Contents

1 Introduction

"In the beginning there was the word."

This work deals, to a great extent, with the so-called finitely presented groups, algebraic structures motivated by languages. We start with a finite alphabet, the set of algebraic *generators* (for example, $a, b, c, \ldots$). The (binary) operation is writing (concatenation) and the elements are words obtained from generators by writing (for example, $w = aabbb = a^2b^3$). Writing is *associative*: $(uv)w = u(vw)$, so we simply put uvw. The associativity gives the words with the writing operation the algebraic structure of a *semigroup*.

There is an empty word, e, a unique algebraic *identity* for which $we = ew = w$. (The same role is played by 0 for number addition, or 1 for number multiplication). A semigroup with identity is called a *monoid*. For every generator a, there is an *inverse* element, a^{-1}, and hence for every word w, there is an inverse word w^{-1}. For example, the inverse of ab is $b^{-1}a^{-1}$. Writing its inverse next to a word is equivalent to erasing the word: $ww^{-1} = w^{-1}w = e$. The inverses assure that a semigroup with identity is actually a *group*. Since writing is usually *noncommutative* (that is, for some words u, w, we have $uw \neq wu$), these groups are often noncommutative (also called *nonabelian* after Abel). However, if we have only one symbol, say a, then we have a commutative group with elements $e, a, aa, \ldots$, and their inverses $e, a^{-1}, a^{-2} = a^{-1}a^{-1}, \ldots$. This group is the same as (algebraically isomorphic to) the group of integers $1, 2, 3, \ldots, 0, -1, -2, \ldots$ under addition. For example, $(aa)(a^{-1}a^{-1}a^{-1})$ corresponds to $2 + (-3)$.

So far, we have only finitely generated *free* groups. The only equalities are those imposed by the identity and the inverses. We can also require that our words get identified under some other given identities – they are called (group) *relations*. The relations are of the form $u = w$ or, equivalently, $uw^{-1} = e$, in which case uw^{-1} is called a (group) *relator*. Thus every group G can be described using two sets, the set of its generators X and the set of its relators R. We call the pair $\langle X | \, R \rangle$ a *presentation* of the group G. If we have finitely many generators and finitely many relations, then we have a *finitely presented* group.

It is easy to show that all finitely presented groups are countable. Finitely presented groups can be studied using sophisticated algebraic, topological and logic methods, and their study forms its own discipline within noncommutative algebra – combinatorial group theory. Topologists are interested in these groups because the generators can be interpreted as moves on surfaces, whose iteration forms paths. The inverse, of course, corresponds to taking the opposite direction.

Easy to state problems of combinatorial group theory quickly become very difficult to solve. For example, in 1912, W. Dehn asked whether for every finitely presented group there is an algorithm which for any two given arbitrary words of the group, u and w, decides whether they are equal: $u = w$? If they are equal, their equality must follow from the group relations and nothing else. This problem became known as the *word problem* and it was shown more than forty years later, by P. Novikov in 1955 and independently by W. Bonne in 1959, that it is undecidable.

There are also infinitely generated and infinitely presented groups, which are much more complicated than the finitely presented ones. Especially interesting among them are finitely generated groups with algorithmic set of relators. A striking result about these groups is the Higman's embedding theorem which states that every such group can be embedded into a finitely presented one. This theorem was proved in 1960 and also gave another solution for the word problem.

While the decidability results could be established earlier using the intuitive notion of an algorithm, the undecidability results required the development of formal computability theory, the mathematical theory of algorithms. For example, Dehn was able to show, using the intuitive notion of an algorithm, that surface groups have a decidable word problem. The precise notion of an algorithm was developed in the 1930's and 1940's when different, but equivalent, definitions were given. Among the best-known are those proposed by Turing (1936), Church (1941), and Markov (1954). Intuitively, problems which can be solved algorithmically (using Turing machines) are called *decidable*. Provided that we have some "external knowledge" (that is, we are able to consult an oracle, which can only answer certain types of questions), we can define generalized algorithms for solving undecidable problems. Computational complexity of undecidable problems can be measured using *Turing degrees*, which reflect the level of the "external knowledge" (or the "power of oracle") needed to solve these problems.

Decision problems studied in computability theory often arise naturally as consequences of important problems in other areas of mathematics. The best example related to our work is the *isomorphism problem* for finitely presentable groups. That is, is there an algorithm that decides for any two finitely presented groups, given *via* their generators and relators, whether they are isomorphic? The undecidability of the isomorphism problem follows from the undecidability of the word problem. The existence of an algorithm that solves the isomorphism problem would have had some strong consequences in other areas of mathematics. For example, in 3-dimensional topology many important problems could be decided, in particular, the algorithmic Poincaré Conjecture, which is related to the open (for more than one hundred years) Poincaré Conjecture.

In this work, we use techniques of computability theory to study computational properties of groups and their orders. In particular, we are concerned with computable groups and their linear orders. Such an order is left if it is left-invariant under the group operation: $x < y$ implies $zx < zy$ for every x, y, z in G. Similarly, we define a right order and a bi-order. One of the main goals of our work is to study computational complexity of left orders for some classes of computable groups. It is well-known that the existence of a computable left order on a finitely presented group forces the group to have a decidable word problem. Moreover, if there is a computable bi-order, then also the conjugacy problem must be decidable. This makes the study of left orders important for combinatorial group theory, and consequently, for other fields of mathematics.

Recently, the questions concerning left-orderability of fundamental groups of 3-manifolds have been studied by 3-dimensional topologists, including D. Rolfsen, B. Wiest and S. Boyer. For example, they showed that many classes of 3-manifold groups admit bi-orders. The property of orderability of groups has strong consequences in topology. The primary interest for studying this property by topologists is to find geometrical properties of 3-manifolds that are consequences of the fact that their first homotopy groups are orderable. Although orderability has been studied for such groups for some time, the questions concerning these relationships have not yet been answered completely.

Our general result gives a criterion for the Turing degree spectrum of left orders to include all Turing degrees. We provide a relativized result, which states that for a fully left-orderable group, the space of its left orders realizes all Turing degrees above an arbitrary degree $\mathbf{d}$. A group is *fully left-orderable* if every partial left order can be extended to a (total) left order. We also study, following work of A. Sikora [66], topological properties of these spaces. For example, spaces of left orders for some classes of finitely presentable groups admit an embedding of the Cantor set. We prove this for a particular class of one-relator groups that arise in topology. For torsion-free abelian groups the space of left orders is homeomorphic to the Cantor set. We study relations between topological and computational properties of spaces of left orders. For instance, we show that for a finite greater than 1 rank free group, the Turing degree spectrum of left-orders includes all Turing degrees. It is still open whether the space of all left-orders (bi-orders) on such a group is homeomorphic to the Cantor set (see Sikora's Conjecture in [66], 2004). We provide a criterion for the space of left orders on a countable group to be homeomorphic to the Cantor set.

In the remainder of this introduction we describe the other chapters. To make this work self-contained, we introduce in chapter 2 some general facts and definitions from computable model theory and combinatorial group theory, as well as some basic notions from topology.

In chapter 3, we focus on the computational content of some classes of groups. While all previous research included only commutative (abelian) or metabelian (also called 2-step nilpotent) groups, here we investigate highly noncommutative groups. We start, in section 3.1, with the definition of the degree of the isomorphism type of a structure, first introduced by C. Jockusch and L. Richter [60]. We review facts concerning degrees of the isomorphism type of some structures. Let $\mathcal{A}$ be a countable structure in a computable language. The set of Turing degrees of all isomorphic copies of

$\mathcal{A}$ is called the *Turing degree spectrum* of $\mathcal{A}$, in symbols $DgSp(\mathcal{A})$. The least element in $DgSp(\mathcal{A})$, if it exists, is the *degree of the isomorphism type* of $\mathcal{A}$. Since such a set of Turing degrees may not have a least element, the isomorphism types of many structures fail to have a degree. Richter [60] studied two kinds of theories. First, there are theories whose models have isomorphism types of arbitrary Turing degrees. This case occurs whenever a theory T has a computable sequence of finite models, which forms an antichain under embeddability, and satisfies certain additional conditions. A method for constructing a model of a theory T satisfying these conditions is called a *combination method* for T. Examples of such theories include abelian groups, partially ordered sets, trees with an edge relation, lattices, and homogeneous graphs. The second class contains theories with models whose isomorphism types are poor in degrees, that is, the only possible degree of the isomorphism type of a countable model is $\mathbf{0}$. However, Richter also showed that all known theories rich in degrees also have countable models whose isomorphism types have no degree.

In section 3.2, we investigate Turing degrees for some classes of "highly nonabelian" groups. Using the combination method, we show that there is a countable centerless group $\mathcal{A}$, obtained as the free product of the sequence of nontrivial groups of finite orders, whose isomorphism type has an arbitrary Turing degree, hence its Turing degree spectrum $DgSp(\mathcal{A})$ includes all degrees above an arbitrary degree $\mathbf{d}$. A similar argument provides a centerless group with infinitely many finite nonabelian subgroups, whose isomorphisms type does not have a degree. Our main theorems in this chapter, Theorem 24 and Theorem 25, modify the previous conditions in order to allow infinite structures in the sequence of models. As a consequence we obtain an interesting class of nonabelian groups, whose isomorphism type has an arbitrary Turing degree. By our construction, a group in this class is a countable centerless group with infinitely many infinite nonabelian subgroups, which are finite rank Burnside groups of finite exponent.

Chapters 4 through 6 focus on a class of countable groups which are computable and left-orderable. In chapter 4, we provide definitions and basic facts concerning partially and totally left-orderable and bi-orderable groups. We discuss conditions for left-orderability and full left-orderability of groups here, as well as the connection between left-orderability and local-indicability of groups.

We consider the property of orderability for a finitely presented group from the point of view of computability, using the so-called *Markov property*. If a property of finitely presentable groups is Markov, then it is not computably recognizable. Since left-orderability of a finitely presentable group is a Markov property, left-orderability is non-decidable. However, there are classes of finitely presentable groups for which we can decide whether they are left-orderable, for instance, torsion-free abelian groups and nilpotent groups, and (finitely generated) free groups. We will show that for one-relator groups, the property of being left-orderable is also decidable.

We provide some interesting examples of orderable groups. Recently, it has been discovered that many groups which arise naturally in topology are left-orderable, and even bi-orderable. Rolfsen and Wiest [64] showed bi-orderability of the fundamental groups of closed surfaces, with the exceptions of the projective plane and the Klein bottle. Together with Boyer [4], they also considered orderability of 3-manifold groups. In most cases, fundamental groups of 3-manifolds admit left orders. However, an interesting class of finitely presentable torsion-free fundamental groups of 3-manifolds that do not admit any order has recently been found ([14], 2005).

In chapter 5, we turn our attention to computational and topological properties of spaces of left orders. In section 5.1, we consider the complexity of the space of left orders for a computable group. Throughout the text, we use Turing reducibility as the complexity measure. At the beginning of this section, we recall R. Solomon's [70] result concerning the connection between spaces of orders and computably bounded Π^0_1 classes. We also briefly recall Solomon's results for torsion-free abelian groups. We extend these results to finitely generated free groups in section 6.2. In the further part of this section, we introduce the notion of the *Turing degree spectrum of left orders* on a computable group G, $DgSp_G(LO)$. We define it to be the set of Turing degrees of all possible left orders on G. We refer to this notion repeatedly in chapter 6. The key results of this section, Theorems 72, 73, and Corollary 76, deal with Turing degree spectra of left orders for orderable, computable groups and provide direct and easy arguments for both new and earlier results. For a computable group G, Theorem 72 provides a general sufficient condition, expressed in terms of finite sets of generators for partial left orders on G, for the set $DgSp_G(LO)$ to include all Turing degrees. Theorem 73 is a relativized version of Theorem 72 and, under the additional assumption for the degree spectrum to be closed upwards, it implies that $DgSp_G(LO)$ includes all Turing degrees above an arbitrary degree $\mathbf{d}$.

In section 5.2, following the work of A. Sikora [66], we define a topology on the space of left orders $LO(G)$ of a countable group G, in a very natural way, as the collection $\mathcal{S}$ of subsets of $LO(G)$, $\mathcal{S} = \{S_a\}_{a \in G}$, where $S_a = \{P \in LO(G) |\ a \in P\}$. We describe (using positive cones of left orders on G) $LO(G)$ as a topological space to be compact, metrizable, and totally disconnected. The main theorem of this section, Theorem 85, is a criterion for the space of left orders on a countable group G to be homeomorphic to the Cantor set. Theorem 85 captures the similarities between topological and computability-theoretic results since the sets of generators for partial left orders of G need to satisfy virtually the same conditions as in Theorems 72 and 73. In particular, this theorem implies the main result of [66] that the space of left orders on a computable torsion-free abelian group of finite rank n greater than 1 is homeomorphic to the Cantor set. We extend this result to the class of computable torsion-free abelian groups of infinite rank in the next chapter.

In chapter 6, we analyze the computational and topological properties of spaces of left-invariant orders for various classes of computable groups. We derive results concerning both the Turing degree spectra of left orders for these groups and the topological properties of their spaces of orders. In section 6.1, we use Theorem 72 to prove that $DgSp_G(LO) = \mathcal{D}$ for a computable infinite rank torsion-free abelian group $G = \mathbb{Z}^\omega$. Here, $\mathcal{D}$ is the set of all Turing degrees. The results of [70] for torsion-free abelian groups are corollaries of this theorem. Moreover, we show that $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set (using Theorem 85), which extends the main result of [66]. We finish section 6.1 with a proof of a more general result (see Proposition 91), which states that $DgSp_G(LO) \supseteq \{\mathbf{a} \in \mathcal{D} |\ \mathbf{a} \geq \mathbf{d}\}$ for a fully left-orderable group G. In section 6.2, we consider a free group F_n of rank n greater than 1. Using the properties of its lower central series, we prove that $DgSp_{F_n}(LO) = \mathcal{D}$. Sikora [66] showed that the space of left orders of F_n admits an embedding of the Cantor set. However, it is still open whether these two topological spaces are homeomorphic. In section 6.3, we prove the existence of the embedding of the Cantor set into the space of left orders for a particular class of one-relator groups arising in topology, the fundamental groups of closed, connected and orientable surfaces of genus g greater than 1. Furthermore, we conjecture that the space of left orders for one-relator groups satisfying some specific conditions has an order in every Turing degree. We conclude the chapter with some preliminary ideas for more general and thus more complicated theory of finitely presented groups.

2 Basic Notions

In this chapter, we recall some notations and definitions used in the further parts of this thesis. We denote the sets of integer, rational, and real numbers by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. We consider first-order computable languages. A *computable language* is a countable language with an algorithmically presented set of symbols and their arities. We denote models by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$,... and their universes by A, B, C,..., respectively. We will focus on countable structures, that is, the structures whose domains could be identified with the set of natural numbers ω.

Let L be a language of $\mathcal{A}$. Define the language

$$L_A = L \cup \{\mathbf{a} | a \in A\}$$

to be L expanded by adding a constant symbol $\mathbf{a}$ for every element $a \in A$. Let

$$\mathcal{A}_A = (\mathcal{A}, a)_{a \in A}$$

be the expansion of $\mathcal{A}$ to L_A. The *atomic (open) diagram* of a structure $\mathcal{A}$ (denoted by $D^a(\mathcal{A})$ or $D^o(\mathcal{A})$) is the set of all atomic and negated atomic sentences of L_A which are true in $\mathcal{A}_A$. The *complete (elementary) diagram* of $\mathcal{A}$ (denoted by $D^c(\mathcal{A})$ or $D^e(\mathcal{A})$) is the set of all sentences of L_A which are true in $\mathcal{A}_A$.

A consistent deductively closed set of sentences in L is called a *theory* in L. Let $\mathcal{A}$ be a model for some language L, then $Th(\mathcal{A})$ denotes its theory. A *(complete) type* of a theory is a maximal consistent set of formulae in a fixed number of variables. An *n-type* is a type in n variables, and a finite type is an n-type for some $n \in \omega$. Let $p(x_1, x_2, ..., x_n)$ be a type in some language L, where

$x_1, x_2, ..., x_n$ are variables. Then a model $\mathcal{A}$ realizes the type p if for some $b_1, b_2, ..., b_n \in A$ we have

$$\mathcal{A} \models \varphi[b_1, b_2, ..., b_n] \text{ for all } \varphi(x_1, x_2, ..., x_n) \in p(x_1, x_2, ..., x_n).$$

A type p is said to be finitely realizable in $\mathcal{A}$ if $\mathcal{A}$ realizes all finite subsets of p. We say a type $p(x_1, x_2, ..., x_n)$ is computable (computably enumerable) if $\{\ulcorner \phi(x_1, x_2, ..., x_n) \urcorner | \phi \in p\}$ is a computable (computably enumerable) set, where $\ulcorner . \urcorner$ is the Gödel numbering of formulae of L.

Let $X, Y \subseteq \omega$. The set X is *Turing reducible* to the set Y, in symbols $X \leq_T Y$, if X is Y-recursive or, equivalently, if the characteristic function of X is Y-computable. Intuitively, it means that given membership information about Y, we can construct an algorithm for deciding questions about X. The *characteristic function* of a set X is given by

$$c_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases}.$$

Moreover, X is said to be *recursive* (or *computable*) *set* if c_X is a computable function, i.e., there exists an algorithm that computes it. We say that X is *computably* (or *recursively*) *enumerable* (abbreviated by *c.e.* or *r.e.*) whenever there exists an algorithm that enumerates the elements of X. Now, the sets X and Y are *Turing equivalent* (denoted by $X \equiv_T Y$) if $X \leq_T Y$ and $Y \leq_T X$. Let

$$\deg(X) = \{Y | X \equiv_T Y\}$$

denote the *Turing degree* of X (also called the *degree of unsolvability* of X). Define $\mathbf{0} = \deg(\emptyset)$ to be the degree of any recursive set, and $\mathbf{0}'$ to be the degree of the halting set.

The *halting problem* decides for an arbitrary $x, y \in \omega$ whether the program P_x on the input y ever halts, where P_x is the Turing program with the code (or Gödel) number x. Let φ_x be the partial function computed by P_x. We say that $\varphi_x(y)$ converges, and denote it by $\varphi_x(y) \downarrow$, if there exists an output z such that $\varphi_x(y) = z$. Then, the *halting set* is defined as $K = \{\langle x, y \rangle | \varphi_x(y) \downarrow\}$. Since the halting problem is undecidable, the halting set K is not recursive. The set K is recursively enumerable, i.e., K can be enumerated by a total computable (recursive) function. Now, let $K^A = \{\langle x, y\rangle |\ \varphi_x^A(y)\downarrow\}$. The set K^A is called the *jump of a set* $A \subseteq \omega$ and it is denoted by A'. Then the *nth jump of* A, $A^{(n)}$, is obtained by iterating the jump n times.

If $\mathbf{x} = \deg(X), X \subseteq \omega$, then $\mathbf{x}^{(n)} = \deg(X^{(n)})$ for $n \geq 1$, where $X^{(n)}$ denotes the nth jump of X. We call a degree $\mathbf{x} \leq \mathbf{0}'$ a *low degree* if $\mathbf{x}' = \mathbf{0}'$, i.e., if the jump $\mathbf{x}'$ has the lowest degree possible, and high if $\mathbf{x}' = \mathbf{0}''$ (the highest possible value). Define $\mathbf{x}^{(\omega)} = \deg(X^{(\omega)})$ to be the degree of the ω-jump of X. The degree $\mathbf{0}^{(\omega)}$ is the natural upper bound for the sequence $(\mathbf{0}^{(n)})_{n\in\omega}$, while there is no least upper bound for any ascending sequence of Turing degrees. Turing degrees $\mathbf{x}$ and $\mathbf{y}$ form a minimal pair if $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ and

$$(\forall \mathbf{z})[(\mathbf{z} \leq \mathbf{x} \wedge \mathbf{z} \leq \mathbf{y}) \Longrightarrow \mathbf{z} = \mathbf{0}].$$

The set of all Turing degrees is denoted by $\mathcal{D}$. It is a partially ordered set $\mathcal{D}$ with cardinality $2^{\aleph_0}$ and forms an upper semilattice, in which the supremum of $deg(X)$ and $deg(Y)$ is $deg(X \oplus Y)$, where $X \oplus Y = \{2n | n \in X\} \cup \{2n+1 | n \in Y\}$ is the join of X and Y $(X, Y \subseteq \omega)$.

We say that X is *enumeration reducible* to Y, in symbols $X \leq_e Y$, if there is a computably enumerable binary relation E such that

$$x \in X \Leftrightarrow (\exists u)[D_u \subseteq Y \wedge E(x, u)],$$

where D_u is the finite set with canonical index u. Intuitively, a set X is enumeration reducible to a set Y if there exists an algorithm whose outputs enumerate X when any enumeration of Y in any order is supplied for the inputs. In other words, $X \leq_e Y$ if and only if for every set S, if Y is c.e. relative to S, then X is c.e. relative to S. This reducibility is reflexive and transitive, so it can be used to define en equivalence relation on classes of functions. The equivalence classes are called *enumeration degrees* (or *degrees of difficulty*) and form a lattice called Medvedev lattice. For additional information on Turing and enumerations degrees see [69], and [62] or [55].

Now, we are ready to recall some important notions from computable model theory, based on [22].

We assume that a formula φ is identified with its Gödel number $\ulcorner\varphi\urcorner$, so the set of formulae is a subset of ω. A set Γ of formulae belongs to a computability-theoretic complexity class $\mathcal{P}$ if the set

$$\{\ulcorner\varphi\urcorner \,|\varphi \in \Gamma\} \in \mathcal{P}.$$

Hence, a theory T belongs to a complexity class $\mathcal{P}$ if the set of Gödel numbers of the sentences of T belongs to $\mathcal{P}$. Moreover, a theory T is decidable (or computable) if T is a computable set of sentences. In other words, if Ax is the set of axioms of a theory T, then T is decidable if there is an algorithm which determines, for every sentence τ of T, whether $Ax \vdash \tau$. Since a computably axiomatizable theory is obviously computably enumerable, a complete computably axiomatizable theory is decidable. In particular, a complete finitely axiomatizable theory is decidable.

Example 1 *The theory of dense linear orders is a complete finitely axiomatizable theory, hence a decidable theory. Other important examples of decidable theories include additive number theory, as well as the theories of algebraically closed fields, Boolean Algebras, abelian groups, free commutative algebras, and the theory of linear order. The examples of undecidable theories include number theory, the theories of simple groups, semigroups, rings, fields, distributive lattices, the theory of partial order, and the theory of the rational field.*

We are interested in the theory of orderable groups. Let us recall the following well known result, which states that the theory of orderable groups is computably but not finitely axiomatizable in the first order language of groups.

Theorem 2 *There is a computable set of sentences Φ in the language of group theory such that for every group G we have*

$$G \models \Phi \text{ if and only if } G \text{ is left-orderable.}$$

Moreover, such a set Φ cannot be finite.

For the proof of Theorem 2 see [72].

Now, let us recall definitions of computable and decidable models.

Definition 3 *A model $\mathcal{A}$ is computable if its domain A is computable and its relations and functions are uniformly computable, or equivalently, $\mathcal{A}$ is computable if A is computable and there is a computable enumeration of A such that the atomic diagram of $\mathcal{A}$, $D^o(\mathcal{A})$, is decidable.*

The standard model of Peano Arithmetic, that is, a structure isomorphic to $\mathcal{N} = (\omega, +, \times, S, 0)$, is computable, yet there is no computable nonstandard model of Peano Arithmetic.

Definition 4 *A model $\mathcal{A}$ is decidable if its domain A is computable and there is a computable enumeration of A such that the complete diagram of $\mathcal{A}$, $D^e(\mathcal{A})$, is decidable, namely, if $(a_i)_{i\in\omega}$ is the computable enumeration of A, then $Th((\mathcal{A}, \mathbf{a}_i)_{i\in\omega})$ is decidable.*

The following theorem is the Effective Completeness Theorem.

Theorem 5 *A decidable theory has a decidable model.*

Obviously, every decidable model is computable, but the converse is not true. For example, $\mathcal{N}$ is a computable, but non-decidable model by Gödel incompleteness theorem. However, if a complete theory admits effective quantifier elimination, then every computable model of the theory is decidable.

Definition 6 *The Turing degree of a model $\mathcal{A}$ (in a finite language L),* $\deg(\mathcal{A})$, *is the least upper bound of Turing degrees of its universe, relations, and functions (or, equivalently, the Turing degree of its atomic diagram $D^o(\mathcal{A})$).*

Hence, $\mathcal{A}$ is computable if and only if $deg(\mathcal{A}) = \mathbf{0}$. For example, every finite structure is computable. A structure for a finite language is computable if its domain is a computable set and its operations and relations are computable, while for infinite languages we need uniform computability of operations and relations. In particular, we will be interested in computable groups.

Definition 7 *A countable group $(G, \cdot)$ is computable if the set $G \subseteq \omega$ is computable and the group-theoretic operation $\cdot : G \times G \to G$ is computable.*

Now let us introduce some necessary notions from combinatorial group theory. For more information see [40].

Definition 8 *Let X be a subset of a group F. Then F is a* free group *with basis X if for any function $\phi : X \to G$, where G is a group, there exists a unique extension of ϕ to a homomorphism $\phi^* : F \to G$.*

Remark 9 *The requirement for the homomorphism $\phi^* : F \to G$ to be unique for the map $\phi : X \to G$ is equivalent to F being generated by X. Moreover, one can easily prove that all bases of a free group F have the same cardinality, which is called the rank of a free group F.*

We will sketch here a construction of a free group with the basis X. Let $W(X)$ denote the set of all finite words on the alphabet $X \cup X^{-1}$, where $X^{-1} = \{x^{-1} \mid x \in X\}$ is the set of all inverses of generators $x \in X$. Define an involution

$$\eta : X \cup X^{-1} \to X \cup X^{-1}$$

by putting $\eta(x) = x^{-1}$ and $\eta(x^{-1}) = x$, which we express as $(x)^{-1} = x^{-1}$ and $(x^{-1})^{-1} = x$. For any words $w_1, w_2 \in W(X)$, define a new word w by concatenating w_1 and w_2, i.e.

$$w = w_1 w_2.$$

Thus, every element of $W(X)$ can be viewed as a concatenation of letters in $X \cup X^{-1}$. Using the above interpretation, one defines the inverse of $w = x_1 x_2 ... x_n$, where $n \geq 1$ and $x_i \in X \cup X^{-1}$ for $1 \leq i \leq n$, as

$$w^{-1} = x_n^{-1} x_{n-1}^{-1} ... x_1^{-1}.$$

A word $w \in W(X)$ is *reduced* if it does not contain any subword of the form xx^{-1}, where $x \in X \cup X^{-1}$. The product $w = w_1 w_2$ of reduced words $w_1, w_2 \in W(X)$ need not be reduced. However, one can show that for any $w \in W(X)$, there is unique reduced word that is obtained from w by deleting all subwords of the form xx^{-1} for $x \in X \cup X^{-1}$. This leads to the definition of the product on the set of reduced words $F(X)$. That is, for $w_1, w_2 \in F(X)$, we define w to be the unique reduced word obtained from $w_1 w_2$. Obviously, the empty word is reduced, and also if w is a reduced word, then so is w^{-1}. The set of reduced words $F(X)$ with the operation defined above and the empty word serving as an identity element forms a group. One can check that $F(X)$ is a free group with basis X. Hence, the following theorem holds.

Theorem 10 *If X is any set, then there exists a free group with X as its basis.*

Now, let G be any group and let

$$X = \{x_g | g \in G\}.$$

By theorem 10, $F(X)$ is a free group with the basis X. Define

$$\varphi : X \to G$$

such that

$$\varphi(x_g) = g.$$

By the definition of $F(X)$, we have that for every map $\varphi : X \to G$ there exists a unique homomorphism

$$\psi : F(X) \to G.$$

Notice that ψ is an epimorphism since

$$(\forall g \in G)(\exists x_g \in X \subseteq F(X))[\psi(x_g) = \varphi(x_g) = g],$$

so

$$F(X)/\ker\psi \cong G.$$

Thus, the following holds.

Proposition 11 *Every group is isomorphic to the quotient of a free group.*

In particular, we have the following result.

Corollary 12 *If a group is generated by a set of n elements (where n is finite or infinite), then it is a quotient group of a free group of rank n.*

Let us now define a presentation for a group. Since every group has a presentation, we can conveniently define groups using their presentations. Moreover, many significant properties of groups can be studied in terms of generators and relations. Let $F(X)$ be a free group with basis X and let $R \subseteq F(X)$. Define

$$\langle\langle R\rangle\rangle := \bigcap_{N \in \mathcal{N}} N,$$

where

$$\mathcal{N} = \{N \trianglelefteq F(X) | R \subseteq N\}.$$

Then $\langle\langle R \rangle\rangle = \bigcap_{N \in \mathcal{N}} N$ is the smallest normal subgroup of $F(X)$ containing R. Denote by $\langle X|R \rangle$ the group obtained as the quotient of $F(X)$ by its normal subgroup $\langle\langle R \rangle\rangle$, that is

$$\langle X|R \rangle := F(X)/\langle\langle R \rangle\rangle.$$

We call $\langle X|R \rangle$ a *presentation for the group* $F(X)/\langle\langle R \rangle\rangle$ and we refer to the elements of X as *generators* and to the elements of R as *relators,* while the equations $r = 1$ for $r \in R$ are called *relations*. Thus, in order to describe any group G, it is sufficient to provide the set X of generators of $F(X)$ and the subset R of $F(X)$ such that

$$G \cong F(X)/\langle\langle R \rangle\rangle,$$

which is always possible by Proposition 11. We will focus on the case when both X and $R \subseteq F(X)$ are finite sets and we will call the group described by $\langle x_1, x_2, ..., x_n \mid r_1, r_2, ..., r_m \rangle$, where $n, m \in \mathbb{Z}_+$ and $r_i \in F(\{x_1, x_2, ..., x_n\})$, $1 \leq i \leq n$, a *finitely presented group.* A group for which there exists a finite presentation is called *finitely presentable*. Let us emphasize the fact that not every group is finitely presentable, for example, $\mathbb{Q}$ does not admit a finite presentation.

Now, let us recall an important decision problem for finitely presentable groups.

Definition 13 *A (finitely generated) group $G = \langle g_1, g_2, ..., g_n | R \rangle$ has a* decidable *(or* recursively solvable*)* word problem *with respect to a system of generators $g_1, g_2, ..., g_n$ if the set U of words $u(x_1, x_2, ..., x_n)$ on the symbols $x_1, x_2, ..., x_n$ for which the equation*

$$u(x_1, x_2, ..., x_n) = 1$$

is satisfied in G upon substituting $g_1, g_2, ..., g_n$ for $x_1, x_2, ..., x_n$, is a recursive subset of the set of all words on $x_1, x_2, ..., x_n$.

The word problem for a (finitely generated) free group F_n is trivially decidable, since given a word w, it represents e of F if and only if the reduced form for w is e. Hence, every (finitely generated) free group has a decidable word problem. All (finitely generated) abelian groups have a decidable word problem, as well as all finitely presented groups with a single defining relator.

Theorem 14 *(M. Rabin [59]) A finitely generated group* $G = \langle g_1, g_2, ..., g_n | R \rangle$ *has a decidable word problem with respect to a system of generators* $g_1, g_2, ..., g_n$ *if and only if* G *has a computable isomorphic copy.*

We will now recall the defnition of a free product of groups. Let $\{G_i \mid i \in I\}$ be a family of groups. A *free product* of these groups is a group G, in symbols

$$*_{i \in I} G_i,$$

which has the following properties:

(i) G contains an isomorphic copy of each G_i,

(ii) for every group B and every family of homomorphisms $f_i : G_i \to B$, $i \in I$, there is a unique homomorphism $h : G \to B$ extending each f_i. We call G_i's the *free factors* of G. It can be shown that a free product exists. The next result is the *Kurosh Subgroup Theorem*, see [40].

Theorem 15 *Let* $G = *_{i \in I} G_i$*. Let* H *be a subgroup of* G*. Then* H *is isomorphic to the free product of a free group together with groups that are conjugates of subgroups of the free factors of* G*.*

It follows that H is isomorphic to the free product of a free group together with groups isomorphic to subgroups of G_i 's.

In further sections we study topology on the spaces of left orders of a group G. Let us recall the notions of topology and topological space. Let X be a set and τ be a family of it subsets which satisfies the following properties:

- $\emptyset, X \in \tau$;
- If $U_\alpha \in \tau$ for all $\alpha \in \Gamma$, then $\bigcup\limits_{\alpha \in \Gamma} U_\alpha \in \tau$;
- If $U_1, U_2, ..., U_n \in \tau$, then $\bigcap\limits_{i=1}^{n} U_i \in \tau$, $n \in \mathbb{Z}_+$.

The family τ satisfying the above conditions is called a *topology* on X, and the pair (X, τ) is called a *topological space*. Each element U of topology τ ($U \in \tau$) is called an *open set* and each subset C of X such that $X \backslash C$ is open ($X \backslash C \in \tau$) is called a *closed set* of X.

Example 16 *Let $X = \mathbb{R}$ be a set of all real numbers and let $\tau = \{U \subseteq X \mid X \backslash U \text{ is finite}\} \cup \{\emptyset\}$. Then τ is a topology on X and (X, τ) is a topological space.*

By the definition of τ, $\emptyset \in \tau$ and since $X \backslash X = \emptyset$ is finite, $X \in \tau$. Now, if $\{U_\alpha\}_{\alpha \in \Gamma}$ is a family of elements of τ, then $X \backslash \bigcup_{\alpha \in \Gamma} U_\alpha = \bigcap_{\alpha \in \Gamma} X \backslash U_\alpha$ is either finite or all of X. Hence, $\bigcup_{\alpha \in \Gamma} U_\alpha \in \tau$. Moreover, it is easy to see that $X \backslash \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} X \backslash U_i$ is either finite or all of X. Therefore, $\bigcap_{i=1}^{n} U_i \in \tau$. So we have just checked that (X, τ) is topological space.

It is usually not easy to describe every element of a given topology τ on X. For example, if we take the standard topology on $X = \mathbb{R}$, we usually use the following descriptive definition of its elements. A set $U \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if for every point $x \in U$ there is an open interval $(a, b) \subseteq U$ such that $x \in (a, b)$, where $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. Therefore, we see that open sets in $\mathbb{R}$ are defined in terms of a smaller collection $\mathcal{B}$ consisting of all open intervals. Obviously, the collection $\mathcal{B}$ is not a topology on $\mathbb{R}$. For example, union of two open intervals need not be an open interval, so the family of open intervals is not closed for unions of its elements, thus $\mathcal{B}$ cannot be a topology. However, the collection $\mathcal{B}$ satisfies the following properties:

(i) For any $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$ (the collection $\mathcal{B}$ covers X);

(ii) For any $B_1, B_2 \in \mathcal{B}$ if $x \in B_1 \cap B_2$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Any collection $\mathcal{B}$ of subsets of X that satisfies conditions **(i)** and **(ii)** is called a *basis*. One can verify that the collection

$$\tau(\mathcal{B}) = \{U \subseteq X \mid (\forall x \in X)\ [(x \in U) \Rightarrow (\exists B \in \mathcal{B})\ [x \in B \subseteq U]]\}.$$

of subsets of X is a topology on X, and it is called the *topology generated* by $\mathcal{B}$. Obviously, $\emptyset \in \tau(\mathcal{B})$ and by the property **(i)** of $\mathcal{B}$ we have $X \in \tau(\mathcal{B})$. Moreover, if $\{U_\alpha\}_{\alpha \in \Gamma}$ is a family of elements of $\tau(\mathcal{B})$ and $x \in \bigcup_{\alpha \in \Gamma} U_\alpha$, then $x \in U_\alpha$ for some $\alpha \in \Gamma$. Therefore, by the definition of $\tau(\mathcal{B})$, there is

$B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha$. Since $U_\alpha \subseteq \bigcup_{\alpha\in\Gamma} U_\alpha$, for any $x \in \bigcup_{\alpha\in\Gamma} U_\alpha$ there is $B \in \mathcal{B}$ such that $x \in B \subseteq \bigcup_{\alpha\in\Gamma} U_\alpha$, which shows that $\bigcup_{\alpha\in\Gamma} U_\alpha \in \tau(\mathcal{B})$. Now, let $U_1, U_2 \in \tau(\mathcal{B})$ and $x \in U_1 \cap U_2$. Then there are $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Therefore, by **(ii),** there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. This shows that $U_1 \cap U_2 \in \tau(\mathcal{B})$, and by induction we have that if $U_1, U_2, ..., U_n \in \tau(\mathcal{B})$, then $\bigcap_{i=1}^{n} U_i \in \tau(\mathcal{B})$. So, $\tau(\mathcal{B})$ is a topology on X.

Defining a topology τ on X by providing its basis $\mathcal{B}$ is one of the most common ways to introduce topology on X. However, it does not need to be the most efficient one as a smaller collection might be sufficient to describe τ on X. Suppose that we are given a collection $\mathcal{S}$ of subsets of X such that for all $x \in X$ there is $S \in \mathcal{S}$ with $x \in S$. Every collection $\mathcal{S}$ with such a property is called a *subbasis*. And let us consider a collection $\mathcal{B}(\mathcal{S})$ consisting of all finite intersections of elements of $\mathcal{S}$, that is,

$$\mathcal{B}(\mathcal{S}) = \{S_1 \cap S_2 \cap ... \cap S_k \mid S_j \in \mathcal{S},\, j = 1, 2, ..., k;\ k \in \mathbb{Z}_+\}.$$

One can easily show that $\mathcal{B}(\mathcal{S})$ is a basis, as it obviously covers X (because $\mathcal{S} \subseteq \mathcal{B}(\mathcal{S})$ and $\mathcal{S}$ covers X by the definition of $\mathcal{S}$). Moreover, if $B_1, B_2 \in \mathcal{B}(\mathcal{S})$, then both of them are finite intersections of elements of the family $\mathcal{S}$, so $B_1 \cap B_2$ is also a finite intersection of elements of $\mathcal{S}$. Therefore, $B_1 \cap B_2 \in \mathcal{B}(\mathcal{S})$, and for any $x \in B_1 \cap B_2$ there is $B_3 = B_1 \cap B_2 \in \mathcal{B}(\mathcal{S})$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We call the collection $\mathcal{B}(\mathcal{S})$ a *basis generated by the subbasis* $\mathcal{S}$. In section 6.2, we use this approach, following work of Sikora, to define a topology on the set of all left-invariant orders $LO(G)$ on a countable group G. We also use another approach to define a basis for a topology on $LO(G)$. Recall, that a function $d : X \times X \to \mathbb{R}_+$ is called a *metric* on X if d satisfies the following conditions:

(i) For any $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$;

(ii) For any $x, y \in X$ we have $d(x, y) = d(y, x)$;

(iii) For any $x, y, z \in X$ we have $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric space*.

Let d be a metric on X, $x_0 \in X$, and $\epsilon > 0$. Then the subset

$$B(x_0, \epsilon) = \{x \in X \mid d(x_0, x) < \epsilon\}$$

is called an *open ball* in X with the center x_0 and radius ϵ. Now, let $\mathcal{C} = \{B(x_0, x) \mid x_0 \in X, \, \epsilon > 0\}$. Notice that the collection $\mathcal{C}$ is a basis. The topology $\tau(\mathcal{C})$ generated by $\mathcal{C}$ is called a *metric topology* on X. We say that a topological space (X, τ) is *metrizable* if there is a metric d on X such that the topology $\tau(\mathcal{C})$ generated by the collection $\mathcal{C}$ of all open balls in X equals τ ($\tau = \tau(\mathcal{C})$). A collection $\mathcal{A} = \{U_\alpha\}_{\alpha \in \Gamma}$ of open subsets of a topological space X is called an *open covering* if for all $x \in X$ there is $U_\alpha \in \mathcal{A}$ such that $x \in U_\alpha$. We say that an open covering $\mathcal{A} = \{U_\alpha\}_{\alpha \in \Gamma}$ is a *finite open covering* if the index set Γ is a finite set (the collection $\mathcal{A}$ is finite). A topological space (X, τ) is said to be *compact* if from any open covering $\mathcal{A}$ of X we can choose a finite subcollection $\mathcal{D}$ ($\mathcal{D} \subseteq \mathcal{A}$) that covers X. In the case of a metric space (X, d), the condition for compactness can be formulated in terms of sequences in X. Recall that in a metric space, a sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ is said to be *convergent* to x_0 if

$$(\forall \epsilon > 0)(\exists N_0 \in \mathbb{Z}_+)(\forall n \geq N_0)[d(x_n, x_0) < \epsilon].$$

We say that a metric space (X, d) is compact if from every sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ of elements of X one can choose a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{Z}_+}$. In the following chapters, we also need the following notion of a totally disconnected topological space. We say that a topological space (X, τ) is *totally disconnected* if for any two distinct points $x, y \in X$ there are open sets U_x, U_y containing x and y, respectively ($x \in U_x$ and $y \in U_y$) such that $X = U_x \cup U_y$ and $U_x \cap U_y = \emptyset$. We observe that in such a space every open subset U is also closed.

3 Turing degrees of nonabelian groups

3.1 Turing degrees of isomorphism types of structures

In this section, we recall a definition of the Turing degree of the isomorphism type of a structure and some well-known facts about Turing degree spectra and degrees of isomorphism types for countable structures. We also discuss the criteria for the isomorphism type of a structure to have an arbitrary Turing degree as well as no degree.

Let $\mathcal{A}$ be a countable structure (its universe is a subset of ω) with a finite number of predicates and functions. We can assign a degree of unsolvability to $\mathcal{A}$ by defining $\deg(\mathcal{A})$ to be the least upper bound of the Turing degrees of the universe, predicates, and functions of $\mathcal{A}$. However, $\deg(\mathcal{A})$ is not an invariant under isomorphisms of $\mathcal{A}$. Namely, two countable classically isomorphic structures may have different Turing degrees. Another complexity measure of a structure that is invariant under isomorphisms, *the degree of the isomorphism type of a structure,* was introduced by Jockusch and Richter [60] by assigning the least element, if it exists, in the class of all degrees of isomorphic copies of the structure.

Let us consider the set of Turing degrees of all isomorphic copies of a structure $\mathcal{A}$. The *Turing degree spectrum* of $\mathcal{A}$, denoted by $DgSp(\mathcal{A})$, is defined as follows:

$$DgSp(\mathcal{A}) = \{\deg(\mathcal{B}) \mid \mathcal{B} \cong \mathcal{A}\}.$$

Recall that a countable structure $\mathcal{A}$ is *automorphically trivial* if there is a finite subset P of the domain A such that every permutation of A, whose restriction to P is the identity, is an automorphism of $\mathcal{A}$. Knight [33] proved that for an automorphically nontrivial structure $\mathcal{A}$, and a Turing degree $\mathbf{x}$ with $\mathbf{x} \geq \deg(\mathcal{A})$, there is a structure $\mathcal{B} \cong \mathcal{A}$ such that $\deg(\mathcal{B}) = \mathbf{x}$. That is, $DgSp(\mathcal{A})$ is closed upwards. On the other hand, for an automorphically trivial structure, all isomorphic copies have the same Turing degree, and in the case of a finite language that degree must be $\mathbf{0}$ [24]. Harizanov, Knight, and Morozov [25] showed that, while for every automorphically trivial structure $\mathcal{A}$, we have $D^e(\mathcal{A}) \equiv_T D^a(\mathcal{A})$, for every automorphically nontrivial structure $\mathcal{A}$, and every set $X \geq_T D^e(\mathcal{A})$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D^e(\mathcal{B}) \equiv_T D^a(\mathcal{B}) \equiv_T X$. Now, we can rewrite the definition of the degree of the isomorphism type of a structure as follows.

Definition 17 *Turing degree of the isomorphism type of a structure $\mathcal{A}$, if it exists, is the least Turing degree in $DgSp(\mathcal{A})$.*

If no least degree exists in $DgSp(\mathcal{A})$, the degree of the isomorphism type of $\mathcal{A}$ is undefined. Obviously, if the Turing degree spectrum of a given structure contains a computable isomorphic copy, then the degree of its isomorphism type is $\mathbf{0}$. Slaman [68] and Wehner [74] independently showed that there exists a structure with isomorphic copies in every nonzero Turing degree, but without a computable isomorphic copy. While the structure in [74] is elementarily equivalent to a computable structure, the structure in [68] is not.

We will focus on degrees of isomorphism types for countable groups. In [32], it was shown that for every abelian p-group (p is a prime number) without a computable isomorphic copy, its isomorphism type does not have a degree. Downey and Knight [16] showed that for any Turing degree $\mathbf{d}$, there is a rank 1 torsion-free abelian group whose isomorphism type has degree $\mathbf{d}$. Also, Downey and Jockusch [16] showed that some rank 1 torsion-free abelian groups do not have a degree of the isomorphism type. Calvert, Harizanov, and Shlapentokh, in [8], proved that there are torsion-free abelian groups of any finite rank whose isomorphism types have arbitrary Turing degrees, as well as those in the same classes whose isomorphism types fail to have a Turing degree. The similar results were obtained for (countable) fields and rings of algebraic numbers and functions [8]. The authors of [27] proved a general result from which it follows that there are 2-step nilpotent groups, also called *metabelian*, with arbitrary Turing degrees of their isomorphism types, as well as metabelian groups without such degrees.

For countable structures that fail to have a degree of their isomorphism types, we can consider *Turing jump degrees,* introduced by Jockusch. The jump degrees have been studied for torsion abelian groups [54] and for rank 1 torsion-free abelian groups [10]. Moreover, Turing degrees and jump degrees of the isomorphism types have been studied for partial and linear orders, trees, Boolean algebras, models of Peano arithmetic, and prime models. For more details, see [23].

Richter [60] established the following general criterion for existence of a structure whose isomorphism type has an arbitrary Turing degree. We will write $\mathcal{A} \hookrightarrow \mathcal{B}$ if $\mathcal{A}$ is embeddable in $\mathcal{B}$.

Theorem 18 *(L. Richter [60]) Let T be a theory in a finite language L such that there is a computable sequence*

$$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$$

of finite structures for L, which are pairwise nonembeddable. Assume that for every set $X \subseteq \omega$, there is a (countable) model $\mathcal{A}_X$ of T such that

$$\mathcal{A}_X \leq_T X,$$

and for every $i \in \omega$,

$$\mathcal{A}_i \hookrightarrow \mathcal{A}_X \Leftrightarrow i \in X.$$

Then for every Turing degree $\mathbf{d}$*, there is a model of T whose isomorphism type has degree* $\mathbf{d}$*.*

The procedure for constructing a structure $\mathcal{A}_X$ satisfying the assumptions of the above theorem is called a *combination method* for T. Richter applied Theorem 18 to prove that for every Turing degree $\mathbf{d}$, there is a countable abelian group whose isomorphism type has degree $\mathbf{d}$. The group $\mathcal{A}_X$ constructed in [60] using the combination method is a countable direct product of cyclic groups of prime order, hence it is an abelian torsion group. On the other hand, Richter [60] showed that a modification of Theorem 18, obtained by replacing Turing reducibility in $\mathcal{A}_X \leq_T X$ by enumeration reducibility $\mathcal{A}_X \leq_e X$, yields a different conclusion, namely, that the isomorphism type of $\mathcal{A}_X$ does not have a degree.

Theorem 19 *(L. Richter [60]) Let T be a theory in a finite language L such that there is a computable sequence*

$$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$$

of finite structures for L, which are pairwise nonembeddable. Assume that for every set $X \subseteq \omega$, there is a (countable) model $\mathcal{A}_X$ of T such that

$$\mathcal{A}_X \leq_e X,$$

and for every $i \in \omega$,

$$\mathcal{A}_i \hookrightarrow \mathcal{A}_X \Leftrightarrow i \in X.$$

Then there is a set Y such that the isomorphism type of $\mathcal{A}_Y$ does not have a degree.

As a corollary, we have that there is a countable torsion abelian group whose isomorphism type does not have a degree.

As we can see, all groups whose isomorphism type degrees were studied so far are abelian or metabelian. In the next section, we will consider Turing degrees of isomorphism types for some 'highly non-abelian' groups. In particular, since our groups are obtained using nontrivial free products, they are *centerless*, hence non-nilpotent.

3.2 Turing degrees of isomorphism types of centerless groups

In this section, we will use Theorem 18 to study the Turing degrees of the isomorphism types for various nonabelian groups. Recall, for a group G, its *center,* denoted by $Z(G)$, is defined as follows:

$$Z(G) = \{z \in G | \ (\forall g \in G)[zg = gz]\}.$$

A group G is *centerless* if $Z(G) = \{e\}$. Clearly, nontrivial centerless groups are non-nilpotent. The free product of nontrivial groups produces an infinite centerless group. Now, let us recall the following well-known group theoretic result.

Lemma 20 *Let p and q be prime numbers such that $q \mid (p-1)$. Then there is a nonabelian group of order pq.*

Proof. Let $(\mathbb{Z}_q, +)$ be the additive cyclic group with q elements, where $\mathbb{Z}_q = \{0, 1, \ldots, q-1\}$. Let $(\mathbb{Z}_p^*, \cdot)$ be the multiplicative group of the field $(\mathbb{Z}_p, +, \cdot)$. Consider the embedding τ:

$$\tau : (\mathbb{Z}_q, +) \hookrightarrow (\mathbb{Z}_p^*, \cdot).$$

Let $\mathbb{G}(q,p)$ be the semidirect product of $(\mathbb{Z}_q, +)$ and $(\mathbb{Z}_p, +)$. That is, for $(a,b),(c,d) \in \mathbb{Z}_q \times \mathbb{Z}_p$, we have in $\mathbb{G}(q,p)$:

$$(a,b)(c,d) = (a+c, b+\tau(a)\cdot d).$$

The group $\mathbb{G}(q,p)$ is nonabelian and of order pq. ∎

Theorem 21 *For every Turing degree* $\mathbf{d}$*, there is a centerless group* G *whose isomorphism type has degree* $\mathbf{d}$*. Hence*

$$DgSp(G) = \{\mathbf{x} \in \mathcal{D} | \mathbf{x} \geq \mathbf{d}\}.$$

Furthermore, G *has infinitely many finite nonabelian subgroups.*

Proof. Let

$$p_0 = 3, p_1 = 5, p_2 = 7, ...$$

be an increasing sequence of all primes > 2. For every $i \in \omega$, let q_i be the greatest prime divisor of $p_i - 1$, and let

$$\mathcal{A}_i = \mathbb{G}(q_i, p_i).$$

The sequence $(\mathcal{A}_i)_{i\in\omega}$ consists of pairwise nonembeddable groups, since for primes p_i, q_i, p_j, q_j with $q_i < p_i$ and $q_j < p_j$, we have

$$(p_i q_i \mid p_j q_j) \Leftrightarrow (p_i = p_j \wedge q_i = q_j).$$

Let $X \subseteq \omega$. Let $\mathcal{A}_X$ be the free product of the groups $\mathcal{A}_i$ for $i \in X$:

$$\mathcal{A}_X = *_{i\in X}\mathcal{A}_i.$$

We can arrange that $\mathcal{A}_X$ is a group with domain ω such that

$$\mathcal{A}_X \leq_T X.$$

Clearly, if $i \in X$, then $\mathcal{A}_i \hookrightarrow \mathcal{A}_X$. Conversely, for some $k \in \omega$, let f be an embedding from $\mathcal{A}_k$ into $\mathcal{A}_X$. Consider $f(\mathcal{A}_k)$. Since $f(\mathcal{A}_k)$ is a subgroup of $\mathcal{A}_X$ consisting of the elements of finite order, by the Kurosh Subgroup Theorem (Theorem 15), it is a conjugate of some subgroup of $\mathcal{A}_i$ for $i \in X$. Hence $k = i$, so $k \in X$.

Let

$$\mathcal{A} = \mathcal{A}_X,$$

where $X = D \oplus \overline{D}$ and $D \subseteq \omega$ is a set of Turing degree $\mathbf{d}$. Since $\mathcal{A}$ is automorphically nontrivial, we have that

$$DgSp(\mathcal{A}) = \{\mathbf{x} \in \mathcal{D} | \mathbf{x} \geq \mathbf{d}\},$$

which completes the proof ■

Now, if we choose the set $X \subseteq \omega$ so that the set of functions $\{f : ran(f) = X\}$, where $ran(f)$ denotes the range of f, has no Turing least element, we can obtain the following result.

Theorem 22 *There is a centerless group with infinitely many finite nonabelian groups such that its isomorphism type does not have a Turing degree.*

We can also apply a free product construction, similar to the one in the proof of Theorem 21, to the sequence of cyclic groups of prime order

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7, \ldots,$$

to obtain the following result.

Theorem 23

(i) *For every Turing degree* $\mathbf{d}$*, there is a centerless group* $\mathcal{A}$ *without infinite noncyclic abelian subgroups, whose all finite subgroups are cyclic, such that the isomorphism type of* $\mathcal{A}$ *has degree* $\mathbf{d}$.

(ii) *There is a centerless group* $\mathcal{B}$ *without infinite noncyclic abelian subgroups, whose all finite subgroups are cyclic, such that the isomorphism type of* $\mathcal{B}$ *has no degree.*

We now prove a general result, which is a modification of Theorem 18, but allows infinite structures in the sequence.

Theorem 24 *Let $\mathcal{C}$ be a class of countable structures in language L, closed under isomorphisms. Assume that there is a computable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of computable (possibly infinite) structures for L such that for any set $X \subseteq \omega$, there is a structure $\mathcal{A}_X$ in $\mathcal{C}$ satisfying the following conditions:*

(i) $\mathcal{A}_X \leq_T X$

(ii) *For every $i \in \omega$,*

$$\mathcal{A}_i \hookrightarrow \mathcal{A}_X \Leftrightarrow i \in X.$$

(iii) *Suppose that $\mathcal{A}_X$ is isomorphic to a structure $\mathcal{B}$. Then there is a uniformly effective procedure with oracle $\mathcal{B}$, which for a pair of structures $\mathcal{A}_i$, $\mathcal{A}_j$ such that exactly one of the structures embeds in $\mathcal{B}$, decides which of the two structures embeds in $\mathcal{B}$.*

Then for every Turing degree $\mathbf{d}$*, there is a structure in $\mathcal{C}$ whose isomorphism type has degree* $\mathbf{d}$.

Proof. Let $D \subseteq \omega$ be such that $deg(D) = \mathbf{d}$. We will show that $\mathcal{A}_{D\oplus\overline{D}}$ is a structure in $\mathcal{C}$, whose isomorphism type has Turing degree $\mathbf{d}$. Clearly, (by definition of D and assumption **(i)**),

$$\deg(\mathcal{A}_{D\oplus\overline{D}}) \leq \deg(D \oplus \overline{D}) = \mathbf{d}.$$

Now, let a structure $\mathcal{B}$ be such that $\mathcal{B} \simeq \mathcal{A}_{D\oplus\overline{D}}$. We then have, by the definition of $D \oplus \overline{D}$ and assumption **(ii)** of the theorem, that for every $j \in \omega$:

$$(j \in D \Leftrightarrow \mathcal{A}_{2j} \hookrightarrow \mathcal{B})\text{ , and}$$

$$(j \notin D \Leftrightarrow \mathcal{A}_{2j+1} \hookrightarrow \mathcal{B})\,.$$

Thus, by assumption **(iii)** of the theorem, we conclude that $D \leq_T \mathcal{B}$. Hence $\deg(\mathcal{A}_{D\oplus\overline{D}}) = \mathbf{d}$, and the degree of the isomorphism type of $\mathcal{A}_{D\oplus\overline{D}}$ is $\mathbf{d}$. ■

We can also establish the following result, 'dual' to Theorem 24, which we state without proof.

Theorem 25 *Let $\mathcal{C}$ be a class of countable structures in language L, closed under isomorphisms. Assume that there is a computable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots$ of computable (possibly infinite) structures for L such that for any set $X \subseteq \omega$, there is a structure $\mathcal{A}_X$ in $\mathcal{C}$ satisfying the following conditions:*

(i) $\mathcal{A}_X \leq_e X$

(ii) *For every $i \in \omega$,*

$$\mathcal{A}_i \hookrightarrow \mathcal{A}_X \Leftrightarrow i \in X.$$

(iii) *Suppose that $\mathcal{A}_X$ is isomorphic to a structure $\mathcal{B}$. Then from any enumeration of $\mathcal{B}$ we can effectively enumerate those i for which $\mathcal{A}_i$ embeds in $\mathcal{B}$.*

Then there is a set Y such that the isomorphism type of $\mathcal{A}_Y$ has no degree.

We will now apply the last two results to an interesting class of nonabelian groups. We first introduce the following definition.

Definition 26 *A group G is of a* finite exponent *if there is a finite integer n such that $g^n = e$ for all $g \in G$. If, in addition, there is no positive integer $m < n$ such that $g^m = e$ for all $g \in G$, then we say that G has an* exponent *n.*

W. Burnside [7] was first to consider groups G of finite exponents. In particular, he was interested in the case when G is a finitely generated group of a fixed exponent. He asked whether there exists an infinite but finitely generated group G of finite exponent. This question is now known as the *Burnside Problem.*

Let $F_r = \langle x_1, x_2, ..., x_r \rangle$ be the free group of rank r and let $B(r,n) = F_r/N$, where N is the normal subgroup of F_r generated by $\{g^n |\ g \in F_r\}$. We denote by $B(r,n)$ the *rank r Burnside group of exponent n.* Using this notation, Burnside's question can be formulated as: *For what values of r and n is $B(r,n)$ infinite?*

The group $B(r,n)$ was shown to be finite when $r = 1$, or r is an arbitrary positive integer and $n = 2, 3, 4, 6$ in [7, 65, 21]. It was proved by Novikov and Adjan [50, 51, 52] that $B(r,n)$ is infinite when $r > 1$, n is odd, and $n \geq 4381$. This result was later improved by Adjan [1], who showed that

$B(r,n)$ is infinite if $r > 1$, n is odd, and $n \geq 665$. Moreover, for these values of r and n, Novikov and Adjan's proof implies that any finite or abelian subgroup of $B(r,n)$ is cyclic, as well as that the word problem (and the conjugacy problem) for $B(r,n)$ is decidable. Hence, every such group has a computable copy.

Let the sequence $(\hat{p}_i)_{i\in\omega}$ be an increasing sequence of all prime numbers ≥ 665. Let $\mathcal{G}_i$, $i \in \omega$, be a group such that

$$\mathcal{G}_i \simeq B(r, \hat{p}_i),$$

where $r \geq 2$ is fixed, and $i \in \omega$. Notice that $\mathcal{G}_i$ is computable. The following propositions establish an important property for the sequence of groups $(\mathcal{G}_i)_{i\in\omega}$.

Proposition 27 *Let $i, j \in \omega$. Then $\mathcal{G}_i \hookrightarrow \mathcal{G}_j$ iff $i = j$.*

Proof. First observe that every element of $\mathcal{G}_i = B(r, \hat{p}_i)$ has a fixed prime order. Hence, $\mathcal{G}_i \hookrightarrow \mathcal{G}_j$ if and only if $\mathcal{G}_j$ has an element of order $\hat{p}_i$. Since $\hat{p}_i$ is prime, we have that $\hat{p}_i = \hat{p}_j$, and thus $i = j$. ■

It follows from the Torsion Theorem for Free Products [40] that an element of finite order in a free product of groups is a conjugate of an element of finite order in one of the factors.

Proposition 28 *Let $X \subseteq \omega$, and let $\mathcal{B} \simeq \ast_{j\in X} \mathcal{G}_j$. Then*

$$\mathcal{G}_i \hookrightarrow \mathop{\ast}_{j\in X} \mathcal{G}_j \Leftrightarrow i \in X.$$

Moreover, there is an a uniformly effective procedure with oracle $\mathcal{B}$ which for a pair of structures $\mathcal{G}_i$, $\mathcal{G}_j$ such that exactly one of the structures embeds in $\mathcal{B}$, decides which of the two groups embeds in $\mathcal{B}$. Also, from any enumeration of $\mathcal{B}$, we can effectively enumerate those i for which $\mathcal{G}_i$ embeds in $\mathcal{B}$.

We are now ready to apply Theorems 24 and 25 to the sequence $(\mathcal{G}_i)_{i\in\omega}$. Thus we obtain the following result.

Theorem 29 *For every Turing degree $\mathbf{d}$, there is a centerless group $\mathcal{A}$ whose isomorphism type has degree $\mathbf{d}$, such that $\mathcal{A}$ has infinitely many infinite nonabelian subgroups, which are generated by a fixed finite number of generators, and are of finite exponents. Moreover, every finite or abelian subgroup of $\mathcal{A}$ is cyclic. There is also such a group $\mathcal{B}$ whose isomorphism type has no degree.*

4 Left-orderable, fully left-orderable, and bi-orderable groups

In this chapter, we recall the definitions and some important facts concerning orderable groups. Groups, on which one is able to define an order that is left-invariant (or/and right-invariant) under the group operation, have been studied for a long time. Some results concerning orderable groups, due to Hölder, are from the beginning of the last century. Many algebraically strong properties of groups are implied by the existence of an order on them. Recent interest in left-ordered groups of Rolfsen, Boyer, Wiest [4, 64], and Sikora [66] (in topology), and Solomon [70, 71] (in recursion theory) brought a new perspective to the theory of orderable groups.

Definition 30 *Let G be a multiplicative group. We say that G is* left-orderable *(*partially left-orderable*) if there is an order (a partial order, respectively) $\leq$ on G with the property:*

$$(\forall x, y, z \in G)[x \leq y \Longrightarrow zx \leq zy].$$

Analogously, G is right-orderable *(*partially right-orderable*) if there is an order (a partial order, respectively) $\leq$ on G such that:*

$$(\forall x, y, z \in G)[x \leq y \Longrightarrow xz \leq yz].$$

We call a group G orderable *(or* bi-orderable*) if the relation $\leq$ is both left- and right-invariant under the group operation.*

Given a partial (or a total) order $\leq$ on G, let $<$ be a relation on G defined as follows:

$$(\forall x, y \in G)[x < y \Leftrightarrow (x \leq y) \wedge (x \neq y)].$$

We call $<$ a strict partial (or total) order.

Remark 31 *Obviously, every left-invariant order $\leq$ on a group G induces a right-invariant order $\leq'$ on G simply by setting:*

$$g \leq' h \Leftrightarrow h^{-1} \leq g^{-1} \text{ for } g, h \in G.$$

Therefore, there is a one-to-one correspondence between left-invariant and right-invariant orders on G.

The existence of a left-invariant (right-invariant) order on a group G implies the following algebraic property of left-orderable (right-orderable) groups.

Proposition 32 *Every nontrivial left-orderable group G is torsion-free.*

Proof. Suppose that there is an element $x \in G$ with $x \neq e$ and such that $x^n = e$, $n \in \mathbb{Z}_+$. Then

$$e < x \leq x^2 \leq ... \leq x^{n-1} \leq x^n = e,$$

a contradiction. ∎

It follows from the above proposition that no finite nontrivial group is left-orderable. Moreover, left-orderable groups can be viewed as subgroups of the group of orientation-preserving homeomorphisms of $\mathbb{R}$.

Theorem 33 *If G is a countable group, then the following are equivalent:*

(i) *G is left-orderable.*

(ii) *G is isomorphic to a subgroup of orientation-preserving homeomorphisms of $\mathbb{R}$, $Homeo_+(\mathbb{R})$.*

(iii) *G is isomorphic to a subgroup of orientation-preserving homeomorphisms of $\mathbb{Q}$, $Homeo_+(\mathbb{Q})$.*

The previous theorem can be found in [4].

Note that left-orderable groups admit effective actions by order-preserving bijections on linearly ordered sets. We say that a group G acts (from the left or right) on a set X if there exists a homomorphism $\psi : G \to Sym(X)$, where $Sym(X)$ denotes the set of all bijections of X. We say that G *acts effectively on* X if $\ker(\psi) = \{e\}$.

Theorem 34 *(P. F. Conrad [11]) A group G is left-orderable if and only if it acts effectively on a linearly ordered set X by order-preserving bijections.*

Proof. Since G acts on itself by right multiplications, we take $X = G$. Since G is linearly ordered and the order on G is left-invariant, then the only-if-direction follows.

For the if-part, let us assume that G acts effectively on a linearly ordered set X by order-preserving bijections. That is,

$$(\forall g \in G)(\forall x, y \in X)[x \prec y \Longrightarrow \psi(g)(x) \prec \psi(g)(y)].$$

Let $\preceq$ be some well-ordering of the elements of X. For $g \neq h$ and $g, h \in G$ define

$$g \precsim h \text{ if and only if } \psi(g)(x) \preceq \psi(h)(x),$$

where

$$x = \min\{y \in X | \psi(g)(y) \neq \psi(h)(y)\},$$

and the minimum is taken with respect to the well-ordering $\preceq$ on X. One can easily check that $\precsim$ on G is left-invariant. ■

A left-ordering $\leq$ on a group G is called *Archimedean* if for all $x, y \in G$ such that $e < x \leq y$ we have

$$(\exists n \in \mathbb{Z}_+)[y \leq x^n].$$

Algebraic properties of groups that admit a left-invariant ordering with the Archimedean property are characterized by the following result, which can be found in [11]

Theorem 35 *(O. Hölder [29]) If a left-orderable group G is Archimedean, then the order $\leq$ on G is a bi-order and G is isomorphic (via the order-preserving isomorphism) to a subgroup of the group of additive real numbers $(\mathbb{R}, +)$ with the standard order. In particular, G is abelian.*

The theorem above states that if the order on a group G is Archimedean, then the group does not have algebraically interesting structure.

Since any torsion-free abelian group admits a total bi-order [29] (see also Corollary 44 below), examples of bi-orderable groups can be found among familiar groups.

Example 36 *The additive group of integers* $(\mathbb{Z},+)$ *is bi-orderable with the standard order.*

Example 37 *The additive group of complex numbers* $(\mathbb{C},+)$ *is bi-orderable with the ordering* $\prec$ *that is defined as follows: for* $a_1 + b_1 i$, $a_2 + b_2 i \in \mathbb{C}$ *we set*

$$a_1 + b_1 i \prec a_2 + b_2 i \Leftrightarrow [(a_1 < a_2) \vee (a_1 = a_2 \wedge b_1 < b_2)].$$

In our further discussion, we will identify an order on a group with a subset of its 'positive' elements. This is a more convenient way for working with orderable groups, which also allows us to easier formulate results in the following section.

Definition 38 *Let* $\leq$ *be a partial left order on a group* G*. Define a* positive partial cone P *as follows:*

$$P = \{x \in G | e \leq x\}.$$

Similarly, define a negative partial cone *as*

$$P^{-1} = \{x \in G | x^{-1} \in P\} = \{x \in G | x \leq e\}.$$

The following properties of cones are simple consequences of the properties of $\leq$.

Proposition 39 *Let* P*,* P^{-1} *be subsets of a group* G *defined above. Then*

(i) $PP \subseteq P$ & $P^{-1}P^{-1} \subseteq P^{-1}$ (P, P^{-1} are *sub-semigroups* of G),

(ii) $P \cap P^{-1} = \{e\}$ (P, P^{-1} are called *pure* sub-semigroups),

(iii) $P \cup P^{-1} = G$ if $\leq$ is a total left order on G (P, P^{-1} are *total*).

We denote a positive cone and a negative cone for a total order on G by P^+ and P^-, respectively.

Proof. **(i)** If $x, y \in P$, then $e \leq x$ and $e \leq y$, so $e \leq xy$, i.e. $xy \in P$, thus $PP \subseteq P$. Similarly for P^{-1}.

(ii) Let $x \in P \cap P^{-1}$ and $x \neq e$, then $e < x$ and $e > x$, hence $e < x < e$, a contradiction. Hence, one has $P \cap P^{-1} = \{e\}$.

(iii) Since the relation $\leq$ is total, for all $x \in G$ we have either $e \leq x$ or $e \geq x$, i.e., $x \in P$ or $x \in P^{-1}$, so $P \cup P^{-1} = G$. ■

Proposition 40 *If in a group G there are subsets P and P^{-1} satisfying conditions* (**i**)*,* (**ii**)*, and* (**iii**) *of Proposition 39, then there is a total left order $\leq_1$ on G with the positive cone $P^+ = P$.*

Proof. Let us define a relation $\leq_1$ on G as follows:

$$(\forall a, b \in G)[(a \leq_1 b) \Leftrightarrow (a^{-1}b \in P)].$$

We show that $\leq_1$ is a linear ordering on G, which is left-invariant under the group operation. If $x \leq_1 y$ and $y \leq_1 z$, then

$$x^{-1}y \in P \text{ and } y^{-1}z \in P.$$

Since P satisfies **(i)** (P is a semigroup), then

$$(x^{-1}y)(y^{-1}z) \in P,$$

so

$$(x^{-1}y)(y^{-1}z) = x^{-1}z \in P.$$

Hence $x \leq_1 z$. Therefore, the relation $\leq_1$ is transitive.

If $x \leq_1 y$ and $y \leq_1 x$, then

$$x^{-1}y \in P \text{ and } y^{-1}x \in P.$$

That is,

$$x^{-1}y \in P \text{ and } (x^{-1}y)^{-1} \in P,$$

so

$$x^{-1}y \in P \text{ and } x^{-1}y \in P^{-1}.$$

Hence

$$x^{-1}y \in P \cap P^{-1} = \{e\}$$

by **(ii)**, and we have $x = y$. Moreover, since $e \in P$, for any $x \in G$ we have $x \leq_1 x$, so $\leq_1$ is reflexive. This shows that $\leq_1$ is a partial ordering on G.

By property **(iii)**, we have

$$(\forall x, y \in G)[(x^{-1}y \in P) \vee (x^{-1}y \in P^{-1})].$$

Therefore, we have either $x \leq_1 y$ or $y \leq_1 x$ for $x, y \in G$. So $\leq_1$ is a linear ordering on G.

We will now prove that $\leq_1$ is left-invariant under the group operation. Suppose $x \leq_1 y$ and $z \in G$. Since

$$x^{-1}y \in P$$

and

$$x^{-1}y = (x^{-1}z^{-1})(zy) = (zx)^{-1}(zy) \in P,$$

we have

$$zx \leq_1 zy,$$

so

$$(\forall z \in G)[x \leq_1 y \Rightarrow zx \leq_1 zy].$$

Therefore, $\leq_1$ is a left ordering on G. ■

Remark 41 *If in addition to conditions **(i)**, **(ii)**, and **(iii)** of Proposition 39 we assume that*

$$(\forall x \in G)[x^{-1}Px \subseteq P],$$

in which case P is called a normal cone*, then P defines a bi-invariant order on G. To see this, let $(G, \leq)$ be left-orderable with positive cone P. Let $x, y \in G$ with $x \leq y$. Then $x^{-1}y \in P$. Since P is normal,*

$$(\forall z \in G)[z^{-1}x^{-1}yz = (xz)^{-1}(yz) \in P].$$

Hence for all $z \in G$ we havexz $\leq yz$, so $\leq$ is a bi-order.

Properties **(i)**, **(ii)** and **(iii)** listed in Proposition 39 state that P and P^{-1} are pure and total sub-semigroups of G. Therefore, we may study orders on G by studying total pure sub-semigroups of G.

Let $\{x_1, x_2, ..., x_n\}$ be a nonempty subset of G. Define $sgr(\{x_1, x_2, ..., x_n\})$ to be the minimal sub-semigroup of G containing $\{x_1, x_2, ..., x_n\}$, i.e.,

$$sgr(\{x_1, x_2, ..., x_n\}) = \{a_1 a_2 ... a_k |\ a_j \in \{x_1, x_2, ..., x_n\},\ j = 1, 2, ..., k;\ k \in \omega\},$$

and $sgr(\emptyset) = \emptyset$. Let $P = sgr(\{x_1, x_2, ..., x_n\})$, where $x_1, x_2, ..., x_n$ are nonidentity elements of G. It can easily be shown that a sub-semigroup $P \cup \{e\}$ defines a partial left order on G if and only if $e \notin P$. Since finitely generated sub-semigroups P with $e \notin P$ are determined by the choice of elements $g_1, g_2, ..., g_n \in G$, we can identify each P with its finite set of generators $p = \{g_1, g_2, ..., g_n\}$.

Remark 42 *It is not true that every partial left order (right order) on a group G can be extended to a total left order (right order). It was shown in [11] that a partial right order $P \cup \{e\}$ can be extended to a total right order P^+ on G if and only if for every finite set $\{x_1, x_2, ..., x_n\} \subset G \backslash \{e\}$ there exists a finite sequence of corresponding integers $\epsilon_1, \epsilon_2, ..., \epsilon_n$, where $\epsilon_i = \pm 1$, such that*

$$e \notin sgr(P \cup \{x_1^{\epsilon_1}, x_2^{\epsilon_2}, ..., x_n^{\epsilon_n}\}).$$

Note that if we take P to be the empty set, $P = \emptyset$, the above criterion becomes a necessary and sufficient condition for the existence of a total right order (left order) on G.

We call the groups with the property that all partial left orders on them can be extended to total left orders *fully left-orderable*. The following criterion determines whether a given group G is fully left-orderable.

Theorem 43 (*S. A. Todorinov, see [35]*) *A group G is fully left-orderable if and only if for every element $a \in G \setminus \{e\}$ we have the following two conditions satisfied:*

(i) $a^n \neq e$ *for all* $n \in \omega$,

(ii) *For any nontrivial subset S of G and for any nonidentity elements $b \in sgr(S^{-1} \backslash \{e\}, a)$ and $c \in sgr(S^{-1} \backslash \{e\}, a^{-1})$, the semigroup $sgr(S \backslash \{e\}, b, c)$ contains the identity element e.*

Using the above theorem, we can show that many classes of groups, including torsion-free abelian groups and torsion-free nilpotent groups, are fully left-orderable.

Corollary 44 *Every torsion-free abelian group is fully bi-orderable.*

Proof. Let $a \in G$, $a \neq e$. Since G is torsion-free, then $a^n \neq e$ for all $n \in \omega$. Let $S \subseteq G$, $S \neq \{e\}$. An element

$$b \in sgr(S^{-1}\backslash\{e\}, a)$$

iff

$$b = \sum_{i=1}^{j} k_i \overline{s_i} + ka,$$

where $k_i, k \in \mathbb{Z}_+ \cup \{0\}$ and $\sum_{i=1}^{j} k_i + k \neq 0$, and $\overline{s_i} = -s_i$, $s_i \in S\backslash\{e\}$.

Let $c \in sgr(S^{-1}\backslash\{e\}, a^{-1})$. Then

$$c = \sum_{i=1}^{l} m_i \overline{s_i} + ma^{-1},$$

where $m_i, m \in \mathbb{Z}_+ \cup \{0\}$ and $\sum_{i=1}^{l} m_i + m \neq 0$.

If $k = 0$ or $m = 0$, then $b \in sgr(S^{-1}\backslash\{e\})$ or $c \in sgr(S^{-1}\backslash\{e\})$. Hence $-b \in sgr(S\backslash\{e\})$ or $-c \in sgr(S\backslash\{e\})$. Thus,

$$e \in sgr(S\backslash\{e\}, b, c).$$

Therefore, we may assume that $k \neq 0$ and $m \neq 0$ in which case

$$b + \sum_{i=1}^{j} k_i s_i = ka,$$

and

$$c + \sum_{i=1}^{l} m_i s_i = m\overline{a}$$

where $\overline{a} = -a$, which implies that

$$e = m(ka) + k(m\overline{a}) \in sgr(S\backslash\{e\}, b, c).$$

Thus, by Theorem 43, every torsion-free abelian group is fully left-orderable, hence fully bi-orderable.

■

Another useful criterion for orderability of groups is given in the following theorem.

Theorem 45 *(R. Burns, V. Hale [6]) A group is left-orderable if and only if every finitely generated subgroup has a nontrivial quotient which is left-orderable.*

Now, let us recall an important notion for the theory of orderable groups, the notion of locally indicable groups.

Definition 46 *A group G is* locally indicable *if every nontrivial finitely generated subgroup of G has $\mathbb{Z}$ as a quotient.*

Note that, by Theorem 45, every locally indicable group is left-orderable.

Example 47 *The group $\mathbb{Z}^n$ is locally indicable. Every free group is locally indicable.*

Next theorem establishes a connection between locally indicable and bi-orderable groups [11, 6].

Theorem 48 *If G is a bi-orderable group, then G is locally indicable. If G is locally indicable, then G is left-orderable.*

The following example shows that local-indicability of a group G does not imply its bi-orderability.

Example 49 *The group G given by the following presentation $\langle x, y | xyx^{-1}y \rangle$[1] is locally indicable (it was shown in [28] that any torsion-free one-relator group is locally indicable), hence left-orderable, but not bi-orderable. In order to see that G is not bi-orderable, notice that since in G the relation $xyx^{-1}y = e$ holds, then $xy = y^{-1}x$. Hence, every element of G can be written in the form $x^a y^b$, where $a, b \in \mathbb{Z}$. One defines*

$$P^+ = \{x^a y^b \in G | a \in \mathbb{Z}_+ \vee (a = 0 \wedge b \in \mathbb{Z}_+)\}.$$

to be a positive cone for a left order on G. Let us assume that there exists a bi-order $\prec$ on G. Then, since $xyx^{-1} = y^{-1}$ in G, we have that if, for example, $e \prec y$, then $x \prec xy$ (since $\prec$ is left-invariant), so $e = xx^{-1} \prec xyx^{-1} = y^{-1}$ (since $\prec$ is right-invariant), thus $e \prec y^{-1}$, a contradiction.

More generally, one can show that every torsion-free finitely generated group with one-relator admits a left-invariant order, which is not necessarily a bi-order.

[1] The group given by this presentation is isomorphic to the fundamental group of the Klein bottle.

As we mentioned before, interesting examples of computable and orderable groups arise when we deal with topological spaces, in particular, with at most 3-dimensional manifolds. Recall that to every path-connected topological space X, we associate, up to isomorphism, its *fundamental group* $\pi_1(X)$. Usually (in the majority of interesting cases, such as for 3-dimensional manifolds obtained as complements of links in a 3-dimensional sphere S^3), these groups are given by finite presentations. The existence of left orders on fundamental groups of 3-manifolds is a rather common property. For example, fundamental groups of a wide class of 3-manifolds obtained as complements of links in S^3 admit left orders, which are not bi-invariant [4]. Even though it is quite difficult to find classes of 3-manifolds whose fundamental groups do not admit left orders, the following example of a torsion-free group without a left order has recently been found [14].

Example 50 *The group G given by the following presentation $\langle x, y \mid x^2yx^2y^{-1}, y^2xy^2x^{-1}\rangle$ is not left-orderable.*

The proof that this group is not left-orderable can be found in [14], where families of finitely presentable torsion-free fundamental groups with no left-invariant orders were found for certain classes of 3-dimensional manifolds obtained as cyclic branched covers of S^3 along some classes of two-bridge and precel links.

In the case of connected 2-manifolds, the following theorem holds.

Theorem 51 (*D. Rolfsen, B. Wiest [64]*) *If N is any connected surface other than the projective plane $\mathbb{R}P^2$ or Klein bottle, then $\pi_1(N)$ is bi-orderable. For $N =$Klein bottle, $\pi_1(N)$ is left-orderable (see Example 49) and for $N = \mathbb{R}P^2$, $\pi_1(N)$ does not admit any order since it is isomorphic to $\mathbb{Z}_2$.*

Notice that all fundamental groups of orientable, connected 2-manifolds are bi-orderable, in particular, we have the following examples.

Example 52 *The fundamental group of a closed, connected and orientable surface M_g of genus $g > 1$, described using the following presentation*

$$\pi_1(M_g) = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid [x_1, y_1][x_2, y_2] \ldots [x_g, y_g]\rangle,$$

where $[x_i, y_i] = x_i^{-1}y_i^{-1}x_iy_i, 1 \leq i \leq g$, *is bi-orderable*[2]*.*

Example 53 $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} = \langle x, y \mid xyx^{-1}y^{-1} \rangle$ *is a presentation of the fundamental group of a two-dimensional torus* T^2 *and it is bi-orderable with the positive cone defined as follows*

$$P^+ = \{x^a y^b | a \in \mathbb{Z}_+ \vee (a = 0 \wedge b \in \mathbb{Z}_+)\}.$$

The next proposition is a consequence of Theorem 90 in section 6.1.

Proposition 54 *There are uncountably many different bi-orders on* $\mathbb{Z}^n$ *for* $n > 1$*.*

Let us now consider the property of orderability for the class of finitely presented groups from the point of view of computability.

Definition 55 *A property* P *of finitely presentable groups is called a* Markov property *if the following conditions are satisfied:*

(i) *There exists a finitely presented group* G_+ *with property* P*;*

(ii) *There exists a finitely presented group* G_- *such that for any finitely presentable group* H *if* G_- *embeds in* H*, then* H *does not have property* P*.*

Proposition 56 *The property of being left-orderable for finitely presentable groups is a Markov property.*

Proof. Obviously, there exist a finitely presented group that is left-orderable, for example, let $G_+ = \mathbb{Z} \oplus \mathbb{Z}$. For a finitely presented group that is not left-orderable, let us take $G_- = \mathbb{Z}_2$. Let H be a finitely presentable group such that G_- embeds in H. Since G_- has a torsion, then H has a torsion, so H cannot be left-orderable. ∎

The above result is quite interesting for the following reason.

Theorem 57 (*M. Rabin [58]*) *Every Markov property* P *of finitely presentable groups is not decidable.*

[2]Moreover, analyzing the lower central series of $\pi_1(M_g)$ [36], we can show that $BiO(\pi_1(M_g))$ is uncountable and as a topological space admits an embedding of the Cantor set.

Thus, we obtain the following result.

Corollary 58 *The property of being left-orderable for finitely presentable groups is not decidable.*

Recall that groups with presentations of the form

$$\langle x_1, x_2, ..., x_n | r \rangle,$$

where $r \in F(\{x_1, x_2, ..., x_n\})$, are called *one-relator groups*. The theory of one-relator groups is quite well understood. Many decision problems, such as the word problem or the conjugacy problem, have been proven to be decidable for this class of groups. Therefore, one may expect to be able to understand the problem of the existence of an order on them.

Let $r \in F_n$, where F_n denotes the free group with the basis of cardinality $n \in \mathbb{Z}_+$. Then r can be uniquely written in the form $r = s^m$ for some $m \in \mathbb{Z}_+$, where m is a maximal positive integer with this property, and $s \in F_n$. We call s the *root* of r. If $m = 1$, we say that r is not a proper power.

Theorem 59 *Let $G = \langle x_1, x_2, ..., x_n |\ r \rangle$. Then G is torsion-free if and only if r is not a proper power.*

Since every orderable group must be torsion-free, the above theorem[3] provides us with a simple method of excluding the possibility for the existence of left-invariant orders on one-relator groups. It happens that torsion-free one-relator groups are left-orderable, which is a consequence of the following theorem.

Theorem 60 (*S. D. Brodsky [5]*) *Any torsion-free subgroup of a one-relator group is locally indicable.*

Since every locally indicable group is left-orderable (Theorem 48), we have an immediate criterion for orderability of one-relator groups.

Corollary 61 *For one-relator groups, the problem of being left-orderable is decidable.*

[3] The proof of Theorem 59 can be found in [42].

Proof. Consider the following decision problem in a free group F_n. Given $r \in F_n$, decide whether there exist $k \in \mathbb{Z}_+$ and $w \in F_n$ such that $r = w^k$. Since this problem is decidable, the problem whether a one-relator group is torsion-free is decidable. Therefore, by Theorem 60, it follows that it is possible to decide whether a one-relator group is left-orderable. ■

5 Spaces of left orders for computable groups

Let $LO(G)$ be the set of all left-invariant orders on a group G. Using Proposition 40, we can identify $LO(G)$ with the set of all positive cones in G, i.e.,

$$LO(G) = \{P_+ \subseteq G \mid P_+ \text{ is positive cone for a total left order on } G\}.$$

Similarly, we can define $RO(G)$ to be the set of all right-invariant orders on G, and $BiO(G)$ to be the set of all orderings on G that are bi-invariant (left- and right-invariant at the same time). Thus,

$$BiO(G) = LO(G) \cap RO(G).$$

Since we are interested in left-orderable groups, we will always assume (unless otherwise stated) that $LO(G) \neq \emptyset$.

5.1 Computability theoretic analysis of the space of left orders

In this section, we analyze computational properties of spaces of left orders on computable groups. We recall R. Solomon's result describing the complexity of spaces of orders on computable groups in terms of computably bounded Π_1^0 classes [70]. We also cite his results concerning the complexity of the space of orders for a class of computable torsion-free abelian groups. We will later extend our considerations by introducing a new complexity measure for the space of left orders on a computable group G, the Turing degree spectrum of left orders. We will use the results of this section to provide, in chapter 7, recursive description of spaces of left orders for some interesting classes of groups.

Let us first recall the notion of a computably bounded Π^0_1 class, which was used by Solomon to analyze the complexity of the space of orders for orderable, computable groups in [70].

Definition 62 *A binary branching tree is a set $T \subseteq \{0,1\}^{<\omega}$ such that for all $\alpha, \beta \in \{0,1\}^{<\omega}$:*

$$(\alpha \subseteq \beta) \wedge (\beta \in T) \Rightarrow \alpha \in T.$$

A path in T is a function $\varphi : \omega \to \{0,1\}$ such that $\langle \varphi(0), \varphi(1), ..., \varphi(n) \rangle \in T$ for all n.

Definition 63 *The subset $C \subseteq \{0,1\}^{\omega}$ is a* computably bounded Π^0_1 class (c.b. Π^0_1 class) *if there is a computable binary branching tree T such that C is the set of paths in T.*

The following results of Jockusch and Soare characterize c.b. Π^0_1 classes.

Theorem 64 (*C. G. Jockusch Jr., R. I. Soare [31]*) *There is an infinite c.b. Π^0_1 class C such that for all $\alpha, \beta \in C$:*

$$\alpha \neq \beta \Longrightarrow \alpha \text{ and } \beta \text{ are Turing incomparable.}$$

The next result is Low Basis Theorem.

Theorem 65 (*C. G. Jockusch Jr., R. I. Soare [31]*) *Every c.b. Π^0_1 class has a member of low degree or, equivalently, every infinite computable binary branching tree has a low path.*

Given a computable, orderable group G, a natural question to ask is how complicated are the elements of its space of orders. The question was answered by Solomon, who established a connection between $BiO(G)$ and c.b. Π^0_1 classes.

Theorem 66 (*R. Solomon [70]*) *Let G be a bi-orderable computable group. Then there is a c.b. Π^0_1 class C and a bijection $\varphi : BiO(G) \to C$ that preserves Turing degrees.*

A general strategy used in the proof of Theorem 66 is as follows: starting with a computable bi-orderable group G, build a computable binary branching tree T with paths that code all the orders on G. The tree is infinite since G is orderable and the paths of T correspond, up to Turing degree, to the orders of G. Hence the space of orders on G is, up to Turing degree, a c.b. Π^0_1 class.[4]

[4] A rigorous proof of Theorem 66 can be found in [70], Theorem 3.34.

Remark 67 *We know that a stronger result does not hold, namely, it is not true that for every c.b. Π^0_1 class C there exists an orderable computable group G and a Turing degree preserving bijection*

$$\varphi : BiO(G) \to C.$$

To see this, notice that for any orderable group G we have

$$P \equiv_T P^{-1}.$$

However, there is a c.b. Π^0_1 class C such that

$$(\forall \alpha, \beta \in C)[\alpha \neq \beta \Rightarrow \deg(\alpha) \neq \deg(\beta)].$$

In his Ph.D. thesis [70], Solomon also considered the problem of the computational complexity of the space of orders for computable torsion-free abelian groups.

Theorem 68 (*R. Solomon [70]*)

(i) *If G is a computable torsion-free abelian group of rank 1, then G has exactly two orders, both of which are computable.*

(ii) *If G is a computable torsion-free abelian group of finite rank > 1, then G has 2^ω orders and has orders of every Turing degree.*

(iii) *If G is a computable torsion-free abelian group of infinite rank, then G has 2^ω distinct orders and has orders of every degree $\mathbf{a} \geq \mathbf{0}'$.*

We will later show that the results **(ii)** and **(iii)** follow from Theorem 72. Notice that Theorem 68 implies that every computable torsion-free abelian group of finite rank has a computable order since the group has a finite basis. However, this is not the case if the rank is infinite since we need an oracle $\emptyset'$ to find a basis of G.

The following theorem shows that the space of orders $BiO(G)$ for a computable torsion-free abelian group G is not sufficient to represent all c.b. Π^0_1 classes, not even in a weak sense.

Theorem 69 (*R. Solomon [70]*) *There is a c.b.* Π^0_1 *class* C *such that for any computable torsion-free abelian group* G *we have that*

$$\{\deg(\alpha)|\ \alpha \in C\} \neq \{\deg(P)|\ P \in BiO(G)\}.$$

To prove this theorem, let C be an infinite c.b. Π^0_1 class as in Theorem 64, and let G be any computable torsion-free abelian group. By Theorem 68, G has only computable orders, orders of every degree, or orders of every degree above $\mathbf{0}'$. Hence the set of degrees of elements of $BiO(G)$ cannot be equal to the set of degrees of elements of C in any of the cases.

Actually, it is shown in the next theorem that even more is true. First, recall that if $A, B \subset \omega$ and $A \cap B = \emptyset$, then S is a *separating set* for A and B if

$$\text{either } A \subseteq S \wedge S \cap B = \emptyset \text{ or } B \subseteq S \wedge S \cap A = \emptyset.$$

Then, the subset $C \subseteq \{0,1\}^\omega$ is a Π^0_1 *class of separating sets* if there are computably enumerable sets A and B such that C is the class of characteristic functions for the separating sets of A and B.

Theorem 70 (*R. Solomon [70]*) *There is a* Π^0_1 *class of separating sets* C *such that for any computable torsion-free abelian group* G

$$\{\deg(\alpha)|\ \alpha \in C\} \neq \{\deg(P)|\ P \in BiO(G)\}.$$

In 1986, Downey and Kurtz [17] showed that there exists a computable, orderable abelian group G with no computable order. They constructed a computable group isomorphic to $\mathbb{Z}^\omega$, thus isomorphic to a computable group with a computable order. Therefore, Downey and Kurtz asked whether every computable, orderable abelian group is isomorphic to a computable group with a computable order (the isomorphism does not have to be order-preserving). The question was answered positively by Solomon [70] who used a result by Dobritsa [15] that every computable torsion-free abelian group is isomorphic to a computable group with a computable basis. The questions concerning computability of orders can naturally be asked for the class of computable and orderable non-abelian groups.

We will now introduce the following notion of a Turing degree spectrum of left orders on a computable group G.

Definition 71 *The* Turing degree spectrum of left orders *on a computable group G, in symbols $DgSp_G(LO)$, is the set of Turing degrees of all possible left orders on G, that is,*

$$DgSp_G(LO) = \{\deg(P^+) |\ P^+ \in LO(G)\}.$$

Similarly, we define $DgSp_G(RO)$ for right orders and $DgSp_G(BiO)$ for bi-orders on G. Since G is computable, without loss of generality, we may assume that its domain is a subset of ω. A finite set of natural numbers can be coded (indexed) by its *canonical index* in the following way. Let

$$D_0 =_{def} \emptyset.$$

For $m > 0$, let

$$D_m = \{d_0, \ldots, d_{k-1}\},$$

where $d_0 < \ldots < d_{k-1}$ and $m = 2^{d_0} + \ldots + 2^{d_{k-1}}$. A sequence $\{p_i\}_{i \in \omega}$ of finite sets is called a *strong array* if there is a unary computable function ν such that for every $i \in \omega$ we have

$$p_i = D_{\nu(i)}.$$

The following result gives a general criterion, expressed in terms of finite sets of generators for partial left orders, for $DgSp_G(LO)$ to include all Turing degrees. The set of all Turing degrees is denoted by $\mathcal{D}$.

Theorem 72 *Let G be a computable group. Assume that there is a strong array $\{p_i\}_{i \in \omega}$ for which $\mathbb{P} = \{p_i\}_{i \in \omega}$ is a family of finite subsets of $G \backslash \{e\}$ such that each partial left order $P_i = sgr(p_i) \cup \{e\}$ on G could be extended to a total left order on G, and for all elements $p \in \mathbb{P}$, the following two conditions are satisfied:*

(i) $(\exists a \in G \backslash \{e\})\ (\exists q, r \in \mathbb{P})\ [(q \supseteq p) \wedge (r \supseteq p) \wedge (a \in q) \wedge (a^{-1} \in r)]$*;*

(ii) $(\forall a \in G \backslash \{e\})\ (\exists q \in \mathbb{P})\ [(q \supseteq p) \wedge ((a \in q) \vee (a^{-1} \in q))]$.

Then

$$DgSp_G(LO) = \mathcal{D}.$$

Theorem 72 is a special case of the following more general result.

Theorem 73 *Let G be a computable group. Let $D \subseteq \omega$ be a set of Turing degree $\mathbf{d}$. Assume that there is a D-computable enumeration $\{p_i\}_{i\in\omega}$ for which $\mathbb{P} = \{p_i\}_{i\in\omega}$ is a family of finite subsets of $G\backslash\{e\}$ such that each partial left order $P_i = sgr(p_i) \cup \{e\}$ on G can be extended to a total left order on G, and for all elements p of $\mathbb{P}$ we have:*

(i) $(\exists a \in G\backslash\{e\})\ (\exists q, r \in \mathbb{P})\ [(q \supseteq p) \wedge (r \supseteq p) \wedge (a \in q) \wedge (a^{-1} \in r)]$;

(ii) $(\forall a \in G\backslash\{e\})\ (\exists q \in \mathbb{P})\ [(q \supseteq p) \wedge ((a \in q) \vee (a^{-1} \in q))]$.

Then for every Turing degree $\mathbf{x} \geq \mathbf{d}$, there exists $\mathbf{z} \in DgSp_G(LO)$ such that $\mathbf{x} = \mathbf{z} \vee \mathbf{d}$.

Proof. Let $\mathbf{x} \in \mathcal{D}$, $\mathbf{x} \geq \mathbf{d}$, and let X be a set of natural numbers such that $deg(X) = \mathbf{x}$. We will use an extension argument to construct a subsequence $(p_{i_s})_{s\in\omega}$ of finite sets of generators for partial left orders $P_{i_s} = sgr(p_{i_s}) \cup \{e\}$ on G such that the union $\cup_{s\in\omega} P_{i_s}$ defines a left order on G as a positive cone and $\deg(\cup_{s\in\omega} P_{i_s}) = \mathbf{x}$.

Construction

Stage $s = 0$. Set $p_{i_0} = \emptyset$.

Stage $s + 1 = 2k + 1$. At the end of stage s we have p_{i_s} (a finite set of generators for a partial order $P_{i_s} = sgr(p_{i_s}) \cup \{e\}$). Using the oracle D, find the first pair of elements $r_1, r_2 \in \mathbb{P}$ such that

$$r_1 \supseteq p_{i_s} \text{ and } r_2 \supseteq p_{i_s},$$

hence

$$sgr(r_1) \supseteq sgr(p_{i_s}) \text{ and } sgr(r_2) \supseteq sgr(p_{i_s}).$$

Then for some $a \in G\backslash\{e\}$, we have

$$a \in r_1 \text{ and } a^{-1} \in r_2,$$

hence

$$sgr(r_1) \neq sgr(r_2).$$

Choose the least such a. Define $p_{i_{s+1}}$, using oracle X, by

$$p_{i_{s+1}} = \begin{cases} r_1 & \text{if } k \in X, \\ r_2 & \text{if } k \notin X. \end{cases}$$

Stage $s + 1 = 2k + 2$. Let a be the least element in G such that

$$a \notin p_{i_s} \text{ and } a^{-1} \notin p_{i_s}.$$

Define $p_{i_{s+1}}$ to be the first element q in $\mathbb{P}$ such that

$$q \supseteq p_{i_s}$$

and

$$a \in q \vee a^{-1} \in q.$$

End of the construction.

Let

$$f =_{def} \cup_{s\in\omega} P_{i_s}.$$

Clearly, f is the positive cone of a left order on G. Let $\mathbf{z} = \deg(f)$.

Lemma 74 $f \leq_T X$

Proof. Since the construction is computable in D and X, and $D \leq_T X$, the sequence $(p_{i_s})_{s\in\omega}$ is X-computable. Let $a \in G\backslash\{e\}$. To decide whether $a \in f$ or $a^{-1} \in f$, find the least s such that $a \in p_{i_s}$ or $a^{-1} \in p_{i_s}$. ■

Lemma 75 $X \leq_T f \oplus D$

Proof. We prove inductively that $(p_{i_s})_{s\in\omega}$ is an $f \oplus D$-computable sequence, and that $X \leq_T f \oplus D$. Given p_{i_s}, $s = 2k$, D-computably find the corresponding r_1, r_2, and then the corresponding a. Since

$$a \in p_{i_{s+1}} \text{ if } k \in X,$$

and

$$a^{-1} \in p_{i_{s+1}} \text{ if } k \notin X,$$

we can use an $f \oplus D$-oracle to determine whether $k \in X$. Since

$$p_{i_{s+1}} = r_1 \text{ if } k \in X,$$

and

$$p_{i_{s+1}} = r_2 \text{ if } k \notin X,$$

we can also use an $f \oplus D$-oracle to determine $p_{i_{s+1}}$. Now, given $p_{i_{s+1}}$ we can find $p_{i_{s+2}}$ computably in D. ■

Since $X \leq_T f \oplus D$, $f \leq_T X$, and $D \leq_T X$, we have $X \equiv_T f \oplus D$, i.e., $\mathbf{x} = \mathbf{z} \vee \mathbf{d}$. ■

Recall that a set S of Turing degrees is *closed upward* if for all $\mathbf{x}, \mathbf{z} \in \mathcal{D}$ we have:

$$(\mathbf{z} \in S \wedge \mathbf{x} \geq \mathbf{z}) \Rightarrow \mathbf{x} \in S.$$

Corollary 76 *Let the conditions of Theorem 73 be satisfied for a computable group G and a set D of Turing degree $\mathbf{d}$. If, in addition, the set $DgSp_G(LO)$ is closed upward, then*

$$\{\mathbf{x} \in \mathcal{D} \mid \mathbf{x} \geq \mathbf{d}\} \subseteq DgSp_G(LO).$$

Note that we do not require that for all $p, q \in \mathbb{P}$ such that $p \neq q$ we have $sgr(p) \neq sgr(q)$. Let us also emphasize that the family $\mathbb{P}$ in Theorem 73 does not have to include all possible finite subsets of G such that the partial left orders on G defined by them can be extended to total left orders on G. In particular, there is no need to assume that G is fully left-orderable. This motivates the following definition.

Definition 77 *A family $\mathbb{P} = \{p_s\}_{s \in \omega}$ of finite subsets of $G \backslash \{e\}$ is called* complete *if $\mathbb{P}$ consists of all finite subsets $p \subset G \backslash \{e\}$ such that each partial left order $P = sgr(p) \cup \{e\}$ can be extended to a total left order P^+ on G.*

5.2 Topology on the space of left orders

In this section, we give a simple topological description of the space of left orders $LO(G)$ on a countable group G. We provide new proofs of the results obtained in [66] in terms of positive cones so that the proofs fit better into our framework. Finally, we provide a new criterion for the space of left orders on G to be homeomorphic to the Cantor set. In the next chapter, we will use this result to prove the existence of the homeomorphism between the space of left orders on $\mathbb{Z}^\omega$ and the Cantor set.

Recall that we can identify the elements of $LO(G)$ with total, pure sub-semigroups of G. That is,

$$LO(G) = \{P_+ \subseteq G |\ P_+ \text{ is a total and pure sub-semigroup of } G\}.$$

Following the work of A. S. Sikora [66] let us define the topology on $LO(G)$ as follows. For each $a \in G$, define

$$S_a = \{P \in LO(G) |\ a \in P\}.$$

Let

$$\mathcal{S} = \{S_a\}_{a \in G}.$$

Obviously, the family $\mathcal{S}$ covers $LO(G)$, i.e.,

$$\bigcup_{a \in G} S_a = LO(G).$$

Thus, the collection $\mathcal{S}$ is a subbasis for a topology on $LO(G)$. Let $\tau_\mathcal{S}$ be the topology generated by $\mathcal{S}$.

Theorem 78 *(A. Sikora [66]) If G is a countable group, then the topological space $(LO(G), \tau_\mathcal{S})$ is metrizable.*

Proof. We define a metric ρ on $LO(G)$ such that the topology induced by ρ on $LO(G)$ is the same as the topology $\tau_\mathcal{S}$ defined above. Let $\{G_i\}_{i \in \omega}$ be a family of finite subsets of G which satisfies the following properties:

(i) $G_0 \subset G_1 \subset ... \subset G_n \subset ... \subset G$;

(ii) $\bigcup_{i\in\omega} G_i = G$.

Since G is a countable group, such a collection always exists, and any collection of finite subsets of G that satisfies conditions **(i)** and **(ii)** is called a *complete filtration.* Let $\rho : LO(G) \times LO(G) \longrightarrow \mathbb{R}_+$ be a function defined as follows:

$$\rho(P^+, Q^+) = \begin{cases} \frac{1}{2^r} & \text{if } r = \max\{i \in \omega |\ G_i \cap P^+ = G_i \cap Q^+,\ G_i \cap P^- = G_i \cap Q^-\} < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

We show now that ρ is a metric on $LO(G)$. From the definition of ρ, it follows that ρ is a non-negative and symmetric function. If $\rho(P^+, Q^+) = 0$, then

$$\max\{i \in \omega |\ G_i \cap P^+ = G_i \cap Q^+, G_i \cap P^- = G_i \cap Q^-\} = \infty,$$

so for all $i \in \omega$ we have that

$$(P^+ \cap G_i = Q^+ \cap G_i) \wedge (P^- \cap G_i = Q^- \cap G_i),$$

hence

$$(\bigcup_{i\in\omega} G_i) \cap P^+ = (\bigcup_{i\in\omega} G_i) \cap Q^+.$$

Since the filtration $\{G_i\}_{i\in\omega}$ is complete,

$$P^+ \cap G = Q^+ \cap G,$$

but since $P^+, Q^+ \subseteq G$, we have

$$P^+ = G \cap P^+ = G \cap Q^+ = Q^+.$$

Obviously, if $P_+ = Q_+$, then for all $i \in \omega$:

$$(P^+ \cap G_i = Q^+ \cap G_i) \wedge (P^- \cap G_i = Q^- \cap G_i).$$

Hence

$$r = \max\{i \in \omega |\ G_i \cap P^+ = G_i \cap Q^+, G_i \cap P^- = G_i \cap Q^-\} > n$$

for all $n \in \omega$. Thus $r = \infty$, and so

$$\rho(P^+, Q^+) = 0.$$

We now prove that ρ satisfies the triangle inequality. Let $P^+, Q^+, R^+ \in LO(G)$ and assume that

$$\rho(P^+, R^+) = \frac{1}{2^n},\ \rho(R^+, Q^+) = \frac{1}{2^m},\ \text{and}\ \rho(P^+, Q^+) = \frac{1}{2^r}.$$

We need to show that

$$\min\{n, m\} \leq r,\ \ \min\{n, r\} \leq m,\ \text{and}\ \min\{m, r\} \leq n.$$

Let us consider the first inequality. Suppose that $r < \min\{n, m\}$. Then, without loss of generality, we may assume that $r < n \leq m$. Hence

$$\begin{aligned} (G_n \cap P^+ &= G_n \cap R^+) \wedge (G_n \cap P^- = G_n \cap R^-) \\ &\text{and} \\ (G_m \cap R^+ &= G_m \cap Q^+) \wedge (G_m \cap R^- = G_m \cap Q^-). \end{aligned}$$

Since $n \leq m$, then $G_n \subseteq G_m$, so we also have

$$(G_n \cap R^+ = G_n \cap Q^+) \wedge (G_n \cap R^- = G_n \cap Q^-).$$

Therefore,

$$(G_n \cap P^+ = G_n \cap Q^+) \wedge (G_n \cap P^- = G_n \cap Q^-),$$

hence

$$\max\{i \in \omega |\ (G_i \cap P^+ = G_i \cap Q^+) \wedge (G_i \cap P^- = G_i \cap Q^-)\} \geq n,$$

so $\rho(P^+, Q^+) \leq \frac{1}{2^n}$, but $\rho(P^+, Q^+) = \frac{1}{2^r}$, thus $\frac{1}{2^r} \leq \frac{1}{2^n}$, so $r \geq n$, a contradiction. Thus,

$$\rho(P^+, Q^+) = \frac{1}{2^r} \leq \frac{1}{2^{\min\{n,m\}}} \leq \frac{1}{2^n} + \frac{1}{2^m} = \rho(P^+, R^+) + \rho(R^+, Q^+).$$

The case when at least one of the numbers $\rho(P^+, R^+)$, $\rho(R^+, Q^+)$, $\rho(P^+, Q^+)$ equals 0 is obvious, so we have proved that ρ is a metric on $LO(G)$.

Now, let $\tau_{(LO(G),\rho)}$ be the topology induced by ρ on $LO(G)$. We need to show that

$$\tau_{(LO(G),\rho)} = \tau_{\mathcal{S}}.$$

Since $\tau_{(LO(G),\rho)}$ is generated by the following basis collection

$$\mathcal{B} = \{B_\rho(P^+, \varepsilon)| \ P^+ \in LO(G), \varepsilon > 0\},$$

it is sufficient to show that $B_\rho(P^+, \varepsilon) \in \tau_{\mathcal{S}}$ for any $P^+ \in LO(G), \varepsilon > 0$. Notice that $B_\rho(P^+, \varepsilon) \in \tau_{\mathcal{S}}$ if and only if for every positive cone $Q^+ \in B_\rho(P^+, \varepsilon)$ there exists $U \in \tau_{\mathcal{S}}$ with $Q^+ \in U$ and such that $U \subseteq B_\rho(P^+, \varepsilon)$. Let $Q^+ \in B_\rho(P^+, \varepsilon)$, then $\rho(P^+, Q^+) < \varepsilon$. Let

$$r = \min\{i \in \omega| \ \frac{1}{2^i} < \varepsilon\}.$$

Then

$$(P^+ \cap G_r = Q^+ \cap G_r) \wedge (P^- \cap G_r = Q^- \cap G_r).$$

Let

$$G_r(P^+) = G_r \cap P^+ \cup \{a \in G| \ a^{-1} \in G_r \cap P^-\}.$$

Notice that $G_r(P^+) \neq \emptyset$. Let

$$U = \bigcap_{a \in G_r(P^+)} S_a,$$

so $U \in \tau_{\mathcal{S}}$ since it is a finite intersection of generators of $\tau_{\mathcal{S}}$. We show that $Q^+ \in U$ and $U \subseteq B_\rho(P^+, \varepsilon)$. Notice that

$$G_r(P^+) \subseteq P^+,$$

since $G_r \cap P^+ \subseteq P^+$and $\{a \in G| \ a^{-1} \in G_r \cap P^-\} \subseteq P^+$. To see the latter, notice that $G_r \cap P^- \subseteq P^-$ and $G = P^+ \cup P^-$. Moreover, since

$$(P^+ \cap G_r = Q^+ \cap G_r) \wedge (P^- \cap G_r = Q^- \cap G_r),$$

then

$$G_r(P^+) = G_r(Q^+).$$

And since for every $a \in G_r(P^+)$ we have $Q^+ \in S_a$, then

$$Q^+ \in \bigcap_{a \in G_r(Q^+)} S_a = \bigcap_{a \in G_r(P^+)} S_a = U.$$

To see that $U \subseteq B_\rho(P^+, \varepsilon)$, let $R^+ \in U$, and since

$$U = \bigcap_{a \in G_r(P^+)} S_a,$$

so for all $a \in G_r(P^+)$ we have $R^+ \in S_a$. Hence for all $a \in G_r(P^+)$ we have $a \in R^+$, and thus

$$G_r(P^+) \subseteq R^+.$$

Moreover, since

$$G_r(P^+) = G_r \cap P^+ \cup \{a \in G |\ a^{-1} \in G_r \cap P^-\},$$

it follows that

$$G_r \cap P^+ = G_r \cap R^+ \text{ and } G_r \cap P^- = G_r \cap R^-.$$

Then, by the definition of r, we have

$$\max\{i \in \omega |\ (G_i \cap P^+ = G_i \cap R^+) \wedge (G_i \cap P^- = G_i \cap R^-)\} \geq r,$$

so

$$R^+ \in B_\rho(P^+, \varepsilon).$$

Therefore,

$$\tau_{(LO(G),\rho)} \subseteq \tau_{\mathcal{S}}.$$

We will now show that $\tau_{\mathcal{S}} \subseteq \tau_{(LO(G),\rho)}$. Since $\tau_{\mathcal{S}}$ is generated by $\mathcal{S}$ as its subbasis, it is enough to show that every generator $S \in \mathcal{S}$ is an element of $\tau_{(LO(G),\rho)}$. Let $S \in \mathcal{S}$. Then $S = S_a$ for some $a \in G$. Let $P^+ \in LO(G)$ be such that $P^+ \in S_a$, so $a \in P^+$. We will show that for any $\varepsilon > 0$ we have

$$B_\rho(P^+, \varepsilon) \subseteq S_a.$$

Since $\bigcup_{i \in \omega} G_i = G$, there is $r \in \omega$ such that $a \in G_r$. Let $\varepsilon = \frac{1}{2^r}$. We argue that $B_\rho(P^+, \varepsilon) \subseteq S_a$. Let $Q^+ \in B_\rho(P^+, \varepsilon)$, then $\rho(P^+, Q^+) < \frac{1}{2^r}$. Hence

$$\max\{i \in \omega |\ (G_i \cap P^+ = G_i \cap Q^+) \wedge (G_i \cap P^- = G_i \cap Q^-)\} > r,$$

so, in particular,

$$G_r \cap P^+ = G_r \cap Q^+,$$

thus $a \in Q^+$. Therefore, we have $Q^+ \in S_a$, which completes the proof. ■

Remark 79 *The metric ρ defined on $LO(G)$ does not depend on the choice of a complete filtration since we have shown that $\tau_{(LO(G),\rho)} = \tau_{\mathcal{S}}$ for an arbitrary complete filtration.*

Recall that a topological space is *totally disconnected* if and only if two distinct points of the space are contained in two disjoint open sets covering the space. In the case of a countable group G, the following theorem characterizes the topological space $(LO(G), \tau_{\mathcal{S}})$.

Theorem 80 *(A. Sikora [66]) $(LO(G), \tau_{\mathcal{S}})$ is a compact, totally disconnected topological space.*

Proof. For any $a \in G \backslash \{e\}$, one has

$$LO(G) = S_a \cup S_{a^{-1}}$$

and

$$S_a \cap S_{a^{-1}} = \emptyset,$$

since if $P^+ \in LO(G)$, then

$$\text{either } a \in P^+ \text{ or } a^{-1} \in P^+,$$

but it cannot happen that both $a \in P^+$ and $a^{-1} \in P^+$. Therefore,

$$\text{either } P^+ \in S_a \text{ or } P^+ \in S_{a^{-1}},$$

and no $P^+ \in LO(G)$ is an element of the set $S_a \cap S_{a^{-1}}$.

Let $P^+, Q^+ \in LO(G)$ and assume that $P^+ \neq Q^+$. Then there exists an element $a \in G \backslash \{e\}$ such that

$$a \in P^+ \wedge a \notin Q^+.$$

Since

$$G = Q^+ \cup Q^-,$$

$a \in Q^-$, so $a^{-1} \in Q^+$. Thus, we have

$$a \in P^+ \text{ and } a^{-1} \in Q^+,$$

and hence

$$P^+ \in S_a \text{ or } Q^+ \in S_{a^{-1}}.$$

Since $S_a, S_{a^{-1}} \in \tau_{\mathcal{S}}$ and $S_a \cup S_{a^{-1}} = LO(G)$, and $S_a \cap S_{a^{-1}} = \emptyset$, the topological space $(LO(G), \tau_{\mathcal{S}})$ is totally disconnected.

We will show that $(LO(G), \tau_S)$ is compact. Since $LO(G)$ with topology $\tau_{\mathcal{S}}$ is metrizable, it suffices to show that $LO(G)$ is *sequentially compact*, i.e., that every sequence in $LO(G)$ has a convergent subsequence. Let $G_0 \subset G_1 \subset ...$ be a complete filtration in G by its finite subsets and let ρ the metric as in the proof of Theorem 78. Consider the metric space $(LO(G), \rho)$ and let $P_0^+, P_1^+, ...$ be a sequence of points in $LO(G)$. Since G_0 is finite, there are infinitely many terms of the sequence $\{P_k^+\}_{k=0}^{\infty}$ with the following property

$$P^+_{k_0^0} \cap G_0 = P^+_{k_1^0} \cap G_0 = ...$$

and

$$P^-_{k_0^0} \cap G_0 = P^-_{k_1^0} \cap G_0 = ...$$

Otherwise, all but finitely many terms of the sequence $\{P_k\}_{k=0}^{\infty}$ satisfy the following property

$$G_0 \cap P_j^+ \neq G_0 \cap P_l^+ \text{ or } G_0 \cap P_j^- \neq G_0 \cap P_l^-$$

for $j \neq l$, i.e.,

$$(\exists N_0 \in \omega)(\forall j, l \geqslant N_0)\ [(l \neq j) \Rightarrow (G_0 \cap P_j^+ \neq G_0 \cap P_l^+) \vee (G_0 \cap P_j^- \neq G_0 \cap P_l^-)],$$

which is impossible since G_0 is finite. Now, one chooses an infinite subsequence $\{P^+_{k_j^1}\}_{j=0}^{\infty}$ of the sequence $\{P^+_{k_j^0}\}_{j=0}^{\infty}$ with the property

$$P^+_{k_0^1} \cap G_1 = P^+_{k_1^1} \cap G_1 = ...$$

and

$$P^-_{k_0^1} \cap G_1 = P^-_{k_1^1} \cap G_1 = ...$$

Again, since G_1 is finite, such a subsequence exists. Proceeding in the same way for each term of the sequence $\{G_i\}_{i=0}^{\infty}$, one obtains a family of sequences

$$\mathcal{F} = \{\{P^+_{k_j^l}\}_{j=0}^{\infty} |\ l = 0, 1, 2, ...\}.$$

Define a new subsequence $\{Q_i^+\}_{i=0}^{\infty}$ of the sequence $\{P_k^+\}_{k=0}^{\infty}$ as follows

$$Q_i^+ := P_{k_i^i}^+,$$

where $i = 0, 1, 2, ...$

Claim 81 *$\{Q_i^+\}_{i=0}^{\infty}$ is convergent.*

Define Q^+ as follows:

$$a \in Q^+ \Leftrightarrow (\exists N_0 \in \omega)(\forall n \in \omega)\ [n \geqslant N_0 \Rightarrow a \in Q_n^+].$$

First, we show that $Q^+ \in LO(G)$. Let $a, b \in Q^+$, then $ab \in Q^+$, since by the definition of Q^+ we have

$$a \in Q^+ \Leftrightarrow (\exists N_0 \in \omega)(\forall n \in \omega)\ [n \geqslant N_0 \Rightarrow a \in Q_n^+],$$

and, analogously,

$$b \in Q^+ \Leftrightarrow (\exists K_0 \in \omega)(\forall n \in \omega)\ [n \geqslant K_0 \Rightarrow b \in Q_n^+].$$

Define

$$M_0 = \max\{N_0, K_0\}.$$

Since Q_i^+'s are positive cones, then

$$(\forall n \geqslant M_0)\ [ab \in Q_n^+].$$

Therefore, we have

$$Q^+Q^+ \subseteq Q^+,$$

so Q^+ is a semigroup. We show that for every $a \in G \backslash \{e\}$, we have either

$$a \in Q^+ \text{ or } a^{-1} \in Q^+,$$

and there is no element $a \in G \backslash \{e\}$ such that both $a \in Q^+$ and $a^{-1} \in Q^+$. If there is $a \in G \backslash \{e\}$ such that $a \in Q^+$ and $a^{-1} \in Q^+$, then there is $N_0 \in \omega$ such that

$$(\forall n \geq N_0)\ [a, a^{-1} \in Q_n^+],$$

but Q_n^+ is a positive cone, so $a = a^{-1} = e$, a contradiction. We will now show that Q^+ is total. Let $a \in G \backslash \{e\}$. If

$$(\exists N_0 \in \omega)(\forall n \in \omega)\ [n \geqslant N_0 \Rightarrow a \in Q_n^+],$$

then $a \in Q^+$. If

$$(\forall N_0 \in \omega)(\exists n \in \omega)\ [n \geqslant N_0 \wedge a \notin Q_n^+],$$

then let $N_0 = 0$, hence

$$(\exists N_1 \geq 0)\ [a \notin Q_{N_1}^+].$$

Since $a \in G$, there is $r \in \omega$ such that $a \in G_r$, so $a \notin Q_{N_1}^+ \cap G_r$ and $a \in Q_{N_1}^- \cap G_r$. Since

$$(\forall k \geq N_1)\ [(Q_k^+ \cap G_r = Q_{N_1}^+ \cap G_r) \wedge (Q_k^- \cap G_r = Q_{N_1}^- \cap G_r)]$$

by the construction of $\{Q_i^+\}_{i=1}^{\infty}$, for every $k \geq N_1$ we have $a \notin Q_k^+$. Hence we showed that

$$(\exists K_0 = N_1)(\forall n \in \omega)\ [n \geqslant K_0 \Rightarrow a \notin Q_n^+],$$

so

$$(\exists K_0 = N_1)(\forall n \in \omega)\ [n \geqslant K_0 \Rightarrow a^{-1} \in Q_n^+],$$

hence $a^{-1} \in Q^+$. Therefore, Q^+ is a well-defined positive cone. Now, we show that $\{Q_i^+\}_{i=0}^{\infty}$ converges to Q^+. Let $\varepsilon > 0$ be given. We show that there is $K_0 \in \omega$ such that for all $n \geq K_0$ one has $\rho(Q^+, Q_n^+) < \varepsilon$. Define

$$r = \min\{i \in \omega |\ \frac{1}{2^i} < \varepsilon\}$$

and let $K_0 = r$.

Claim 82 *For all $n \geq K_0$ we have $\rho(Q^+, Q_n^+) < \varepsilon$.*

By the definition of $\{Q_i^+\}_{i=0}^{\infty}$, we have that for all $i \geq r$:

$$(Q_i^+ \cap G_r = Q_r^+ \cap G_r) \wedge (Q_i^- \cap G_r = Q_r^- \cap G_r).$$

Then

$$(\forall i \geq r)(\forall a \in G_r(Q_r^+))\ [a \in Q_i^+],$$

where $G_r(Q_r^+)$ defined as in the proof of Theorem 78. Thus $G_r(Q_r^+) \subseteq Q^+$, since all elements of $G_r(Q_r^+)$ belong to all but a finite number of terms of $\{Q_i^+\}_{i=0}^{\infty}$. This implies that for $i \geq r$ we have

$$(Q_i^+ \cap G_r = Q^+ \cap G_r) \wedge (Q_i^- \cap G_r = Q^- \cap G_r),$$

which, in turn, implies that

$$(\forall i \geq r)\ [\max\{j \in \omega |\ (Q_i^+ \cap G_j = Q^+ \cap G_j) \wedge (Q_i^- \cap G_j = Q^- \cap G_j)\} \geq r],$$

so $\rho(Q^+, Q_i^+) \leq \frac{1}{2^r} < \varepsilon$. ■

Recall that any nonempty, metrizable, compact, perfect, and totally disconnected set is the Cantor set. A set is *perfect* if every point in the set is a limit point. It was shown in the above theorems that $LO(G)$ is metrizable, compact and totally disconnected for any countable group G. To show that $LO(G)$ is homeomorphic to the Cantor set, we need to show that it is perfect, which is equivalent to the condition in the corollary below.

Corollary 83 (*A. Sikora [66]*) *Let G be a countable group. Then $LO(G)$ is homeomorphic to the Cantor set if and only if $LO(G) \neq \emptyset$ and for any sequence $a_1, a_2, ..., a_n \in G$, the set $S_{a_1} \cap S_{a_2} \cap ... \cap S_{a_n}$ is either empty or infinite.*

Since $BiO(G)$ is a subset of $LO(G)$, $BiO(G)$ naturally inherits the subspace topology induced from $LO(G)$.

Proposition 84 (*A. Sikora [66]*) *$BiO(G)$ is a closed subset of $LO(G)$. Hence $BiO(G)$ is homeomorphic to the Cantor set if and only if $BiO(G) \neq \emptyset$ and for any sequence a_1, a_2, ..., $a_n \in G$, the set*

$$BiO(G) \cap S_{a_1} \cap S_{a_2} \cap ... \cap S_{a_n}$$

is either empty or infinite.

It is not true that if $BiO(G)$ is infinite, then $BiO(G)$ is homeomorphic to the Cantor set. The example of a family of groups with a countable infinity of orders can be found in [47]. We will use Corollary 83 to prove a new criterion for the space of left orders of a countable group G to be homeomorphic to the Cantor set. Recall (from the previous section) that a family $\mathbb{P} = \{p_s\}_{s\in\omega}$ of finite subsets of $G\backslash\{e\}$ is *complete* if $\mathbb{P}$ consists of all finite subsets $p \subset G\backslash\{e\}$ such that each partial left order $P = sgr(p) \cup \{e\}$ can be extended to a total left order P^+ on G.

Theorem 85 *Let G be a countable group. Then there exists a complete family $\mathbb{P} = \{p_s\}_{s\in\omega}$ satisfying conditions* (**i**) *and* (**ii**) *of Theorem 72, that is, for all elements p of $\mathbb{P}$:*

(i) $(\exists a \in G\backslash\{e\})\ (\exists q, r \in \mathbb{P})\ [(q \supseteq p) \wedge (r \supseteq p) \wedge (a \in q) \wedge (a^{-1} \in r)]$ *and*

(ii) $(\forall a \in G\backslash\{e\})\ (\exists q \in \mathbb{P})\ [(q \supseteq p) \wedge ((a \in q) \vee (a^{-1} \in q))]$

if and only if $LO(G)$ is homeomorphic to the Cantor set C.

Note that the family $\mathbb{P} = \{p_s\}_{s\in\omega}$ in the above result does not have to be a strong array as in Theorem 72.

Proof. Suppose we have a complete family $\mathbb{P} = \{p_s\}_{s\in\omega}$ of finite sets that satisfies conditions (**i**) and (**ii**) of Theorem 72. To show that the space $(LO(G), \tau_{\mathcal{S}})$ is homeomorphic to the Cantor set, we need to show that for all finite subsets $\{g_1, g_2, ..., g_k\} \subset G, k \in \omega$, the set $\bigcap_{j=1}^{k} S_{g_j}$ is either empty or infinite (Corollary 83).

Suppose that $\bigcap_{j=1}^{k} S_{g_j} \neq \emptyset$. Then there is a left order $P^+ \in LO(G)$ such that

$$\{g_1, g_2, ..., g_k\} \subseteq P^+,$$

hence

$$sgr(\{g_1, g_2, ..., g_k\}) \subseteq P^+.$$

Since $\mathbb{P} = \{p_s\}_{s\in\omega}$ is complete,

$$p = \{g_1, g_2, ..., g_k\} \in \mathbb{P}.$$

Using property (**i**) of the family $\mathbb{P}$, we can find $a \in G\backslash\{e\}$ and finite subsets $q, r \in \mathbb{P}$, such that $p \subset q$ and $p \subset r = p$ and $a \in p$ and $a^{-1} \in r$. Thus, let

$$q = p \cup \{a\} \text{ and } r = p \cup \{a^{-1}\}.$$

Notice that any left order Q^+ on G extending the partial left order $sgr(q) \cup \{e\}$ cannot extend $sgr(r) \cup \{e\}$, and, analogously, any left order R^+ extending the partial left order $sgr(r) \cup \{e\}$ cannot extend $sgr(q) \cup \{e\}$. The existence of extensions $q, r \in \mathbb{P}$ for any $a \in G \backslash \{e\}$ is guaranteed by property (**ii**) of the family $\mathbb{P}$. Moreover, for all $P^+ \in LO(G)$:

$$(sgr(p) \subseteq P^+) \Rightarrow (P^+ \in \bigcap_{j=1}^{k} S_{g_j}).$$

Therefore, for any finite set $p \subset G \backslash \{e\}$ such that $\bigcap_{g \in p} S_g \neq \emptyset$, one has that $\bigcap_{g \in p} S_g$ is infinite.

Suppose now that $LO(G)$ is homeomorphic to the Cantor set. Let $\mathcal{A} = \{p_j\}_{j \in \omega}$ be the family of all finite subsets of G. We define the family $\mathbb{P}$ as follows:

$$(\forall p \in \mathcal{A})[p \in \mathbb{P} \Leftrightarrow \bigcap_{g \in p} S_g \neq \emptyset].$$

We will now show that $\mathbb{P}$ satisfies condition (**i**) of Theorem 72. For $p \in \mathbb{P}$ we have $\bigcap_{g \in p} S_g \neq \emptyset$, so since $LO(G)$ is homeomorphic to the Cantor set, $\bigcap_{g \in p} S_g$ contains infinitely many elements $P^+ \in LO(G)$ such that $sgr(p) \subseteq P^+$.

Let $Q^+, R^+ \in \bigcap_{g \in p} S_g$ and $Q^+ \neq R^+$. Then

$$sgr(p) \subseteq Q^+ \text{ and } sgr(p) \subseteq R^+,$$

and there is an $a \in G \backslash \{e\}$ such that

$$a \in Q^+ \text{ and } a^{-1} \in R^+.$$

Notice that $a \notin p$. Otherwise, $a \in R^+$ since $R^+ \in \bigcap_{g \in p} S_g$, so $a, a^{-1} \in R^+$. Since R^+ is a pure sub-semigroup, we have $a = e$, which contradicts our choice of $a \in G \backslash \{e\}$.

Now, define the subsets

$$q = p \cup \{a\} \text{ and } r = p \cup \{a^{-1}\}.$$

Obviously, $p \subset q$ and $p \subset r$. Moreover, since $sgr(q) \subseteq Q^+$ and $sgr(r) \subseteq R^+$,

$$\bigcap_{g \in q} S_g \neq \emptyset \text{ and} \bigcap_{g \in r} S_g \neq \emptyset,$$

hence $q, r \in \mathbb{P}$. This completes the proof of condition (**i**).

Condition (**ii**) for the family $\mathbb{P}$ follows from the fact that if $\bigcap_{g \in p} S_g \neq \emptyset$, then $sgr(p) \subseteq P^+$ for some $P^+ \in \bigcap_{g \in p} S_g$, thus for any $a \in G \backslash \{e\}$ one has

$$a \in P^+ \text{ or } a^{-1} \in P^+,$$

as $P^+ \cup P^- = G$. Hence we define

$$q = p \cup \{a\} \text{ or } q = p \cup \{a^{-1}\},$$

depending on whether $a \in P^+$ or $a^{-1} \in P^+$. Obviously, in each case $p \subset q$, and since $sgr(q) \subset P^+$, we have that $q \in \mathbb{P}$, which completes the proof of condition (**ii**).

The completeness of the family $\mathbb{P}$ follows directly from its definition. ■

Notice that, as we mentioned before, the statement in Theorem 85 about $LO(G)$ being homeomorphic to the Cantor set does not imply that $DgSp_G(LO) = \mathcal{D}$ (as in the statement of Theorem 72) since the family $\mathbb{P}$ constructed in the proof of Theorem 85 does not necessarily use a recursive predicate $\bigcap_{g \in p} S_g \neq \emptyset$. Therefore, one may not be able to construct a family $\mathbb{P}$ that is a strong array and satisfies the conditions (**i**) and (**ii**) of Theorem 72. However, using the fact that $LO(G)$ is homeomorphic to the Cantor set together with the assumption that the condition in Remark 42 is computable, one can derive the conclusion of Theorem 72. This applies when, for example, we use Theorem 85 for a computable, finite rank, torsion-free, abelian group. In particular, Theorems 85 and 72 allow us to obtain some of the results of [70] and [66] as corollaries.

6 Turing degrees of left orders on computable groups

In the following sections, we use the results of the previous chapter to investigate computational and also topological properties of spaces of left orders for computable torsion-free abelian groups, finitely generated free groups, and some classes of one-relator groups.

6.1 Orders on $\mathbb{Z}^n$ and $\mathbb{Z}^\omega$

We consider orders on a finite rank, computable, torsion-free, abelian group G. Notice that the basis of G can be found algorithmically, and since each left order on G is also a bi-order, $LO(G) = BiO(G)$. Since every finite rank computable torsion-free abelian group is computably isomorphic to $\mathbb{Z}^n$, without loss of generality, we may assume that $G = \mathbb{Z}^n$ with the chosen ordered set of generators $\{\mathbf{e}_i\}_{i=1}^n$, where

$$\mathbf{e}_i = \underbrace{(0, 0, ..., 0, 1, 0, ..., 0)}_{1 \text{ in the } ith \text{ position}}.$$

Moreover, every order on G can be uniquely extended to an order on its divisible closure D, and also every order on G can be viewed as a restriction of an order on D. Obviously, $D = \mathbb{Q}^n$ is the divisible closure of G. We will start this section with a complete topological description of the space of orders on $\mathbb{Q}^n$. We will follow an argument of Proposition 1.7 of [66].

Since we can view $\mathbb{Q}^n$ as a subset of $\mathbb{R}^n$ (here $\mathbb{R}^n$ is equipped with the standard topology), $\mathbb{Q}^n$ is given the topology induced from $\mathbb{R}^n$. Therefore, given $P^+ \in LO(\mathbb{Q}^n)$, we regard it also as a subset $\mathbb{R}^n$. One shows that the boundary of the closure of P^+ in $\mathbb{R}^n$ is a hyperplane

$$H_{n-1} = \partial(Cl_{\mathbb{R}^n}(P^+)).$$

This hyperplane splits $\mathbb{R}^n$ into two closed half-spaces H_n^+ and H_n^- such that

$$H_n^+ \cup H_n^- = \mathbb{R}^n \text{ and } H_n^+ \cap H_n^- = H_{n-1}.$$

Moreover,

$$\text{either } P^+ \subset H_n^+ \text{ or } P^+ \subset H_n^-.$$

In the case when $P^+ \cap H_{n-1} = \{\mathbf{0}\}$, where $\mathbf{0} = (0, 0, ..., 0) \in \mathbb{R}^n$, the order P^+ is determined by the hyperplane H_{n-1} and one of the half-spaces H_n^+ or H_n^-. Now, if $P^+ \cap H_{n-1} \neq \{\mathbf{0}\}$, then $P^+ \cap H_{n-1}$ defines an order on $\mathbb{Q}^n \cap H_{n-1}$. Suppose

$$k = \dim_{\mathbb{Q}}(\mathbb{Q}^n \cap H_{n-1}),$$

$(k \leq n-1)$. Then

$$Cl_{\mathbb{R}^n}(\mathbb{Q}^n \cap H_{n-1}) = H_k,$$

$(H_k \approx \mathbb{R}^k)$ and $P^+ \cap H_{n-1}$ induces an order on $\mathbb{Q}^n \cap H_{n-1}$. Therefore, there is a $(k-1)$-dimensional subspace

$$H_{k-1} = \partial(Cl_{H_k}(P^+ \cap H_{n-1})) \subset H_k \subset \mathbb{R}^n,$$

which splits H_k into two sets (half-subspaces), H_k^+ and H_k^-, such that $P^+ \cap H_{n-1}$ is contained in one of them, and

$$H_k^+ \cup H_k^- = H_k \text{ and } H_k^+ \cap H_k^- = H_{k-1}.$$

Hence, in the case when $P^+ \cap H_{k-1} = \{\mathbf{0}\}$, the order P^+ is determined by the hyperplane H_{n-1}, one of the half-spaces H_n^+ or H_n^- and, additionally, by a subspace $H_k \subset H_{n-1}$, its subspace H_{k-1} and one of the half-subspaces H_k^+ or H_k^-. If $P^+ \cap H_{k-1} \neq \{\mathbf{0}\}$, then $P^+ \cap H_{k-1}$ defines an order on $\mathbb{Q}^n \cap H_{k-1}$. We can continue as in the previous case, obtaining $(l-1)$-dimensional subspace

$$H_{l-1} = \partial(Cl_{H_l}(P^+ \cap H_{n-1})) \subset H_l = Cl_{\mathbb{R}^n}(\mathbb{Q}^n \cap H_{k-1}) \subset \mathbb{R}^n.$$

In the case when $H_{l-1} \cap P^+ = \{\mathbf{0}\}$, the following sets are sufficient to describe P^+ :

i) H_{n-1},

ii) H_n^+ or H_n^-,

iii) $H_k(\subset H_{n-1})$ and $H_{k-1} \subset H_k$,

iv) H_k^+ or H_k^-,

v) $H_l(\subset H_{k-1})$ and $H_{l-1} \subset H_l$,

vi) H_l^+ or H_l^-.

Since the dimension of $\mathbb{Q}^n$ is finite, every $P^+ \in LO(\mathbb{Q}^n)$ can be determined by such a collection of subspaces and half-subspaces of $\mathbb{R}^n$.

Example 86 *To define an order P^+ on $\mathbb{Q}^2$, we choose a line $\ell : ax + by = 0$ in $\mathbb{R}^2$, $a^2 + b^2 \neq 0$. If $\mathbb{Q}^2 \cap \ell = \{\mathbf{0}\}, \mathbf{0} = (0, 0)$, then one of two sets,*

$$\{(x, y) \in \mathbb{Q}^2 |\ ax + by > 0\} \cup \{\mathbf{0}\}$$

or

$$\{(x, y) \in \mathbb{Q}^2 |\ ax + by < 0\} \cup \{\mathbf{0}\},$$

can be chosen to determine P^+. In the case when $\mathbb{Q}^2 \cap \ell \neq \{\mathbf{0}\}$, each of the following subsets of $\mathbb{Q}^2$ can be chosen to determine P^+:

$$\{(x, y) \in \mathbb{Q}^2 |\ ax + by > 0\} \cup \{(x, y) |\ ax + by = 0, x \geq 0\}$$

or

$$\{(x, y) \in \mathbb{Q}^2 |\ ax + by > 0\} \cup \{(x, y) |\ ax + by = 0, x \leq 0\}$$

or

$$\{(x, y) \in \mathbb{Q}^2 |\ ax + by < 0\} \cup \{(x, y) |\ ax + by = 0, x \geq 0\}$$

or

$$\{(x, y) \in \mathbb{Q}^2 |\ ax + by < 0\} \cup \{(x, y) |\ ax + by = 0, x \leq 0\}.$$

That is, every line $\ell : ax + by = 0$ in $\mathbb{R}^2$ determines either two (in the case when the slope of ℓ is irrational) or four (in the case when the slope of ℓ is rational or ∞) different orders on $\mathbb{Q}^2$, hence on $\mathbb{Z}^2$.

Let $A = \{g_1, g_2, ..., g_j\} \subset \mathbb{Z}^n \backslash \{\mathbf{0}\} \subset \mathbb{Q}^n, j \in \omega$. We denote by

$$sgr_{\mathbb{Q}}(A) = sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) = \{\mathbf{q} \in \mathbb{Q}^n | \ \mathbf{q} = \sum_{i=1}^{j} q_i g_i, \ \sum_{i=1}^{j} q_i^2 \neq 0, \ q_i \geq 0, \ q_i \in \mathbb{Q}, \ i = 1, 2, ..., j\}$$

the sub-semigroup of $\mathbb{Q}^n$ generated by A.

Proposition 87 *Let $A = \{g_1, g_2, ..., g_j\} \subset \mathbb{Z}^n \backslash \{\mathbf{0}\}$. Then there is an algorithm which decides whether the sub-semigroup $sgr_{\mathbb{Q}}(A) \subset \mathbb{Q}^n$ defines a partial order*

$$P = sgr_{\mathbb{Q}}(A) \cup \{\mathbf{0}\}$$

on $\mathbb{Q}^n$.

Proof. We notice that a sub-semigroup $sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) \subset \mathbb{Q}^n$ defines a partial order

$$P = sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) \cup \{\mathbf{0}\}$$

on $\mathbb{Q}^n$ if and only if

$$sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) \cap sgr_{\mathbb{Q}}(\{g_1^{-1}, g_2^{-1}, ..., g_j^{-1}\}) = \emptyset.$$

This condition is equivalent to the following:

$$(\forall a \in \mathbb{Q}^n) \ [a \in sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) \implies a^{-1} \notin sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\})].$$

Since $\mathbb{Q}^n$ is fully orderable, the last condition is equivalent to:

$$(\exists P^+ \in LO(\mathbb{Q}^n)) \ [sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) \subset P^+].$$

Using the topological description of orders on $\mathbb{Q}^n$, this condition is equivalent to the existence of an affine hyperplane $H \subset \mathbb{Q}^n$ ($\mathbf{0} \notin H$) such that the convex hull,

$$Conv_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) = \{\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{Q}^n | \ \mathbf{x} = \sum_{i=1}^{j} t_i g_i, \ \sum_{i=1}^{j} t_i = 1, \ t_i \geq 0, \ t_i \in \mathbb{Q}, \ i = 1, 2, ..., j\},$$

spanned by $\{g_1, g_2, ..., g_j\}$, can be separated from the point $\mathbf{0} \in \mathbb{Q}^n$ by H. That is, if

$$H = \{\mathbf{x} \in \mathbb{Q}^n | \ \langle \mathbf{n}, \mathbf{x} \rangle = \sum_{i=1}^{n} n_i x_i = z_0\},$$

where $\mathbf{n} = (n_1, n_2, ...n_n) \in \mathbb{Q}^n$ is a normal vector to H and $z_0 \in \mathbb{Q}$, then for all $g \in G$,

$$(g \in A) \Rightarrow (g \in H^+),$$

where

$$H^+ = \{\mathbf{x} \in \mathbb{Q}^n | \ \langle \mathbf{n}, \mathbf{x} \rangle = \sum_{i=1}^{n} n_i x_i \leq z_0\}$$

and

$$\mathbf{0} \in H^- = \{\mathbf{x} \in \mathbb{Q}^n | \ \langle \mathbf{n}, \mathbf{x} \rangle = \sum_{i=1}^{n} n_i x_i > z_0\}.$$

Therefore, our problem reduces to the following Linear Programming (LP) problem:

Find

$$z_0 \in \mathbb{Q} \text{ and } \mathbf{n} \in \mathbb{Q}^n$$

such that

$$\begin{aligned} \langle \mathbf{n}, g_k \rangle &= \sum_{i=1}^{n} n_i g_i^k \leq z_0, \text{ where } g_k = (g_1^k, g_2^k, ..., g_n^k) \in A, \ k = 1, 2, ..., j \\ & \text{and} \\ z_0 &< 0. \end{aligned}$$

This LP problem can, in turn, be reduced to the following LP problem.

Maximize:

Objective function:		$f(\mathbf{n}, z_0) = -z_0$	
Subject to constraints:		$\sum_{i=1}^{n} n_i g_i^k - z_0 \leq 0$	where $g_k = (g_1^k, g_2^k, ..., g_n^k) \in A$, $k = 1, 2, ..., j$
	and	$-z_0 \leq 1$	

Since the existence of the solution to the above LP problem can be determined algorithmically using the *symplex method* (we can actually use *Mathematica* software to solve it), we have an algorithm that decides whether a given finite set of generators $A \subset \mathbb{Z}^n \backslash \{\mathbf{0}\}$ determines a partial order $sgr_{\mathbb{Q}}(A) \cup \{\mathbf{0}\}$ on $\mathbb{Q}^n$. ■

In [70], it was shown that the degree spectrum of left orders on $\mathbb{Z}^n$ includes all Turing degrees (see Corollary 89) by providing a direct argument in the case $n = 2$ and using direct decomposition

$\mathbb{Z}^n = \mathbb{Z}^2 \oplus \mathbb{Z}^{n-2}$ for $n \geq 3$. Solomon used a lexicographic extension of each order on $\mathbb{Z}^2$ to $\mathbb{Z}^n$ (thus preserving Turing degrees) and constructed orders on $\mathbb{Z}^n$ of an arbitrary Turing degree. This, however, could be achieved by applying Theorem 72 directly. We will see that this result is also a corollary of Theorem 88 below. Moreover, the result of [66] about $LO(\mathbb{Z}^n)$ being homeomorphic to the Cantor set for any $n \geq 2$ (see Corollary 90) is a direct consequence of Theorem 85, and it can also be considered a special case of the following result.

Theorem 88 *Let $\mathbb{Z}^\omega = \oplus_{i\in\omega}\mathbb{Z}$ be a direct sum of infinite number of copies of $\mathbb{Z}$. Then*

(i) $DgSp_G(LO) = \mathcal{D}$.

(ii) *$LO(G)$ with the topology defined in section 5.2 is homeomorphic to the Cantor set.*

Proof. We construct a complete family $\mathbb{P}$ of finite subsets of $\mathbb{Z}^\omega \backslash \{\mathbf{0}\}$, which satisfies the conditions of Theorem 72. Let us start by fixing an enumeration

$$\varphi : (\mathbb{Z}^\omega \backslash \{\mathbf{0}\})^{<\omega} \to \omega$$

of all finite subsets of $\mathbb{Z}^\omega$. Since there is a Turing degree preserving bijection between $LO(\mathbb{Z}^\omega)$ and $LO(\mathbb{Q}^\omega)$, we can consider orders on $\mathbb{Q}^\omega$ instead of orders on $\mathbb{Z}^\omega$. Consider $sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\})$, the sub-semigroup of $\mathbb{Q}^\omega$ generated by $\{g_1, g_2, ..., g_j\} \subset \mathbb{Z}^\omega \backslash \{\mathbf{0}\}$. Since $A = \{g_1, g_2, ..., g_j\}$ is finite and each $g_i = (g_i^1, g_i^2, ...) \in \mathbb{Z}^\omega$, $1 \leq i \leq j$, there is

$$n = n(\{g_1, g_2, ..., g_j\}) = \max\{l(g_i)|\ 1 \leq i \leq j\},$$

where

$$l(g_i) = \max\{k \in \omega|\ g_i^k \neq 0\}.$$

Thus,

$$sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\}) \subset \mathbb{Q}^n.$$

We can apply Proposition 87 to decide whether the sub-semigroup $sgr_{\mathbb{Q}}(\{g_1, g_2, ..., g_j\})$ defines a partial left order on $\mathbb{Q}^n \subset \mathbb{Q}^\omega$. Since each partial left order on $\mathbb{Q}^\omega$ can be extended to a total left order P^+ on $\mathbb{Q}^\omega$ (as $\mathbb{Q}^\omega$ is fully orderable), using our function φ, we recursively define the family $\mathbb{P}$ as follows:

$$p_0 = \varphi^{-1}(k_0),$$

where

$$k_0 = \min\{j \in \omega |\ \mathbf{0} \notin Conv_{\mathbb{Q}}(\varphi^{-1}(j)) \subset \mathbb{Q}^n,\ n = n(\varphi^{-1}(j))\}$$

Suppose that $p_0, p_1, ..., p_{m-1}$ have been already constructed. Therefore, there exist indices

$$k_0 < k_1 < ... < k_{m-1}$$

such that $p_j = \varphi^{-1}(k_j)$, $j = 0, 1, ..., m-1$. Define

$$p_m = \varphi^{-1}(k_m),$$

where

$$k_m = \min\{j \in \omega \backslash \{0, 1, ..., k_{m-1}\} |\ \mathbf{0} \notin Conv_{\mathbb{Q}}(\varphi^{-1}(j)) \subset \mathbb{Q}^n,\ n = n(\varphi^{-1}(j))\}.$$

Now, we need to check that the family $\mathbb{P}$ we constructed satisfies conditions **(i)** and **(ii)** of Theorem 72. Let $p \in \mathbb{P}$, and notice that the set

$$S(p) = \{s \in \mathbb{P} \backslash \{p\} |\ p \subset s\} \neq \emptyset,$$

as there is $P^+ \in LO(\mathbb{Q}^\omega)$ such that $sgr_{\mathbb{Q}}(p) \subset P^+$. For each $g \in \mathbb{Z}^\omega \backslash \{\mathbf{0}\}$, define

$$q_g = p \cup \{g\} \text{ and } r_g = p \cup \{g^{-1}\}.$$

Define

$$a = \min\{g \in \mathbb{Z}^\omega \backslash \{\mathbf{0}\} |\ (g \notin p) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(q_g)) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(r_g))\}.$$

We now show that the element $a \in \mathbb{Z}^\omega \backslash \{\mathbf{0}\}$ defined above exists. First, we note that if the set

$$S = \{g \in \mathbb{Z}^\omega \backslash \{\mathbf{0}\} |\ (g \notin p) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(q_g)) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(r_g))\} \neq \emptyset,$$

then S is computable since, obviously, $g \notin p$ is a computable predicate, and both $\mathbf{0} \notin Conv_{\mathbb{Q}}(q_g)$ and $\mathbf{0} \notin Conv_{\mathbb{Q}}(r_g)$ are computable predicates by Proposition 87. Now, $S \neq \emptyset$ for the following reason. Since $p \in \mathbb{P}$, (from the proof of Proposition 87) there is $z_0 \in \mathbb{Q}$ and $\mathbf{n} \in \mathbb{Q}^n$ such that

$$Conv_{\mathbb{Q}}(p) \subset H^+ = \{\mathbf{x} \in \mathbb{Q}^n |\ \langle \mathbf{n}, \mathbf{x} \rangle = \sum_{i=1}^{n} n_i x_i \leq z_0\},$$

where $z_0 < 0$. Since $\mathbf{n} \in \mathbb{Q}^n$, the intersection of the hyperplane $H_n = \{\mathbf{x} \in \mathbb{Q}^n | \langle \mathbf{n}, \mathbf{x} \rangle = \sum_{i=1}^{n} n_i x_i = 0\}$ and $\mathbb{Z}^n \backslash \{\mathbf{0}\}$ is nonempty. Let $g \in (H_n \cap \mathbb{Z}^n) \backslash \{\mathbf{0}\}$, and define

$$q_g = p \cup \{g\} \text{ and } r_g = p \cup \{g^{-1}\}.$$

Obviously, since $H_n \cap Conv_{\mathbb{Q}}(p) = \emptyset$, we have $g \notin p$. Moreover,

$$\mathbf{0} \notin Conv_{\mathbb{Q}}(q_g) \text{ and } \mathbf{0} \notin Conv_{\mathbb{Q}}(r_g).$$

Otherwise, there exist $\lambda_1, \lambda_2 \in \mathbb{Q} \cap [0, 1]$, and $\mathbf{x}, \mathbf{y} \in Conv_{\mathbb{Q}}(p)$ such that

$$\lambda_1 \mathbf{x} + (1 - \lambda_1) g = \mathbf{0}$$

or

$$\lambda_2 \mathbf{y} + (1 - \lambda_2) g^{-1} = \mathbf{0},$$

hence

$$\mathbf{0} = \langle \mathbf{n}, \lambda_1 \mathbf{x} + (1 - \lambda_1) g \rangle = \lambda_1 \langle \mathbf{n}, \mathbf{x} \rangle + (1 - \lambda_1) \langle \mathbf{n}, g \rangle = \lambda_1 \langle \mathbf{n}, \mathbf{x} \rangle \leq \lambda_1 z_0$$

or

$$\mathbf{0} = \left\langle \mathbf{n}, \lambda_2 \mathbf{y} + (1 - \lambda_2) g^{-1} \right\rangle = \lambda_2 \langle \mathbf{n}, \mathbf{y} \rangle + (1 - \lambda_2) \left\langle \mathbf{n}, g^{-1} \right\rangle = \lambda_2 \langle \mathbf{n}, \mathbf{y} \rangle \leq \lambda_2 z_0$$

Since $z_0 < 0$, it follows $\lambda_1 = 0$ or $\lambda_2 = 0$, so in both cases, we obtain that $g = \mathbf{0}$, contradicting the choice of $g \in (H^n \cap \mathbb{Z}^n) \backslash \{\mathbf{0}\}$. This shows that $S \neq \emptyset$ which, in turn, implies that

$$a = \min\{g \in \mathbb{Z}^\omega \backslash \{\mathbf{0}\} | \ (g \notin p) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(p_g)) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(q_g))\}$$

is well-defined. We define

$$q = p \cup \{a\} \text{ and } r = p \cup \{a^{-1}\}.$$

By definition, we have $p \subset q$ and $p \subset r$ and $a \in q$ and $a^{-1} \in r$, and also $q, r \in \mathbb{P}$ as both of them (according to Proposition 87) define the partial orders $sgr_{\mathbb{Q}}(q) \cup \{e\}$ and $sgr_{\mathbb{Q}}(r) \cup \{e\}$ on $\mathbb{Q}^\omega$, which could be extended to linear orders Q^+ and R^+ on $\mathbb{Q}^\omega$, respectively. The property **(ii)** of the family $\mathbb{P}$ follows immediately from the fact that for any extension P^+ of the partial order $sgr_{\mathbb{Q}}(p) \cup \{e\}$ we have

$$P^+ \cup P^- = \mathbb{Q}^\omega \text{ and } P^+ \cap P^- = \{\mathbf{0}\}.$$

Hence the set

$$S(p) = \{s \in \mathbb{P} \backslash \{p\} |\; p \subset s\} \neq \emptyset.$$

Since for any $a \in \mathbb{Z}^\omega \backslash \{\mathbf{0}\}$ we have $a \in P^+$ or $a^{-1} \in P^+$,

$$\text{either } q = (p \cup \{a\}) \in S(p) \text{ or } q = (p \cup \{a^{-1}\}) \in S(p)$$

as either $sgr_{\mathbb{Q}}(p \cup \{a\}) \subset P^+$ or $sgr_{\mathbb{Q}}(p \cup \{a^{-1}\}) \subset P^+$. Therefore, the family $\mathbb{P}$ of finite subsets of $\mathbb{Z}^\omega \backslash \{\mathbf{0}\}$ satisfies all requirements of Theorem 72, hence $DgSp_{\mathbb{Z}^\omega}(LO) = \mathcal{D}$.

To prove the second part of the theorem, notice that the family $\mathbb{P}$ of finite subsets of $\mathbb{Z}^\omega \backslash \{\mathbf{0}\}$ is complete as it consists of all finite subsets $p \subset \mathbb{Z}^\omega \backslash \{\mathbf{0}\}$ such that $sgr_{\mathbb{Q}}(p) \subset P^+$ for some $P^+ \in LO(\mathbb{Q}^\omega)$. Moreover, since for all elements of $\mathbb{P}$ both conditions of Theorem 72 are satisfied (as we have just shown), we have by Theorem 85 that $LO(\mathbb{Z}^\omega)$ is homeomorphic to the Cantor set. ■

The following two known results can now be obtained as corollaries of the proof of Theorem 88.

Corollary 89 *(R. Solomon [70]) $DgSp_{\mathbb{Z}^n}(LO) = \mathcal{D}$ for $n \geq 2$.*

Corollary 90 *(A. Sikora [66]) $LO(\mathbb{Z}^n)$ with the topology defined in section 5.2 is homeomorphic to the Cantor set for any $n \geq 2$.*

The techniques used in the proof of Theorem 88 motivate the following result. Recall that a group G is *fully left-orderable* if all partial left orders on G can be extended to total left orders.

Proposition 91 *Let $D \subseteq \omega$ be a set of Turing degree $\mathbf{d}$. Suppose G is a computable fully left-orderable group with the following properties:*

(i) *No left order on G is determined uniquely by a finite subset $A \subset G \backslash \{e\}$;*

(ii) *For all finite subsets $A \subset G \backslash \{e\}$ and for all $a \in G$, the problem*

$$a \in sgr(A)$$

is $\mathbf{d}$-decidable;

(iii) *$DgSp_G(LO)$ is closed upward.*

Then

$$DgSp_G(LO) \supseteq \{\mathbf{x} \in \mathcal{D} |\ \mathbf{x} \geq \mathbf{d}\}.$$

Proof. We construct a family $\mathbb{P} = \{p_i\}_{i\in\omega}$ of finite subsets of $G\backslash\{e\}$ which satisfies the assumptions of Theorem 73. Let us start by fixing some enumeration $\varphi : (G\backslash\{e\})^{<\omega} \to \omega$ of all finite subsets of $G\backslash\{e\}$. We now consider $sgr(\{g_1, g_2, ..., g_j\})$, the sub-semigroup of G generated by $\{g_1, g_2, ..., g_j\} \subset G\backslash\{e\}$. By assumption **(ii)**, we can decide with degree $\mathbf{d}$ whether $sgr(\{g_1, g_2, ..., g_j\}) \cup \{e\}$ defines a partial left order on G. Since G is fully left-orderable, each partial order on G can be extended to a total left order P^+ on G. Using the function φ, we define the family $\mathbb{P}$ as follows:

$$p_0 = \varphi^{-1}(k_0),$$

where

$$k_0 = \min\{j \in \omega |\ e \notin sgr(\varphi^{-1}(j))\}.$$

Suppose that $p_0, p_1, ..., p_{n-1}$ have been already constructed. Thus, there exist indices $k_0 < k_1 < ... < k_{m-1}$ such that $p_j = \varphi^{-1}(k_j)$, $j = 0, 1, ..., m-1$. Define

$$p_m = \varphi^{-1}(k_m),$$

where

$$k_m = \min\{j \in \omega\backslash\{0, 1, ..., k_{m-1}\} |\ e \notin sgr(\varphi^{-1}(j))\}.$$

Now, we need to check that the family $\mathbb{P}$, which we have just constructed, satisfies conditions **(i)** and **(ii)** of Theorem 73. Let $p \in \mathbb{P}$, and notice that the set

$$S(p) = \{s \in \mathbb{P}\backslash\{p\} |\ p \subset s\} \neq \emptyset,$$

as there is $P^+ \in LO(G)$ such that $sgr(p) \subset P^+$. For each $g \in G\backslash\{e\}$, let

$$q_g = p \cup \{g\} \text{ and } r_g = p \cup \{g^{-1}\}.$$

Define

$$a = \min\{g \in G\backslash\{e\} |\ (g \notin p) \wedge (e \notin sgr(q_g)) \wedge (e \notin sgr(r_g))\}.$$

We now show that an element $a \in G\backslash\{e\}$ defined above exists. First, we note that if the set

$$S = \{g \in G\backslash\{e\}|\ (g \notin p) \wedge (e \notin sgr(q_g)) \wedge (e \notin sgr(r_g))\} \neq \emptyset,$$

then S is clearly **d**-computable. This is obvious as $g \notin p$ is a decidable predicate and both predicates, $e \notin sgr(q_g)$ and $e \notin sgr(r_g)$, are **d**-computable because of assumption **(ii)**. Now, $S \neq \emptyset$ for the following reason. By assumption **(i),** no left order $P^+ \in LO(G)$ can be determined uniquely by a finite subset $p \subset G\backslash\{e\}$, so there are at least two distinct left orders $Q^+, R^+ \in LO(G)$ such that $sgr(p) \subset Q^+ \cap R^+$. Hence there is $g \in G\backslash\{e\}$, $g \notin p$, such that

$$g \in Q^+ \text{ and } g \notin R^+.$$

Since $R^+ \cup R^- = G$, we have $g^{-1} \in R^+$ as $g \in R^-$. Hence, both partial left orders, $sgr(q_g) \cup \{e\} = sgr(p \cup \{g\}) \cup \{e\}$ and $sgr(r_g) \cup \{e\} = sgr(p \cup \{g^{-1}\}) \cup \{e\}$, can be extended to distinct linear left orders on G. And therefore,

$$e \notin sgr(q_g) \text{ and } e \notin sgr(r_g),$$

so

$$S = \{g \in G\backslash\{e\}|\ (g \notin p) \wedge (e \notin sgr(q_g)) \wedge (e \notin sgr(r_g))\} \neq \emptyset.$$

This shows that the family $\mathbb{P} = \{p_i\}_{i \in \omega}$ satisfies property **(i)** of Theorem 73. Property **(ii)** of the family $\mathbb{P}$ follows immediately from the fact that for any extension P^+ of the partial order $sgr(p) \cup \{e\}$ we have

$$P^+ \cup P^- = G \text{ and } P^+ \cap P^- = \{e\}.$$

Hence the set

$$S(p) = \{s \in \mathbb{P}\backslash\{p\}|\ p \subset s\} \neq \emptyset,$$

and since for any $a \in G\backslash\{e\}$ we have $a \in P^+$ or $a^{-1} \in P^+$, then

$$\text{either } q = (p \cup \{a\}) \in S(p) \text{ or } q = (p \cup \{a^{-1}\}) \in S(p),$$

as either $sgr_{\mathbb{Q}}(p \cup \{a\}) \subset P^+$ or $sgr_{\mathbb{Q}}(p \cup \{a^{-1}\}) \subset P^+$. Therefore, the family $\mathbb{P}$ of finite subsets of $G\backslash\{e\}$ satisfies all requirements of Theorem 73 for a computable fully left-orderable group G and a set D of Turing degree $\mathbf{d}$. Since (by assumption **(iii)**) $DgSp_G(LO)$ is closed upward, we have

$$DgSp_G(LO) \supseteq \{\mathbf{x} \in \mathcal{D} | \mathbf{x} \geq \mathbf{d}\}$$

by Corollary 76. ■

We can show that the following result is a consequence of Proposition 91.

Corollary 92 *(R. Solomon [70]) Let G be a computable torsion-free abelian group of infinite rank. Then*

$$\{\mathbf{x} \in \mathcal{D} |\ \mathbf{x} \geq \mathbf{0}'\} \subseteq DgSp_G(LO).$$

Proof. As an abelian torsion-free group G is fully orderable. Since there is a degree preserving bijection between $LO(G)$ and $LO(D)$, where D is a divisible closure of G, we can restrict our consideration to the case when G is a computable divisible torsion-free abelian group of infinite rank. It can be shown [70] that for G one can find a basis $\{\mathbf{e}_1, \mathbf{e}_2, ...\}$ in $\mathbf{0}'$ degree. Given $g_1, g_2, ...g_k \in G$, each g_i, $i = 1, 2, ...k$, can be uniquely expressed as

$$g_i = q_1^i \mathbf{e}_1 + q_2^i \mathbf{e}_2 + q_{n_i}^i \mathbf{e}_{n_i},$$

where $q_{n_i}^i \neq 0$, $q_j^i \in \mathbb{Q}, j = 1, 2, ...n_i$. Let $n = \max\{n_i |\ i = 1, 2, ..., k\}$. Then we have $\{g_1, g_2, ...g_k\} \subseteq \mathbb{Q}^n$. Now, using Proposition 87, we can decide with $\mathbf{0}'$-oracle whether $a \notin sgr\{g_1, g_2, ...g_k\}$ for any $a \in G$. Obviously, no left order on G can be uniquely determined by a finite subset A of nonidentity elements of G since $A \subset \mathbb{Q}^n$ and no finite subset of $\mathbb{Q}^n$ can uniquely determine a left order on $\mathbb{Q}^n$, and so on G. Since $DgSp_G(LO)$ is closed upward, by Proposition 91, we have that

$$\{\mathbf{x} \in \mathcal{D} |\ \mathbf{x} \geq \mathbf{0}'\} \subseteq DgSp_G(LO),$$

which completes the proof. ■

6.2 Orders on finitely generated free groups

In this section, we are use free differential calculus of R. Fox [18, 19] to investigate Turing degrees on free groups of finite rank. Let $F_n = \langle x_1, x_2, ..., x_n | - \rangle$ be a free group generated by $\{x_1, x_2, ..., x_n\}$.

The results of [18] and [19], which we will briefly discuss, allow us to construct bi-orders on F_n. First, we review some standard notation from algebra. Let G be a multiplicative group, and let $a, b \in G$. As usual, define the commutator of a and b as:

$$[a, b] = a^{-1}b^{-1}ab,$$

and define

$$a^b = b^{-1}ab \text{ and } a^{-b} = b^{-1}a^{-1}b.$$

If H, $K \leq G$ are subgroups of G, then by $[H, K]$ we denote a subgroup of G generated by the set $\{[h, k] |\, h \in H,\, k \in K\}$:

$$[H, K] = \langle \{[h, k] |\ h \in H,\, k \in K\} \rangle \,.$$

Definition 93 *Let $\gamma_1(G) = G$ and for $i = 1, 2, 3, \ldots$ we set*

$$\gamma_{i+1}(G) = [\gamma_i(G), G].$$

The descending sequence of subgroups

$$\gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \cdots \geq \gamma_i(G) \geq \cdots$$

is called the lower central series of *G.*

A convenient tool for dealing with the lower central series of a group is free differential calculus introduced by Fox in [18]. Recall that a *derivation on a group ring* $\mathbb{Z}[G]$ is a $\mathbb{Z}$-linear map

$$D : \mathbb{Z}[G] \to \mathbb{Z}[G],$$

which satisfies the following condition:

$$(\forall g, h \in G)[D(gh) = D(g) + gD(h)].$$

The existence and uniqueness of a derivation for a free group of finite rank $n > 1$, F_n, is assured by the following theorem.

Theorem 94 *(R. Fox [18]) For each generator x_i of F_n, there is a corresponding derivation*

$$D_i : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$$

defined as follows:

$$D_i(f) = \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_i}.$$

It is called a free derivative *with respect to x_i and has the property:*

$$\frac{\partial x_k}{\partial x_i} = \delta_{i,k} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

For a construction of orders on F_n, we will recall important properties of the lower central series of F_n. Let $\varepsilon : \mathbb{Z}[F_n] \to \mathbb{Z}$ be an augmentation homomorphism defined as follows. For a free polynomial

$$f = \sum_{i=1}^{k} a_i w_i \in \mathbb{Z}[F_n],$$

where $a_i \in \mathbb{Z}$, $w_i \in F_n$, $k \in \mathbb{Z}_+$, let

$$\varepsilon(f = \sum_{i=1}^{k} a_i w_i) = \sum_{i=1}^{k} a_i.$$

Let S be a free semigroup generated by the set $\{x_1, x_2, ..., x_n\}$. For any element $f \in \mathbb{Z}[F_n]$ and any $a = a_1 a_2 ... a_k \in S$, where $a_j \in \{x_1, x_2, ..., x_n\}$, $j = 1, 2, ..., k$, let us denote

$$\varepsilon(\frac{\partial^k f}{\partial a_1 \partial a_2 ... \partial a_k}) \in \mathbb{Z} \text{ by } D_a^0(f).$$

Remark 95 *In [19], it was proved that an element w of F_n lies in $\gamma_k(F_n)$ if and only if $D_a^0(w) = 0$ for all $a = a_1 a_2 ... a_j$ of length j smaller than k. Therefore, the predicate $w \in \gamma_k(F_n)$ is decidable.*

Using free derivative, one can show that for a free group F_n of a finite rank $n > 1$, we have the property described in the following theorem.

Theorem 96 (*W. Magnus [43]*) *Let F_n be a free group of rank n, then*

$$\bigcap_{i=1}^{\infty} \gamma_i(F_n) = \{e\},$$

where $\gamma_1(F_n) \geq \gamma_2(F_n) \geq ... \geq \gamma_n(F_n) \geq ...$ is the lower central series of F_n.

For the purpose of constructing orders on finitely generated free groups, we need another important property of the lower central series of F_n, which can be proved using free calculus. Recall that the *Möbius function* μ is a number theoretic function defined for all $n \in \omega$, and has its values in $\{-1, 0, 1\}$ depending on the factorization of n into prime factors:

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is a square-free positive integer with an even number of distinct prime factors,} \\ -1 & \text{if } n \text{ is a square-free positive integer with an odd number of distinct prime factors,} \\ 0 & \text{if } n \text{ is not square-free.} \end{cases}$$

Theorem 97 *(M. Hall [20]) $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$, where*

$$k_i = \frac{1}{i} \sum_{d|i} \mu(\frac{i}{d}) n^d,$$

$i = 1, 2, ...,$ and μ is the Möbius function.

It is important to mention that the above isomorphism is uniformly computable since the basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ can be found algorithmically, and the groups $\gamma_i(F_n)$'s are computable (see Remark 95). The algorithm for constructing a basis for $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ is given in [20].

Example 98 *If $n = 2$ and $i = 1$, then $\gamma_1(F_2)/\gamma_2(F_2)$ has a basis given by the set $H_1 = \{x_1, x_2\}$, where $F_2 = \langle x_1, x_2 | - \rangle$. In the case when $i = 2$, $\gamma_2(F_2)/\gamma_3(F_2)$ has a basis given by $H_2 = \{[x_1, x_2]\}$, and for $i = 3$ we have a basis $H_3 = \{[[x_1, x_2], x_1], [[x_1, x_2], x_2]\}$, etc.*

Thus, in general, we denote the basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$ by H_i, and call it the *Hall basis.* Notice that

$$|H_i| = rank(\gamma_i(F_n)/\gamma_{i+1}(F_n)) = k_i$$

by Theorem 97.

Proposition 99 *There is an algorithm, which given $g \in \gamma_i(F_n) \backslash \gamma_{i+1}(F_n)$ finds a projection of g onto $\gamma_i(F_n)/\gamma_{i+1}(F_n)$.*

Proof. We show that there is an algorithm, which given $g \in \gamma_i(F_n)$ and $g \notin \gamma_{i+1}(F_n)$ computes $(\alpha_1, \alpha_2, ..., \alpha_{k_i}) \in \mathbb{Z}^{k_i}$ such that

$$g \equiv \prod_{j=1}^{k_i} b_j^{\alpha_i} \bmod \gamma_{i+1},$$

where $H_i = \{b_1, b_2, ..., b_{k_i}\}$ is the Hall basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$, $i = 1, 2, ...$ Let $g \in F_n$ be such that $g \in \gamma_i(F_n)$ and $g \notin \gamma_{i+1}(F_n)$ and let

$$\varphi : \mathbb{Z}^{k_i} \to \omega$$

be an effective enumeration of elements of $\mathbb{Z}^{k_i}$. Since H_i is a basis of $\gamma_i(F_n)/\gamma_{i+1}(F_n)$, there is unique $(\alpha_1, \alpha_2, ..., \alpha_{k_i}) \in \mathbb{Z}^{k_i}$ such that $g \equiv \prod_{j=1}^{k_i} b_j^{\alpha_i} \bmod \gamma_{i+1}$. The algorithm lists elements of $\mathbb{Z}^{k_i}$:

$$\varphi^{-1}(0), \varphi^{-1}(1), ...$$

For each element $\varphi^{-1}(j) = (a_1^j, a_2^j, ..., a_{k_i}^j) \in \mathbb{Z}^{k_i}$ the algorithm tests whether

$$g \equiv \prod_{l=1}^{k_i} b_l^{a_l^j} \bmod \gamma_{i+1}$$

by checking whether $g^{-1} \prod_{l=1}^{k_i} b_l^{a_l^j} \in \gamma_{i+1}(F_n)$. Since the predicate $w \in \gamma_{i+1}(F_n)$ is decidable for any $w \in F_n$ (see Remark 95), we can decide whether $g \equiv \prod_{l=1}^{k_i} b_l^{a_l^j} \bmod \gamma_{i+1}$. The procedure must terminate after finitely many steps since there is a unique $j \in \omega$ such that $\varphi^{-1}(j) = (\alpha_1, \alpha_2, ..., \alpha_{k_i})$. ■

Exploiting an idea introduced in [53], we construct orders on F_n using orders on quotients of the successive terms of the lower central of F_n. We observe that different choices of orders on quotients of the lower central series of F_n induce different bi-orderings on F_n. Since we can assign to them desired Turing degrees using the result of Corollary 89, we will be able to produce on F_n orders of given Turing degrees.

In the proof of Theorem 102 below, we need to be able to identify elements of a given term of the lower central series of F_n, hence we need to introduce the following notion of weight of an element of a group.

Definition 100 *Let G be a group, and let $\gamma_1(G) \geq \gamma_2(G) \geq ... \geq \gamma_n(G) \geq ...$ be its lower central series. A function $\varpi : G \to \mathbb{Z}_+$ defined by*

$$\varpi(w) = \begin{cases} \max\{i \in \mathbb{Z}_+ |\ w \in \gamma_i(G)\} & \text{if } w \neq e, \\ \infty & \text{if } w = e, \end{cases}$$

is called the weight function*.*

Lemma 101 *The weight function $\varpi : F_n \to \mathbb{Z}_+$ is computable.*

Proof. Assume that $w \neq e$. Since we have $\bigcap\limits_{i=1}^{\infty} \gamma_i(F_n) = \{e\}$, there is $i \in \mathbb{Z}_+$ such that

$$w \in \gamma_i(F_n) \text{ and } w \notin \gamma_{i+1}(F_n),$$

so $S = \{i \in \mathbb{Z}_+ | w \in \gamma_i(G)\} \neq \emptyset$ and S is finite, so the function ϖ is well-defined. Moreover, the predicate $w \in \gamma_i(F_n)$ is decidable (see Remark 95). Hence the function ϖ is computable. ■

To simplify notation, we denote by $\gamma_k (= \gamma_k(F_n))$ the kth term of the lower central series of F_n.

Theorem 102 *Let F_n be a free group of rank $n > 1$. Then $DgSp_{F_n}(LO) = \mathcal{D}$.*

Proof. We construct a family $\mathbb{P} = \{p_i\}_{i\in\omega}$ of finite subsets of F_n, which satisfies the assumptions of Theorem 72 as follows. We start by fixing an enumeration $\varphi : (F_n \backslash \{e\})^{<\omega} \to \omega$ of all finite subsets of $F_n \backslash \{e\}$. Consider $sgr(\{w_1, w_2, ..., w_j\})$, the sub-semigroup of F_n generated by $\{w_1, w_2, ..., w_j\} \subset F_n \backslash \{e\}$. We show that there is an algorithm that gives a sufficient condition for $sgr(\{w_1, w_2, ..., w_j\})$ to extend to an order on F_n. Since each $w_i \neq e$, $i = 1, 2, ..., j$, it follows that $\varpi(w_i) < \infty$ and

$$w_i \in \gamma_{\varpi(w_i)}(F_n) \text{ and } w_i \notin \gamma_{\varpi(w_i)+1}(F_n),\ i = 1, 2, ..., j.$$

Let

$$I = \{\varpi(w_i) |\ i = 1, 2, ..., j\} = \{k_1, k_2, ..., k_l\},\ 1 \leq l \leq j.$$

We can assume $k_1 < k_2 < ... < k_l$. Define

$$s_{k_m} = \{w \in \{w_1, w_2, ..., w_j\} \mid \varpi(w) = k_m\},\ 1 \leq m \leq l.$$

Let g_i denote the projection π of w_i onto $\gamma_{\varpi(w_i)}(F_n)/\gamma_{\varpi(w_i)+1}(F_n)$, where $1 \leq i \leq j$. By Proposition 99, each g_i can be found effectively, and $g_i \neq \mathbf{0}$ since $w_i \notin \gamma_{\varpi(w_i)+1}(F_n), i = 1, 2, ..., j$. Now, let

$$\overline{s_{k_m}} = \{g_i = \pi(w_i) \mid w_i \in s_{k_m}, i = 1, 2, ..., j\},\ 1 \leq m \leq l.$$

Since

$$\overline{s_{k_m}} \subset \mathbb{Z}^{k(m)} \cong \gamma_{k_m}/\gamma_{k_m+1}, \text{ where } k(m) = \frac{1}{k_m} \sum_{d|k_m} \mu(\frac{k_m}{d}) n^d,$$

by applying Proposition 87, we can decide whether the sub-semigroup $sgr(\overline{s_{k_m}})$ defines a partial order

$$P_{k(m)} = sgr(\overline{s_{k_m}}) \cup \{\mathbf{0}\}$$

on $\mathbb{Z}^{k(m)}$. If each $P_{k(m)}$, $1 \leq m \leq l$, is a partial order on $\mathbb{Z}^{k(m)}$, then

$$Q(w_1, w_2, ..., w_j) = sgr(\bigcup_{m=1}^{l} \overline{s_{k_m}}) \cup \{e\}$$

is a partial order on $\mathbb{Z}^\omega = \bigoplus_{i \in \omega} \mathbb{Z}^{n_i}$, where $n_i = \frac{1}{i} \sum_{d|i} \mu(\frac{i}{d}) n^d$. Since $\mathbb{Z}^\omega$ is fully orderable, the partial order $Q(w_1, w_2, ..., w_j)$ can be extended to a total order $Q^+ = Q^+(w_1, w_2, ..., w_j)$ on $\mathbb{Z}^\omega$. Every order Q^+ on $\mathbb{Z}^\omega$ induces an order Q_i^+ on its subgroup $\mathbb{Z}^{n_i}$. It follows from the result of Šimbireva [67] and Neumann [53] (see Theorem 104) that given orders Q_i^+ on each $\mathbb{Z}^{n_i} \cong \gamma_i(F_n)/\gamma_{i+1}(F_n)$, $i = 1, 2, ...$, one can define an order on F_n by taking as its positive cone

$$P^+ = \{w \in F_n | \ (\exists i \in \mathbb{Z}_+) \ [\pi(w) \in Q_i^+]\},$$

where $\pi(w)$ denotes the image of the projection of w onto $\gamma_i(F_n)/\gamma_{i+1}(F_n)$. Therefore, if $Q(w_1, w_2, ..., w_j)$ defines a partial order on $\mathbb{Z}^\omega$, then $sgr(\{w_1, w_2, ..., w_j\})$ extends to a bi-order on F_n. Notice that we can decide whether $Q(w_1, w_2, ..., w_j)$ is a partial order on $\mathbb{Z}^\omega$ by Proposition 87. This implies that we have a sufficient condition which allows us to decide whether a given set of words $\{w_1, w_2, ..., w_j\} \subset F_n$ induces an order on F_n.

Now, let

$$O(\{w_1, w_2, ..., w_j\}) = \bigcup_{m=1}^{l} \overline{s_{k_m}},$$

where $\overline{s_{k_m}}$ are defined above. To construct a family $\mathbb{P}$ satisfying the assumptions of Theorem 72, we use the function $\varphi : (F_n \backslash \{e\})^{<\omega} \to \omega$ as follows:

$$p_0 = \varphi^{-1}(k_0),$$

where

$$k_0 = \min\{j \in \omega | \ \mathbf{0} \notin Conv_{\mathbb{Q}}(O(\varphi^{-1}(j)))\}.$$

Suppose that $p_0, p_1, ..., p_{n-1}$ have been already constructed. Therefore, there exist indices $k_0 < k_1 < ... < k_{n-1}$ such that $p_j = \varphi^{-1}(k_j)$, $j = 0, 1, ..., n-1$. Define

$$p_n = \varphi^{-1}(k_n),$$

where

$$k_n = \min\{j \in \omega \backslash \{0, 1, ..., k_{n-1}\} | \ \mathbf{0} \notin Conv_{\mathbb{Q}}(O(\varphi^{-1}(j)))\}.$$

We need to check whether the family $\mathbb{P}$ satisfies conditions **(i)** and **(ii)** of Theorem 72. Let $p \in \mathbb{P}$, and notice that the set

$$S(p) = \{s \in \mathbb{P}\backslash\{p\} |\ p \subset s\} \neq \emptyset,$$

as there is $P^+ \in BiO(F_n)$ such that $sgr(p) \subset P^+$. For each $w \in F_n\backslash\{e\}$, define

$$q_w = p \cup \{w\} \text{ and } r_w = p \cup \{w^{-1}\}.$$

Let

$$a = \min\{w \in F_n\backslash\{e\} |\ (w \notin p) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(O(q_w))) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(O(r_w)))\}$$

be the word with the minimal index in some algorithmic enumeration of elements of F_n.We show now that the element $a \in F_n\backslash\{e\}$ defined above exists. First, we note that if

$$S = \{w \in F_n\backslash\{e\} |\ (w \notin p) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(O(q_w))) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(O(r_w)))\} \neq \emptyset,$$

then the set S is computable. This is obvious since $w \notin p$ is a decidable predicate, and the predicates $\mathbf{0} \notin Conv_{\mathbb{Q}}(O(q_w))$ and $\mathbf{0} \notin Conv_{\mathbb{Q}}(O(r_w))$ are decidable by Proposition 87.

We will now show that $S \neq \emptyset$. Since $p = \{w_1, w_2, ..., w_j\} \in \mathbb{P}$, there are

$$\overline{s_{k_m}} \subset \mathbb{Z}^{k(m)},\ 1 \leq m \leq l,$$

such that

$$O(p) = \bigcup_{m=1}^{l} \overline{s_{k_m}}.$$

Each $sgr(\overline{s_{k_m}})$ defines a partial order on $\mathbb{Z}^{k(m)}$. Hence, as in the proof of Theorem 88, there is an element g with

$$g = \sum_{i=1}^{k(m)} \alpha_i e_i \in \mathbb{Z}^{k(m)},$$

such that both

$$sgr(\overline{s_{k_m}} \cup \{g\}) \text{ and } sgr(\overline{s_{k_m}} \cup \{g^{-1}\})$$

can be extended to orders on $\mathbb{Z}^{k(m)}$. Therefore,

$$sgr(O(p) \cup \{g\}) \text{ and } sgr(O(p) \cup \{g^{-1}\})$$

can both be extended to distinct orders on $\mathbb{Z}^\omega = \bigoplus_{i\in\omega}\mathbb{Z}^{n_i}$. Let $H_{k_m} = \{b_1, b_2, ..., b_l\}$ be the Hall basis of $\gamma_{k_m}/\gamma_{k_m+1}$, where $l = |H_{k_m}| = k(m)$. Define

$$w = \prod_{i=1}^{l} b_i^{\alpha_i} \in F_n.$$

Clearly, we have

$$\pi(w) = g = \sum_{i=1}^{l} \alpha_i e_i \in \mathbb{Z}^{k(m)},$$

so $sgr(q_w) \cup \{e\}$ and $sgr(r_w) \cup \{e\}$ could be both extended to distinct orders on F_n. This shows that $S \neq \emptyset$, which implies that

$$a = \min\{w \in F_n \backslash \{e\}|\ (w \notin p) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(O(q_w))) \wedge (\mathbf{0} \notin Conv_{\mathbb{Q}}(O(r_w)))\}$$

is well-defined. We define

$$q = p \cup \{a\} \text{ and } r = p \cup \{a^{-1}\}.$$

By definition, $p \subset q$ and $p \subset r$ and $a \in q$ and $a^{-1} \in r$, and also q, $r \in \mathbb{P}$ as both of them define partial left orders $sgr(q) \cup \{e\}$ and $sgr(r) \cup \{e\}$ on F_n, which could be extended to linear orders Q^+ and R^+ on F_n, respectively.

Property **(ii)** of the family $\mathbb{P}$ of Theorem 72 follows immediately from the fact that for any extension P^+ of $sgr(p)$ we have

$$P^+ \cup P^- = F_n \text{ and } P^+ \cap P^- = \{e\}.$$

Since for any $w \in F_n$ either $w \in P^+$ or $w^{-1} \in P^+$, we have that

$$\text{either } q = (p \cup \{w\}) \in S(p) \text{ or } q = (p \cup \{w^{-1}\}) \in S(p),$$

where $S(p) = \{s \in \mathbb{P}\backslash\{p\}|p \subset s\} \neq \emptyset$, as either $sgr(p \cup \{w\}) \subset P^+$ or $sgr(p \cup \{w^{-1}\}) \subset P^+$. Therefore, the family $\mathbb{P}$ of finite subsets of $F_n\backslash\{e\}$ satisfies all requirements of Theorem 72. Hence $DgSp_{F_n}(LO) = \mathcal{D}$ for $n \geq 2$. ■

In [66], Sikora conjectured that the spaces of left- and bi-orders for a free group of rank n are homeomorphic to the Cantor set. He showed the existence of an embedding of the Cantor set in the space of orders of F_n. We will show that a similar result holds for one-relator groups in the next section.

Conjecture 103 *(A. Sikora [66]) For a free group of a finite rank $n > 1$, F_n, the spaces $LO(F_n)$ and $BiO(F_n)$ are homeomorphic to the Cantor set.*

To prove the existence of an embedding of the Cantor set into the space of bi-orders for free groups and for one-relator groups later, we need the following theorem.

Theorem 104 *(E. P. Šimbireva [67], B. H. Neumann [53]) Let $\gamma_1(G) \geq \gamma_2(G) \geq ... \geq \gamma_n(G) \geq ...$ be the lower central series of a group G and assume that $\bigcap_{i=1}^{\infty} \gamma_i(G) = \{e\}$. Then any ordering on $\gamma_i(G)/\gamma_{i+1}(G)$ induces a bi-ordering on G.*

The ordering induced on G can be defined as follows. Let P_i^+ be a positive cone on $\gamma_i(G)/\gamma_{i+1}(G)$ for $i \in \mathbb{Z}_+$. Since $\bigcap_{i=1}^{\infty} \gamma_i(G) = \{e\}$, then for any $g \in G$ there is $i \in \mathbb{Z}_+$ such that $g \in \gamma_i(G)$ and $g \notin \gamma_{i+1}(G)$. Define a positive cone P^+ on G by:

$$g \in P^+ \iff g\gamma_{i+1}(G) \in P_i^+.$$

Let $SBiO(G)$ denote the set of all bi-orders on a group G that can be obtained using the above construction. We call elements of $SBiO(G)$ the *standard orders* on G. The following result by Sikora together with Theorem 104 allows us to prove the existence of an embedding of the Cantor set into the space of bi-orders on free groups and one-relator groups.

Theorem 105 *(A. Sikora [66])*

(i) *$SBiO(G)$ is a closed subset of $BiO(G)$.*

(ii) *If $G \ncong \mathbb{Z}$ and each factor $\gamma_i(G)/\gamma_{i+1}(G)$ is finitely generated, then $SBiO(G)$ is either empty or homeomorphic to the Cantor set.*

Theorem 96 allows us to apply Theorem 104 in the case of a rank $n > 1$ free group, since we have that $\bigcap_{i=1}^{\infty} \gamma_i(F_n) = \{e\}$. Hence any ordering on the quotient groups, if it exists, would induce a bi-ordering on F_n. The existence of bi-orders on γ_i/γ_{i+1}'s is provided by Theorem 97 since we have $\gamma_i(F_n)/\gamma_{i+1}(F_n) \cong \mathbb{Z}^{k_i}$ with $k_i = \frac{1}{i}\sum_{d|i} \mu(\frac{i}{d})n^d$, $i = 1, 2, ...$ Therefore, since $SBiO(F_n) \neq \emptyset$, the space of standard orders of F_n is homeomorphic to the Cantor set by Theorem 105. Since for any group G we have $SBiO(G) \subseteq BiO(G) \subseteq LO(G)$, the space of left orders on F_n admits an embedding of the Cantor set.

6.3 Extension to one-relator groups

As mentioned before, the theory of one-relator groups is well understood. Many problems are decidable for this class of finitely presentable groups. For example, it is decidable whether such a group admits a left-invariant order (see Corollary 61). Furthermore, there exists an algorithm that finds a center of a given one-relator group [47]. We would like to apply the technique described in the previous section to construct bi-orders on one-relator groups. To apply Theorem 104 to a group G, we need to know whether G is residually nilpotent and whether the quotients of successive terms of the lower central series are free abelian groups.

Recall that a group G is *residually nilpotent* if for any $g \in G\backslash\{e\}$ there is a normal subgroup N of G such that $g \notin N$ and G/N is nilpotent. The property of a group to be residually nilpotent can be stated in terms of its lower central series as follows. A group G is *residually nilpotent* if

$$G_\omega = \bigcap_{i=1}^{\infty} \gamma_i(G) = \{e\}.$$

Let $r \in F_n\backslash\{e\}$, and let $\omega(r)$ be the weight of r (see Definition 100).

Definition 106 *An element $r \in F_n$ is said to be primitive (with respect to $\{\gamma_i(F_n)\}_{i=1}^{\infty}$) if $r \neq e$ and r is not an kth power modulo $\gamma_{\omega(r)+1}(F_n)$ for any integer $k > 1$, that is $r \neq s^k \operatorname{mod} \gamma_{\omega(r)+1}(F_n)$ for every $s \in \gamma_{\omega(r)}(F_n)$ and $k \in \mathbb{Z}_+$.*

We state below an important and simplified (for the purposes of this section) version of the result by Labute [36] concerning the structure of the quotients of successive terms of the lower central series for one-relator groups.

Theorem 107 *(J. P. Labute [36]) Let $G \simeq \langle x_1, x_2, ..., x_n | \ r\rangle$ and assume that r is a primitive element of F_n. Then for all $i \geq 1$, we have that $\gamma_i(G)/\gamma_{i+1}(G)$ is a free abelian group of rank g_i, where*

$$g_i = \frac{1}{i}\sum_{d|i} \mu(\frac{i}{d})[\sum_{0 \leq j \leq [\frac{d}{k}]} (-1)^j \frac{d}{d+j-kj}\binom{d+j-kj}{j} n^{d-kj}],$$

and $k = \varpi(r)$ is the weight of r and μ is the Möbius function.

In particular, the above theorem implies the following result.

Proposition 108 *Let $G = \langle x_1, x_2, ..., x_n | \ r\rangle$ ($\ncong \mathbb{Z}$) be a residually nilpotent group and assume that $r \in F_n$ is a primitive element. Then $SBiO(G)$ is homeomorphic to the Cantor set.*

Proof. We show that $SBiO(G) \neq \emptyset$ for a one-relator group $G = \langle x_1, x_2, ..., x_n | \ r\rangle$ with $r \in F_n$ being a primitive element. Since $G_\omega = \{e\}$ and $\gamma_i(G)/\gamma_{i+1}(G)$ is a free abelian group (by Theorem 107), by Theorem 104, we have $SBiO(G) \neq \emptyset$. Applying the result of Theorem 105, it follows that $SBiO(G)$ is homeomorphic to the Cantor set. ■

It follows from Proposition 108 that the number of different orders on any one-relator group

$$G = \langle x_1, x_2, ..., x_n | r\rangle,$$

where r is a primitive element of F_n and G is residually nilpotent, equals 2^ω. This motivates questions related to Turing degree spectra of $LO(G)$ analogous to the ones studied in the proceeding sections.

Fundamental groups of closed, connected and orientable surfaces of genus $g > 1$ are known to be residually nilpotent [3]. Moreover, it has been shown by Cochran and Orr [9] that all groups of surfaces (even non-orientable) are either nilpotent (in the case of $\mathbb{R}P^2$) or residually nilpotent. However, the general question concerning the existence of an algorithm which decides whether a given one-relator group G is residually nilpotent remains open. Although, for example, it is known that the group

$$G(2) = \langle x, y | \ [[x, y], y]\rangle$$

is residually nilpotent [46] (and since the element $[[x, y], y] \in F_2$ is primitive, $G(2)$ admits an embedding of the Cantor set into its space of left orders by Proposition 108 and the fact that $SBiO(G) \subseteq BiO(G) \subseteq LO(G)$ for any group G), it is not known whether one-relator groups generalizing $G(2)$ and described by the presentations of the form:

$$G(n) = \langle x, y \mid c_n\rangle, \text{ where } c_n = \underbrace{[[...[[x, y], y], ..., y], y]}_{n \text{ brackets}}, \ n > 2$$

are residually nilpotent. In [46], the authors formulated the following conjecture that generalizes the above question:

Conjecture 109 *(S. A. Melikhov, R. V. Mikhailov [46]) Finitely presented groups $\langle X \mid R \rangle$, where R is a finite subset of a Hall set H relative to the basis X, are residually nilpotent.*

Natural examples of groups with similar presentations are fundamental groups of the so-called torus links of type $(2, 2n)$, where $2n$ is the number of crossings. These presentations are:

$$G_{(2,2n)} = \langle x, y \mid [x, (xy)^n] \rangle .$$

It was shown by Murasugi [48] that one-relator groups with at least three distinct generators have trivial center ($Z(G) = \{e\}$), and in the case of nonabelian one-relator groups with two generators, if $Z(G) \neq \{e\}$, then $Z(G) \cong \mathbb{Z}$. In [45], it was shown that one-relator residually nilpotent groups with $Z(G) \neq \{e\}$ fall into one of the three categories:

- abelian groups;
- groups with presentations of the form: $\left\langle x, y \mid x^\alpha = y^\beta \right\rangle$, where α, β are powers of the same prime number p;
- groups with presentations of the form: $\left\langle x, y \mid [x, y^\beta] \right\rangle$, where β is a power of a prime number p.

Since $G_{(2,2n)} = \left\langle a, b \mid [a, b^\beta] \right\rangle$, we have that $G_{(2,2n)}$ is residually nilpotent if and only if $\beta = p^k = n$. Thus, $SBiO(G_{(2,2n)})$ is homeomorphic to the Cantor set if the relator $[a, b^\beta] \in F_2$ is primitive. However,

$$[a, b^\beta] \equiv [a, b]^\beta \bmod \gamma_3(G_{(2,2n)}),$$

so we cannot use Theorem 107 to determine whether the successive quotients of the lower central series of $G_{(2,2n)}$ are torsion-free abelian groups, unless $\beta = 1$ (in which case $G_{(2,2n)} \cong \mathbb{Z} \oplus \mathbb{Z}$). The quotients $\gamma_i(G_{(2,2n)})/\gamma_{i+1}(G_{(2,2n)})$ could probably be determined using the algorithm given in [19] for finding the presentation matrix of quotients of the successive terms of the lower central series. This and possibly some other approach could be used to answer the following question.

Problem 110 *For which values of integers n, k and a prime integer p, is the space of standard orders $SBiO(G)$ homeomorphic to the Cantor set if G is a one-relator group with the presentation of the form:*

- $\langle x, y \mid x^\alpha = y^\beta \rangle$ *, where $\alpha = p^k$ and $\beta = p^n$ or*
- $\langle x, y \mid [x, y^\beta] \rangle$ *, where $\beta = p^n$?*

It is known that a one-relator group is left-orderable if and only if it is torsion-free. This is a simple consequence of the theorem by Brodsky [5], which states that all torsion-free one-relator groups are locally indicable and so, by Theorem 48, left-orderable. It is also known that all bi-orderable groups are locally indicable. However, there are locally indicable groups which are not bi-orderable. Since all the groups given in Problem 110 are torsion-free, hence locally indicable, and so left-orderable, we can ask the following interesting question.

Problem 111 *For which values of integers n, k and a prime integer p, is the space of bi-orders $BiO(G)$ nonempty? Moreover, for which values of n, k, p, are $BiO(G)$ and $LO(G)$ homeomorphic to the Cantor set, if G is a one-relator group with the presentation of the form:*

- $\langle x, y \mid x^\alpha = y^\beta \rangle$ *, where $\alpha = p^k$ and $\beta = p^n$ or*
- $\langle x, y \mid [x, y^\beta] \rangle$ *, where $\beta = p^n$?*

We would like to emphasize here the fact that for non-abelian groups all the inclusions:

$$SBiO(G) \subseteq BiO(G) \subseteq LO(G)$$

are usually proper. For example, for the fundamental group of the Klein bottle $G = \langle a, b \mid [a, b] = b^{-2} \rangle$, we have $BiO(G) = \emptyset$ while $LO(G) \neq \emptyset$ (see Example 49). It is also possible that $SBiO(G) = \emptyset$ while $BiO(G) \neq \emptyset$. We provide such examples later, but first we notice that there is a significant class of one-relator groups G with a trivial center such that $SBiO(G)$ is homeomorphic to Cantor set.

Theorem 112 (*G. Baumslag [3]*) *Let*

$$G = \langle x_1, y_1, x_2, y_2, ..., x_g, y_g | \prod_{i=1}^{g} [x_i, y_i] \rangle$$

be the fundamental group of the orientable surface S_g with negative Euler characteristics,

$$\chi(S_g) = 2 - 2g < 0.$$

Then G is residually nilpotent.

Now, we can conclude that the space of standard orders for the orientable surface groups described above is homeomorphic to the Cantor set if we show that the element $\prod_{i=1}^{g} [x_i, y_i]$ is primitive with respect to the free group $F_{2g} = \langle x_1, y_1, x_2, y_2, ..., x_g, y_g | - \rangle$.

Corollary 113 *$SBiO(G)$ is homeomorphic to the Cantor set for any orientable surface group*

$$G = \langle x_1, y_1, x_2, y_2, ..., x_g, y_g | \prod_{i=1}^{g} [x_i, y_i] \rangle.$$

Proof. We need to show that $r = \prod_{i=1}^{g} [x_i, y_i]$ is a primitive element. Since

$$r = \prod_{i=1}^{g} [x_i, y_i] \in \gamma_2(F_{2g}) \text{ and } r \notin \gamma_3(F_{2g}),$$

so $\varpi(r) = 2$. Moreover,

$$r \equiv \sum_{i=1}^{g} [x_i, y_i] \bmod \gamma_3(F_{2g}),$$

so $r\gamma_3(F_{2g})$ equals to the sum of distinct elements of the Hall basis for $\gamma_2(F_{2g})/\gamma_3(F_{2g})$. Therefore, r is not a power of any element $s\gamma_3(F_{2g})$, where $s \in \gamma_2(F_{2g})$, so r is primitive. Thus, by Theorem 107, $\gamma_i(G)/\gamma_{i+1}(G)$ is a free abelian group of a finite rank bigger than 1 for all $i \in \mathbb{Z}_+$. Therefore, $SBiO(G)$ is homeomorphic to the Cantor set. ■

Remark 114 *Notice that in the case of orientable surface groups, the above Proposition and its Corollary generalize the results obtained by Rolfsen and Wiest in [64]. However, it is still not known whether $SBiO(G) = BiO(G)$.*

The result of Corollary 113 is not valid in the case when the surface is not orientable. For closed non-orientable surfaces of genus g, N_g, we have the following presentation

$$\pi_1(N_g) = \langle x_1, x_2, ..., x_g | \prod_{i=1}^{g} x_i^2 \rangle.$$

Since

$$\prod_{i=1}^{g} x_i^2 \in \gamma_1(F_g),$$

where $F_g = \langle x_1, x_2, ..., x_g | - \rangle$, and

$$\prod_{i=1}^{g} x_i^2 \notin \gamma_2(F_g) = [F_g, F_g],$$

and, moreover,

$$\prod_{i=1}^{g} x_i^2 \equiv (\prod_{i=1}^{g} x_i)^2 \bmod \gamma_2(F_g),$$

$\prod_{i=1}^{g} x_i^2$ is not primitive, so theorem 107 cannot be applied. It is easy to show that $\gamma_1(G)/\gamma_2(G) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$, so $SBiO(\pi_1(N_g)) = \emptyset$. However, the groups $\pi_1(N_g)$ are torsion-free and, as it has been shown in [9], they are residually nilpotent. Furthermore, it was proved by Rolfsen and Wiest [64] that fundamental groups of non-orientable surfaces are bi-orderable (except the fundamental group of the Klein bottle and the fundamental group of the projective plane). Hence we have just obtained an interesting family of examples of groups for which $SBiO(G) = \emptyset$ and $BiO(G) \neq \emptyset$. Recall that for the fundamental group of the Klein bottle G, one has $LO(G) \neq \emptyset$ and $SBiO(G) = BiO(G) = \emptyset$.

Corollary 115 *Let $G = \langle x_1, x_2, ..., x_n | \, r \rangle$ be any one-relator group with the primitive relator r in F_n. If $BiO(G) = \emptyset$, then $\bigcap_{i=1}^{\infty} \gamma_i(G) \neq \{e\}$.*

Corollary 115 provides a criterion for determining whether a one-relator group G is not residually nilpotent. This is generally a difficult question.

We have shown that the space of orders for one-relator groups admits an embedding of the Cantor set, provided the assumptions of Proposition 108 are satisfied. We would like to consider the Turing degree spectra of the space of orders on this class of groups.

Conjecture 116 *Let $G = \langle x_1, x_2, ..., x_n | \, r \rangle$ be a one-relator group with the relator $r \in F_n$, which is primitive. If G is residually nilpotent, then $DgSp_G(LO) = \mathcal{D}$.*

We believe that it can be shown that Conjecture 116 is true possibly along the following lines: we need to show that the weight function for G is well-defined and computable. We also need to provide an algorithm that finds, uniformly in $i \in \mathbb{Z}_+$, a basis of $\gamma_i(G)/\gamma_{i+1}(G)$, in which case the projection map $\psi_i : \gamma_i(G) \to \gamma_i(G)/\gamma_{i+1}(G)$ would be computable. This could be achieved by applying the algorithm described in [19]. In order to show that $DgSp_G(LO) = \mathcal{D}$, we use the methods developed in section 7.2 for free groups to produce a family $\mathbb{P} = \{p_i\}_{i\in\omega}$, which satisfies the assumptions of Theorem 72. These and some other results mentioned above will be addressed in my future research projects.

Furthermore, we should be able to generalize the above results to finitely presented groups. We plan to study spaces of orders for residually nilpotent torsion-free finitely presented groups. The theory of such groups is more complicated than the theory of one-relator groups. For example, the word problem and the conjugacy problem are not decidable for this family of groups. Thus, we would restrict our attention to a subfamily of finitely presented groups known as groups of links in S^3. Each member of this family is the fundamental group of 3-manifold that is obtained by removing a tubular neighborhood of a link $\mathcal{L}$ from S^3. Groups of links constitute a rich family of finitely presentable groups which, due to development of knot theory and low-dimensional topology, brought a considerable interest in combinatorial group theory. As a result of this interest, many important decision problems (including the word problem and the conjugacy problem) were proved to be decidable within this class of groups. Moreover, since the fundamental group of a link complement is a strong topological invariant, important results, which describe the structure of the lower central series and the structure of the successive quotients of its terms, were obtained by Murasugi [49], Traldi and Massey [44], Maeda [41] and Labute [38]. In [19], the authors developed an algorithm for finding a presentation for the abelian group obtained as the quotient of the successive terms of the lower central series. The algorithm is rather complex and not easy to apply. Making these results more applicable is one of my future research goals.

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