathbb{P}$ satisfying the assumptions of Theorem 2, we use the computable enumeration φ of finite subsets of $F_n\backslash\{e\}$ as follows. Let $p_0 = \varphi(k_0)$, where

$k_0 = \min\{j \in \omega | \ \mathbf{0} \notin sgr(O(\varphi(j)))\}$. Suppose that $p_0, p_1, ..., p_{n-1}$ have already been constructed. Hence, there exist indices $k_0 < k_1 < ... < k_{n-1}$ such that $p_j = \varphi(k_j)$, $j = 0, 1, ..., n-1$. Define $p_n = \varphi(k_n)$, where

$$k_n = \min\{j \in \omega \backslash \{0, 1, ..., k_{n-1}\} | \ \mathbf{0} \notin sgr(O(\varphi(j)))\}.$$

Clearly, the family $\mathbb{P}$ satisfies condition (i) of Corollary 5. Hence we need to check that $\mathbb{P}$ satisfies the conditions (ii) and (iii). Let $p \in \mathbb{P}$, and note that

$$E(p) = \{s \in \mathbb{P} \backslash \{p\} | \ p \subset s\} \neq \emptyset,$$

since there is $P \in BiO(F_n)$ such that $sgr(p) \subset P$. For every $w \in F_n \backslash \{e\}$, define

$$q_w = p \cup \{w\} \text{ and } r_w = p \cup \{w^{-1}\}.$$

Let

$$a = \min\{w \in F_n \backslash \{e\} \mid w \notin p \wedge \mathbf{0} \notin sgr(O(q_w)) \wedge \mathbf{0} \notin sgr((O(r_w))\}$$

be the element with the least index in some algorithmic enumeration of the elements of F_n. We now show that the element $a \in F_n \backslash \{e\}$ defined above exists. First, we observe that the set $X(p)$ is computable, where

$$X(p) = \{w \in F_n \backslash \{e\} \mid w \notin p \wedge \mathbf{0} \notin sgr(O(q_w)) \wedge \mathbf{0} \notin sgr(O(r_w))\}.$$

This is obvious since $w \notin p$ is a decidable predicate, and the predicates $\mathbf{0} \notin sgr(O(q_w))$ and $\mathbf{0} \notin sgr(O(r_w))$ are decidable by Proposition 10.

We will now show that $X(p) \neq \emptyset$. Since $p = \{w_1, w_2, ..., w_j\} \in \mathbb{P}$, there are

$$\overline{s_{k_i}} \subset \mathbb{Z}^{d(i)}, \ 1 \leq i \leq l, \text{ and}$$

$$O(p) = \bigcup_{i=1}^{l} \overline{s_{k_i}}.$$

Each $sgr(\overline{s_{k_i}})$ defines a partial order on $\mathbb{Z}^{d(i)}$. Hence, as in the proof of Proposition 10, there is an element $g \in \mathbb{Z}^{d(i)}$ with

$$g = \sum_{j=1}^{d(i)} \alpha_j e_j,$$

such that both $sgr(\overline{s_{k_i}} \cup \{g\})$ and $sgr(\overline{s_{k_i}} \cup \{g^{-1}\})$ can be extended to orders on $\mathbb{Z}^{d(i)}$. Therefore, $sgr(O(p) \cup \{g\})$ and $sgr(O(p) \cup \{g^{-1}\})$ can both be extended to distinct orders on $\mathbb{Z}^{\omega} = \bigoplus_{i \in \omega} \mathbb{Z}^{n_i}$. Let $H_{k_i} = \{b_1, b_2, ..., b_{d(i)}\}$ be the Hall basis of $\gamma_{k_i}/\gamma_{k_i+1}$. Define

$$w = \prod_{i=1}^{d(i)} b_i^{\alpha_i} \in F_n.$$

Clearly, we have

$$\pi(w) = \sum_{i=1}^{d(i)} \alpha_i e_i,$$

so $sgr(q_w)$ and $sgr(r_w)$ can be both extended to distinct orders on F_n. This shows that $X(p) \neq \emptyset$. Now, we define

$$q = p \cup \{a\} \text{ and } r = p \cup \{a^{-1}\}.$$

By definition, $p \subset q \wedge p \subset r$ and $a \in q \wedge a^{-1} \in r$. Furthermore, q, $r \in \mathbb{P}$ since both q and r define partial left orders $sgr(q) \cup \{e\}$ and $sgr(r) \cup \{e\}$ on F_n, which can be extended to left orders Q and R on F_n, respectively.

Property (**iii**) of the family $\mathbb{P}$ in Theorem 2 follows immediately from the fact that for any extension P of $sgr(p)$,

$$(P \cup P^{-1} = F_n) \wedge (P \cap P^{-1} = \{e\}).$$

Since for every $w \in F_n$, either $w \in P$ or $w^{-1} \in P$, we have that

$$\text{either } q = (p \cup \{w\}) \in E(p) \text{ or } q = (p \cup \{w^{-1}\}) \in E(p),$$

where $E(p) = \{s \in \mathbb{P} \backslash \{p\} \mid p \subset s\} \neq \emptyset$ because either $sgr(p \cup \{w\}) \subset P$ or $sgr(p \cup \{w^{-1}\}) \subset P$. ■

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