Structures and Partial Computable Automorphisms

by

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Dedication

This work is dedicated the memory of Mr. Richard Creighton.

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Abstract

We consider only structures for finite languages. Such a structure is *computable* if its domain is computable and its relations and operations are computable. For a structure, the set of all *partial automorphisms* forms an inverse semigroup under function composition. The set of finite partial automorphisms is an inverse subsemigroup of that structure. We consider inverse subsemigroups of the inverse semigroup of all partial automorphisms that contain all of the finite partial automorphisms, and determine what information we can recover about the original structures.

We show that for *equivalence structures*, the isomorphism type of the structure may be recovered from the isomorphism type of any such semigroup, and the first-order theory of the structure may be recovered from the first-order theory of any such semigroup. From the first-order theory of the inverse semigroup of finite partial automorphisms of an equivalence structure, we may actually recover the original structure up to isomorphism.

For *partial orders*, we show that from the isomorphism type of any inverse subsemigroup of partial automorphisms that contains all the finite partial au-

tomorphisms, we may recover the isomorphism type of the partial order up to reversal of the order. The result holds with first-order theory in place of isomorphism type.

For a computable structure, we also consider the inverse semigroup of all partial automorphisms which are *partial computable*. We establish that for certain computable *Boolean algebras* and computable *relatively complemented distributive lattices*, the isomorphism of these inverse semigroups implies that the corresponding structures are also isomorphic, even by a computable isomorphism. For computable equivalence structures, the elementary equivalence of these inverse semigroups implies that the structures are computably isomorphic.

Chapter 1

Introduction

The concept of an *isomorphism* is fundamental in the field of universal algebra. By isomorphism, we mean a one-to-one correspondence between the elements of two different structures that preserves all the relations and operations of the structures. When two structures are isomorphic they are, from an algebraic point of view, essentially the same structure. Of course, there may be many isomorphisms between structures. This is of particular interest when we consider isomorphisms from a structure to itself, which are called *automorphisms*. Any structure has at least one automorphism (namely, the identity). We may think of any nontrivial automorphisms of a structure as symmetries of that structure.

A *group* is a set together with an associative binary operation and a distinguished element that acts as the identity with respect to that operation, where each element has an inverse with respect to the operation and the identity. The set of automorphisms of a structure forms a group under the operation of function composition. We are interested in the connection between the structures and their symmetries. With this in mind, we ask what information about a structure we can recover from its automorphism group.

A *Boolean algebra* is a structure with two binary operations (meet and join), a single unary operation (complement) and distinguished greatest and least elements. A typical example of a Boolean algebra is the set of all subsets of some set S. Here, meet corresponds to intersection, join to union, complement to set complement relative to S. The set S itself is the greatest element in the Boolean algebra, and the empty set is the least element. Boolean algebras are an abstraction that capture the properties of such a structure; the precise definition will be given later. We are especially interested in Boolean algebras because there are many known results concerning the recovering of such structures from their automorphism groups. A brief survey of results along these lines is given in the next chapter. The basic idea is to interpret as much of the original structure as possible into the automorphism group.

A *partial isomorphism* is a one-to-one partial mapping (possibly not defined for some elements) from one structure to another that preserves all relations and operations. Once again, we are primarily interested in looking at partial isomorphisms from a structure to itself, which are called *partial automorphisms*. As with automorphisms, partial automorphisms may be thought of as symmetries. However, an automorphism corresponds to a symmetry of the entire structure, or a global symmetry, whereas partial automorphisms represent symmetries between different parts of the structure, or local symmetries. The reasons for studying the local symmetries are manifold. One reason is that there are many structures with no nontrivial global symmetries that do have local symmetries. One example of such a structure is the natural numbers under the usual ordering. Another reason is that, in general, looking at the partial symmetries of a structure yields much more information about the structure than confining oneself to global symmetries. The new results presented here are concerned with recovering as much information as possible about a structure from its partial isomorphisms.

The set of partial automorphisms of a structure does not usually form a group under function composition, but rather an *inverse semigroup.* A *semigroup* is a set together with an associative binary operation. There is not necessarily a universal identity element in a semigroup, but for each element there may be a local identity, which acts as an identity for just that element. There may be, in turn, local inverses associated with the local identities. An inverse semigroup is a semigroup in which every element has such a local inverse. In examining what can be recovered from a structure's partial automorphisms then, we will be primarily dealing with inverse semigroups. We are mostly interested in certain inverse subsemigroups of the semigroup of all partial automorphism. The inverse semigroup of all finite partial automorphisms (for a structure $\mathcal{M}$, denoted by $I_{fin}(\mathcal{M})$) will be fundamental to our study. Within $I_{fin}(\mathcal{M})$ of any structure $\mathcal{M}$ we will be able to interpret the universe M of $\mathcal{M}$ and to define the application of elements of the semigroup on M. This is also true of any semigroup of partial automorphisms that contains $I_{fin}(\mathcal{M})$. This interpretation is essential to our work, and therefore we restrict ourselves to those semigroups that contain $I_{fin}(\mathcal{M})$. Lipacheva [10] has studied semigroups of finite partial automorphisms and found a general condition for two structures to have such semigroups isomorphic. We extend her result.

We say that a structure is *computable* if its domain is computable and there is a uniform algorithmic procedure that can decide whether any relation holds of any tuple of elements of the structure, and find the value of any of the structure's operations on any appropriate tuple. For computable structures, we are typically interested in *computable isomorphisms*—those isomorphisms that can be calculated algorithmically—rather than classical isomorphisms. It is possible for two computable structures to be isomorphic, but for this isomorphism to not be realized by any computable function. When working in the context of computable structures, we are usually more interested in computable isomorphisms, because we wish to restrict our universe to only those objects and functions which are algorithmic. (We will use the symbol $\cong_c$ to signify that structures are computably isomorphic.) When talking about local symmetries of computable objects then, we will be interested in computable local symmetries, which are realized by *partial computable automorphisms.* These are the partial automorphisms f for which there is an algorithm that, when given any element x in the domain of f as input, will output $f(x)$, and when given any element not in the domain of f as input will fail to halt. The set of all partial computable functions of a structure (denoted by $I_c(\mathcal{M})$ for a structure $\mathcal{M}$) also forms an inverse semigroup under function composition. Since any finite function is a partial computable function, we will have that $I_{fin}(\mathcal{M}) \subseteq I_c(\mathcal{M})$ for any structure $\mathcal{M}$.

A notion of equivalence of structures weaker than isomorphism is that of *elementary equivalence*. Two structures are said to be elementarily equivalent if the same first-order sentences are true of both structures. There are many examples of structures which are elementarily equivalent but not isomorphic. For example, the partial orders of type ω (the natural numbers) and $\omega + \zeta$ (the natural numbers followed by a copy of the integers) are elementarily equivalent, but not isomorphic.

Here, we are mostly interested in what we can deduce from elementary equivalence or isomorphism of finite partial automorphism semigroups, or of partial computable automorphism semigroups. We begin by giving interpretations described above of the universe of a general structure and the action of the semigroup on this interpreted copy of the universe. These basic interpretations are necessary for all the following results. We then begin looking at particular types of structures (e.g., equivalence structures, partial orders, Boolean algebras), and attempt to interpret as much of the relations of this structures as possible into the semigroup, with varying results.

We begin by examining equivalence structures. For structures with a single binary equivalence relation, we are able to completely define the relation within the semigroup structure. We are then able to deduce that elementary equivalence of the I_{fin}'s implies isomorphism of the original structures. We have an analogous result in the computable case: elementary equivalence of I_c's of computable equivalence structures implies computable isomorphism of the original structures.

We then examine partially ordered structures. For any partially-ordered structure $\langle M, < \rangle$, we may define its *reverse order* simply by saying that $a < b$ in the reverse order if and only if $b < a$ in the original order. Here, we are only able to define the order within the semigroups modulo reversal, and therefore our conclusions are weaker in this case. We are able to conclude that elementary equivalence of I_{fin}'s implies that the original structures are elementarily equivalent, up to reversal of the ordering of one of the structures. For orderings that happen to be elementarily equivalent to their reversals (such as Boolean algebras), we are able to strengthen the conclusion and say that the original structures are in fact elementarily equivalent.

We next consider Boolean algebras in their natural language, consisting of symbols for meet, join, greatest element, least element, and complement, and a related structure called a *relatively complemented distributive lattice* (RCDL). For these structures we are able to conclude that under the right conditions, isomorphism between I_c's implies computable isomorphism of the original structures.

There are many possibilities for future research in this area. We might consider if similar results can be found for structures with richer algebraic dependence such as groups, rings and fields, etc. We could also try to find results of this type for structures with infinite languages, or try to find a general condition that is equivalent to two structures having elementarily equivalent semigroups of finite partial automorphisms.

Chapter 2

Background on Automorphism Groups

The work presented here deals with recovering properties of structures from the isomorphism type or first-order theory of different semigroups of various types of partial automorphisms of those structures. Similar results have been obtained in the context of automorphism groups of structures. We use $Aut(\mathcal{M})$ to denote the group of automorphisms of a structure $\mathcal{M}$, and $Aut_c(\mathcal{M})$ to denote the group of computable automorphisms of $\mathcal{M}$. The most notable of the results are about Boolean algebras. We give the precise definition of Boolean algebra here:

Definition 2.0.1 *A Boolean algebra is a structure $\mathcal{B}$ in the language $\langle \cap, \cup, ^{-}, 0, 1\rangle$, where $\cap$ and $\cup$ are binary function symbols, $^{-}$ is a unary function symbol, and 0 and 1 are constant symbols such that the following axioms are satisfied for all*

$a, b, c \in B$:

1. $(a \cup b) \cup c = a \cup (b \cup c)$ *and* $(a \cap b) \cap c = a \cap (b \cap c)$ *(associativity)*

2. $a \cup b = b \cup a$ *and* $a \cap b = b \cap a$ *(commutativity)*

3. $a \cup (a \cap b) = a$ *and* $a \cap (a \cup b) = a$ *(absorption).*

4. $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ *and* $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ *(distributivity)*

5. $a \cup \bar{a} = 1$ *and* $a \cap \bar{a} = 0$ *(complementation)*

There is a natural partial order associated with any Boolean algebra. We define:

$$a \subseteq b \Leftrightarrow_{def} a \cap b = a.$$

Again, thinking of a Boolean algebra as a collection of subsets of some set, the partial ordering $\subseteq$ as defined corresponds to set inclusion. Throughout this work, the symbol $\subset$ will always be used for *strict* inclusion.

The above definition makes it convenient to define some terms. We say that a nonzero element a in a Boolean algebra $\mathcal{B}$ is an *atom* if there is no $b \in \mathcal{B}$ such that $0 \subset b \subset a$. An element $b \in \mathcal{B}$ is said to be *atomless* if there is no atom $a \in \mathcal{B}$ such that $a \subset b$. We say that an element $a \in \mathcal{B}$ is *atomic* if there is no $b \in \mathcal{B}$ such that $b \subseteq a$ and b is atomless. A Boolean algebra $\mathcal{B}$ is atomic if it has no atomless elements.

The following result of McKenzie [12] establishes conditions under which the isomorphism type of a countable Boolean algebra can be recovered from the isomorphism type of the associated automorphism group:

Theorem 2.0.2 *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be Boolean algebras, each with at least one atom, such that $\mathcal{B}_0$ has a maximal atomic element. Then*

$$Aut(\mathcal{B}_0) \cong Aut(\mathcal{B}_1) \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1.$$

Morozov [13] considered computable automorphism groups of computable atomic Boolean algebras and proved the following:

Theorem 2.0.3 *Let $\mathcal{B}_0$ be an atomic computable Boolean algebra with a computable set of atoms, and let $\mathcal{B}_1$ be an arbitrary computable Boolean algebra. Then*

$$Aut_c(\mathcal{B}_0) \cong Aut_c(\mathcal{B}_1) \Rightarrow \mathcal{B}_0 \cong_c \mathcal{B}_1.$$

Proof Sketch. We first consider the case when $\mathcal{B}_0$ is finite. Let n be the number of atoms of $\mathcal{B}_0$. An automorphism of a finite Boolean algebra may permute the atoms in any way, but once the images of the atoms are known, the entire automorphism is determined. Of course, any such automorphism is computable, hence $|Aut_c(\mathcal{B}_0)| = n!$, and we must have $|Aut_c(\mathcal{B}_1)| = n!$. Therefore, $\mathcal{B}_1$ must also have n atoms, and hence $\mathcal{B}_0 \cong_c \mathcal{B}_1$ by any injective order-preserving function taking atoms to atoms.

The case when $\mathcal{B}_0$ is infinite involves the interpretation of the structure of the Boolean algebra in its computable automorphism group. In the context of automorphisms of Boolean algebras, by *transposition* we mean an automorphism that permutes exactly two atoms and fixes the rest of the atoms and all atomless elements. Notice that this differs from the usual definition of a transposition in a permutation group, because here there may be more than two elements that are not fixed. For group elements, we have the following notation:

$$[x,y] = x^{-1}y^{-1}xy,$$

and,

$$x^y = y^{-1}xy.$$

We begin with a first-order definition of transpositions via the following formula:

$$tr(x) \Leftrightarrow_{def} (x^2 = 1) \wedge (x \neq 1) \wedge \forall y([x,y]^6 = 1).$$

If $\mathcal{B}$ is a Boolean algebra with an infinite set of atoms, then

$$Aut(\mathcal{B}) \models tr(\tau) \Leftrightarrow \tau \text{ is a transposition.}$$

This will allow us to define the set of atoms within $Aut(\mathcal{B}_i)$ as equivalence classes of pairs of transpositions that have a single atom in common, as follows. A pair of transpositions holds exactly one element in common if and only if they do not commute. For $i = 0, 1$, let

$$A_{\mathcal{B}_i} =_{def} \{(\tau_0, \tau_1) \mid Aut(\mathcal{B}_i) \models tr(\tau_0) \wedge tr(\tau_1) \wedge [\tau_0, \tau_1] \neq 1\}.$$

Two pairs of transpositions will be equivalent if they hold the same common element. We will be able to define the equivalence of pairs with the following first-order formula:

$$E(\tau_0, \tau_1, \pi_0, \pi_1) \Leftrightarrow_{def} \bigwedge_{i,j \in \{0,1\}} (\pi_i \tau_j)^3 = 1 \wedge \bigwedge_{i,j \in \{0,1\}} \pi_i \neq \tau_{1-j}^{\tau_j},$$

Now, $E(\tau_0, \tau_1, \pi_0, \pi_1)$ is equivalent to the statement that the pairs (τ_0, τ_1) and (π_0, π_1) represent the same atom. Define

$$(\tau_0, \tau_1) \sim_E (\pi_0, \pi_1) \Leftrightarrow E(\tau_0, \tau_1, \pi_0, \pi_1).$$

The relation $\sim_E$ is an equivalence, and for $i = 0, 1$, we let $at(\mathcal{B}_i)$ be the factor set $A_{\mathcal{B}_i} / \sim_E$, and denote the equivalence class of (τ_0, τ_1) in $at(\mathcal{B}_i)$ as $\alpha(\tau_0, \tau_1)$.

It makes sense to apply any automorphism in $Aut(\mathcal{B}_i)$ to any atom in $\mathcal{B}_i$, and we wish to define this application within $Aut(\mathcal{B}_i)$. The natural definition of application of $\phi \in Aut(\mathcal{B}_i)$ to an atom $\alpha(\tau_0, \tau_1) \in at(\mathcal{B}_i)$ by $ap(\phi, \alpha(\tau_0, \tau_1)) = \alpha(\tau_0^\phi, \tau_1^\phi)$. This is well-defined, and gives the desired result.

A *two-sorted structure* is a structure whose universe consists of elements of two different types (we may consider them to be two disjoint sets), such that each argument of a relational symbol and each argument and output of a function symbol must be of one specified type. For example,

$$\langle Aut_c(\mathcal{B}); at(\mathcal{B}), ap \rangle$$

is a two-sorted structure, where ap is a binary function symbol of type

$$Aut_c(\mathcal{B}) \times at(\mathcal{B}) \to at(\mathcal{B}).$$

We summarize the above in the following lemma:

Lemma 2.0.4 *The two-sorted structure structure $\langle Aut_c(\mathcal{B}); at(\mathcal{B}), ap \rangle$ is first-order definable in $Aut_c(\mathcal{B})$ for any Boolean algebra $\mathcal{B}$ with an infinite set of atoms.*

Now, for any automorphism $\phi \in \mathcal{B}_1$ that moves an atomless element, $Aut_c(\mathcal{B}_1) \models \neg tr(\phi)$. So if $\mathcal{B}_1$ were to contain a finite number of atoms, there would be only finitely many $\phi \in \mathcal{B}_1$ such that $Aut_c(\mathcal{B}_1) \models tr(\phi)$. But this would contradict $Aut_c(\mathcal{B}_0) \cong Aut_c(\mathcal{B}_1)$. Hence, $\mathcal{B}_1$ has an infinite set of atoms. It follows that:

Lemma 2.0.5 *The two-sorted structures $\langle Aut_c(\mathcal{B}_0); at(\mathcal{B}_0), ap\rangle$ and $\langle Aut_c(\mathcal{B}_1); at(\mathcal{B}_1), ap\rangle$ are isomorphic.*

Denote the restriction of this isomorphism to $at(\mathcal{B}_0)$ by ϕ. It can be shown that ϕ is computable.

The fact that $\mathcal{B}_1$ is also atomic is immediate from Lemma 2.0.5, because if $\mathcal{B}_1$ has an atomless element, then it has a computable automorphism that fixes all the atoms. However, $\mathcal{B}_0$ has no such automorphism.

By $sup(X)$, where X is a set of elements in a Boolean algebra, we mean the least upper bound of the elements under $\subseteq$, if such a bound exists. The following lemma, stated without proof, allows us to finish the proof of the theorem:

Lemma 2.0.6 *The element $sup(X)$ exists in $\mathcal{B}_0$ if and only if $sup\{\phi(\alpha) \mid \alpha \in X$ exists in $\mathcal{B}_1$.*

We can now extend ϕ to an isomorphism

$$\bar{\phi} : \mathcal{B}_0 \to \mathcal{B}_1 \text{ by } \phi(a) = sup\{\phi(\alpha) \mid \alpha \leq a \text{ and } \alpha \text{ is an atom}\}.$$

The function $\bar{\phi}$ may also be shown to be computable. The proof of computability of ϕ and $\bar{\phi}$ is highly technical. See [13] for details. ■

The previous result shows that the key to recovery of a structure from a group of automorphisms of a structure is interpretability of the base structure in the automorphism group. The same strategy will be used in recovery of structures from their semigroups of *partial* automorphisms.

Chapter 3

Basic Interpretations

Our goal is to give results similar in flavor to those of the previous chapter. Instead of considering automorphism groups of structures, however, we begin with semigroups of certain partial automorphisms of structures. Here we develop the necessary framework to be able to interpret the universe of any structure $\mathcal{M}$ within $I_{fin}(\mathcal{M})$ (or any inverse semigroup of partial automorphisms containing all the finite ones), and also to interpret the action of the semigroup on the universe of $\mathcal{M}$.

3.1 Definitions

A *semigroup* is a structure $\mathcal{S} = \langle S, \cdot \rangle$, where $\cdot$ is an associative binary operation. If a and b are elements of a semigroup S, a is an *inverse* of b if:

$$a = a \cdot b \cdot a \wedge b = b \cdot a \cdot b$$

An *inverse semigroup* is a semigroup were every element has a unique inverse. Note that the inverse is first-order definable from the semigroup operation, so a formula involving inverses may still be considered to be a formula in the language of semigroups.

For a structure $\mathcal{M}$ in the language $\mathcal{L}$, a *partial automorphism* f, is an injective partial function from $\mathcal{M}$ to $\mathcal{M}$ such that for any atomic formula θ in the language $\mathcal{L}$, and appropriate tuple $\bar{a}$,

$$\mathcal{M} \models \theta(\bar{a}) \Leftrightarrow \mathcal{M} \models \theta(f(\bar{a})).$$

We admit the possibility that a partial automorphism be the empty function. A *finite partial automorphism* is partial automorphism with finite domain. For any countable structure $\mathcal{S}$, we let $I_{fin}(\mathcal{M})$ be the set of all finite partial automorphisms of $\mathcal{M}$. If $p, q \in I_{fin}(\mathcal{M})$, the composition $p \cdot q$ clearly has finite domain and also preserves satisfaction of atomic formulas, and therefore is a finite partial automorphism. Thus, $I_{fin}(\mathcal{M})$ is a semigroup. Also the inverse partial function p^{-1} exists since p is injective and also is clearly a finite partial automorphism. The function p^{-1} acts as the unique inverse to p in the semigroup $I_{fin}(\mathcal{B})$, which is therefore an inverse semigroup.

For countable structures, we may also consider *partial computable automorphisms*, those partial automorphisms which are partial computable functions. Again, for a given structure $\mathcal{M}$, the set of partial computable automorphisms, denoted $I_c(\mathcal{M})$, forms an inverse semigroup under function composition.

3.2 Semigroups of Partial Automorphisms

We next give general methods for interpreting a structure into a partial automorphism semigroup. The methods work for semigroups of finite partial automorphisms, as well as any inverse semigroup of partial automorphisms that contains all the finite partial automorphisms (including the semigroup of partial computable automorphisms). For the remainder of this section, we let $\mathcal{M}$ be a structure, and $I \supseteq I_{fin}(\mathcal{M})$ be an inverse semigroup of partial automorphisms of the structure. We define a first-order formula in the language of semigroups $S(x)$ as follows:

$$S(x) \Leftrightarrow_{def} x^2 = x,$$

and let

$$S(\mathcal{M}) = \{f \in I \mid f^2 = f\}.$$

Any x satisfying $S(x)$ is called idempotent. An injective function is idempotent exactly when it is the identity on its domain. Therefore, we may naturally identify such functions with a subset of M, namely the domain of the function. In the case that the semigroup we are considering is $I_{fin}(\mathcal{M})$, $S(x)$ defines the finite subsets of M. If we are considering $I_c(\mathcal{M})$, $S(x)$ defines computably enumerable subsets of M.

Naturally, we wish to define the containment relation on pairs of subsets from $S(\mathcal{M})$. We again define a formula in the language of semigroups:

$$x \subseteq y \Leftrightarrow_{def} S(x) \wedge S(y) \wedge x \cdot y = x$$

For x and y such that $I \models S(x) \wedge S(y)$, we now have:

$$(I \models x \subseteq y) \Leftrightarrow (dom(x) \subseteq dom(y)).$$

The empty function is first-order definable as the unique element of I satisfying:

$$x = 0 \Leftrightarrow_{def} \forall y (S(y) \Rightarrow x \subseteq y)$$

We will use Λ to denote the empty function.

Next we will have a formula $A(x)$ that defines the set of *atoms*—those functions whose domain consists of a single element that is mapped to itself:

$$A(x) \Leftrightarrow_{def} x \neq 0 \wedge \forall y [y \subseteq x \Rightarrow (y = x \vee y = 0)].$$

We let $A(\mathcal{M})$ be $\{x \in I \mid A(x)\}$. Every element $a \in M$ corresponds to the finite partial automorphism $\{(a,a)\} \in A(\mathcal{M})$, so we may identify M with the subset $A(\mathcal{M})$ of I. For $a \in M$, we let a^* denote the partial automorphism $\{(a,a)\}$. For $b \in A(\mathcal{M})$, we will denote by $\check{b}$ the unique element in M such that $\check{b}^* = b$. We will in some cases identify an element $a \in \mathcal{M}$ with a^* when it is clear from the context whether we are referring to an element of $\mathcal{M}$ or an element of $I(\mathcal{M})$. Now that we have an interpretation of the universe of $\mathcal{M}$ in I, we wish to interpret the action of partial automorphisms in I on elements of $A(\mathcal{M})$. We define the following formula:

$$ap(f, x) = y \Leftrightarrow_{def} (f \cdot x \cdot f^{-1} = y \wedge A(x)).$$

We will usually use $f(x) = y$ as shorthand for $ap(f, x) = y$.

If we apply a partial automorphism f to an atom $x = a^*$, where $a \in dom(f)$, we expect formula $f(x) = y$ to hold exactly when y is the atom $f(a)^*$. A quick computation shows $f \cdot x \cdot f^{-1}$ is indeed $f(a)^*$ in this case. If $x \notin dom(f)$, then $f(x) = y$ holds if and only if y is the empty function.

Now, the two-sorted structure

$$I^* = \langle I, M \cup \{\Lambda\}; ap, \cdot, ^{-1} \rangle$$

is first-order definable in I.

The above considerations immediately yield the following proposition:

Proposition 3.2.1 *1. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be structures and $I_i, i = 0, 1$, be subsemigroups of $I(\mathcal{M}_0)$ and $I(\mathcal{M}_1)$, respectively, such that $I_i \supseteq I_{fin}(\mathcal{M}_i)$. Then any isomorphism $\lambda : I_0 \to I_1$ can be uniquely extended to an isomorphism of the two-sorted structures I_0^* and I_1^* (there is a bijection $\lambda' : \mathcal{M}_0 \to \mathcal{M}_1$ such that $\langle \lambda, \lambda' \rangle$ is an isomorphism from I_0^* to I_1^*.)*

2. Let $\mathcal{M}$ be a structure and $I \subseteq I(\mathcal{M})$ be such that $I \supseteq I_{fin}(\mathcal{M})$. Then each first-order formula $\phi(\bar{x})$ in the language of the structure I^* with free variables all of type I can effectively be transformed into a formula $\phi'(\bar{x})$ such that

$$I^* \models \phi(\bar{a}) \Leftrightarrow I \models \phi'(\bar{a})$$

for any tuple $\bar{a} \subset I$. ■

3.3 Isomorphic Partial Automorphism Semigroups

We begin with a general characterization of when two structures have isomorphic semigroups of partial automorphisms containing all the finite partial automorphisms. The following is a slight generalization of a result due to Lipacheva [10]:

Theorem 3.3.1 *Let $\mathcal{A} = \langle A; P_0, \dots P_m \rangle$ and $\mathcal{B} = \langle B; Q_0, \dots Q_n \rangle$ be arbitrary structures for finite predicate languages $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. The following are equivalent:*

1. *$I_{fin}(\mathcal{A}) \cong I_{fin}(\mathcal{B})$;*
2. *There exist $I \subseteq I(\mathcal{A})$ and $J \subseteq I(\mathcal{B})$ such that $I \supseteq I_{fin}(\mathcal{A})$, $J \supseteq I_{fin}(\mathcal{B})$ and $I \cong J$;*
3. *There is a bijection $f : A \to B$, such that for each $i \leq n$, the set $f(P_i)$ is definable in $\mathcal{B}$ by a quantifier-free formula, and for each $j \leq m$, $f^{-1}(Q_j)$ is definable in $\mathcal{A}$ by a quantifier-free formula.*

Proof.

Clearly $1 \Rightarrow 2$. To show $2 \Rightarrow 3$, let I and J be as in 2 and let $\lambda : I \to J$. By Proposition 3.2.1, there is an $f : A \to B$, such that the pair of mappings (λ, f) is an isomorphism between the two-sorted models I^* and J^*. For each tuple $\bar{a}$ of elements from A, we let $orb(\bar{a})$, the *orbit* of $\bar{a}$, be the set of tuples

$$\{p(\bar{a}) \mid p \in I \wedge \bar{a} \subseteq dom(p)\}.$$

Note that the orbit of $\bar{a}$ is the set of all tuples $\bar{x}$ of elements of A of the appropriate length satisfying the same quantifier-free formulas as $\bar{a}$. Now, $\mathcal{L}_0$ is a finite predicate language, so there are only finitely many quantifier-free formulas in n variables up to equivalence. Thus, for each fixed $n < \omega$, the set of orbits of n-tuples must be finite. The set $S_i = \{\bar{x} \mid \mathcal{A} \models P_i(\bar{x})\}$ is closed under application of $p \in I$ such that $\bar{x} \subseteq dom(p)$. By definition of orbit, if $\bar{x} \in S_i$, then $orb(\bar{x}) \subseteq S_i$, hence each S_i must be the union of a (finite) collection of orbits. The isomorphism (λ, f) preserves application, and hence orbits. Thus the set $f(S_i)$ is a finite union of orbits of tuples $\bar{b}$ of elements of B, each of which may be distinguished by a quantifier-free formula. Then $f(P_i)$ is definable by the disjunction of those quantifier-free formulas. A symmetric argument shows that for each $j \leq m$, $f^{-1}(Q_j)$ is definable in $\mathcal{A}$ by a quantifier-free formula.

To show $3 \Rightarrow 1$, let $f : A \to B$ be a bijection with the described property. Define $\lambda : I_{fin}(\mathcal{A}) \to I_{fin}(\mathcal{B})$ by $\lambda(p) = f \cdot p \cdot f^{-1}$. We first check that for all p, $\lambda(p) \in I_{fin}(\mathcal{B})$. Clearly $\lambda(p)$ is an injective finite partial function $B \to B$. For $j \leq m$, we have $f^{-1}(Q_j)$ is definable in $\mathcal{A}$ by a quantifier-free formula $\theta(\bar{x})$. So if $\bar{b} \subseteq dom(\lambda(p))$ is a tuple of appropriate length, we have:

$$\mathcal{B} \models Q_j(\bar{b}) \Leftrightarrow \mathcal{A} \models \theta(f^{-1}(\bar{b})) \Leftrightarrow \mathcal{A} \models \theta(p \cdot f^{-1}(\bar{b})) \Leftrightarrow \mathcal{B} \models Q_j(f \cdot p \cdot f^{-1}(\bar{b})),$$

where the second equivalence holds because p is a partial automorphism. Thus, $\lambda(p)$ preserves the relations on $\mathcal{B}$ and is a finite partial automorphism. Clearly, λ respects the semigroup operation. We have:

$$\lambda(p) = \lambda(q) \Rightarrow f^{-1} \cdot f \cdot p \cdot f^{-1} \cdot f = f^{-1} \cdot f \cdot q \cdot f^{-1} \cdot f \Rightarrow p = q,$$

so λ is injective. It is surjective, since for $q \in I_{fin}(\mathcal{B})$, we have $\lambda(f^{-1} \cdot q \cdot f) = q$, where $f^{-1} \cdot q \cdot f \in I_{fin}(\mathcal{A})$, by a symmetric argument to the one that showed $\lambda(p) \in I_{fin}(\mathcal{B})$. ■

Chapter 4

Semigroups of Partial Automorphisms of Equivalence Structures and Partial Orders

4.1 Equivalence Structures

We call an equivalence relation *nontrivial* if it not the same relation as equality. The following theorem tells us what we can deduce about an equivalence structure $\mathcal{M}$, given the isomorphism type or first-order theory of $I_{fin}(\mathcal{M})$, or of any inverse semigroup of partial automorphisms of the structure that contains $I_{fin}(\mathcal{M})$. Essentially, we can recover the isomorphism type or first-order theory of a nontrivial equivalence structure in general from the isomorphism type or first-order theory, respectively, of such a semigroup. In the case where the structure $\mathcal{M}$ is countable, we can even recover the isomorphism type of $\mathcal{M}$ from the first-order theory of $I_{fin}(\mathcal{M})$.

Theorem 4.1.1 *[5]*

Let $\mathcal{M}_0 = \langle M_0, E_0 \rangle$ and $\mathcal{M}_1 = \langle M_1, E_0 \rangle$ be structures for a language with a single binary relation symbol E, where both E_0 and E_1 are nontrivial equivalence relations. Then the following are true:

1. $I_{fin}(\mathcal{M}_0) \cong I_{fin}(\mathcal{M}_1) \Leftrightarrow \mathcal{M}_0 \cong \mathcal{M}_1$.

   Moreover, if I_0 and I_1 are inverse semigroups such that

   $$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i) \text{ for } i = 0, 1,$$

   and $I_0 \cong I_1$, then $\mathcal{M}_0 \cong \mathcal{M}_1$.

2. $I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \Rightarrow \mathcal{M}_0 \equiv \mathcal{M}_1$.

   Moreover, if I_0 and I_1 are inverse semigroups such that

   $$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i) \text{ for } i = 0, 1,$$

   and $I_0 \equiv I_1$, then $\mathcal{M}_0 \equiv \mathcal{M}_1$.

3. *If M_0 and M_1 are both countable, then*

   $$I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \Leftrightarrow \mathcal{M}_0 \cong \mathcal{M}_1.$$

Proof. Let I_i be an inverse semigroup such that $I_{fin}(\mathcal{M}_i) \subseteq I(\mathcal{M}_i)$. We wish to find an interpretation of the equivalence relation E in the language of semigroups. That is, we wish to find a formula E^* in two free variables such that for $i = 0, 1$, and for all $a, b \in M_i$,

$$I_i \models E^*(a^*, b^*) \Leftrightarrow \mathcal{M}_i \models E(a, b).$$

Using the abbreviations defined in section 3.2, the formula $E^*(x, y)$ is as follows:

$$A(x) \wedge A(y) \wedge \forall a, b, c[(A(a) \wedge A(b) \wedge A(c) \wedge a \neq c \wedge$$

$$\exists f[(f(x), f(y)) = (a, b)] \wedge \exists g[(g(x), g(y)) = (b, c)]) \Rightarrow$$

$$(\exists h[(h(x), h(y)) = (a, c)])]$$

By transitivity of the equivalence relation, we have that for all $a, b \in M_i$,

$$\mathcal{M}_i \models E(a, b) \Rightarrow I_i \models E^*(a^*, b^*).$$

Now, suppose $\mathcal{M}_i \models \neg E(a, b)$. Because the equivalence relations are nontrivial, we have $m, n \in M$ such that $\mathcal{M}_i \models E(m, n) \wedge m \neq n$. Let $l \in M$ be such that $\mathcal{M}_i \models \neg E(m, l)$ (by our assumption that $\mathcal{M}_i \models \neg E(a, b)$ we know that there is more than one equivalence class). Now, there are $f, g \in I_{fin}(\mathcal{M}_i)$ such that $m \neq n$, $(f(a), f(b)) = (m, l)$ and $(g(a), (g(b)) = (l, n)$, but no $h \in I_{fin}(\mathcal{M}_i)$ such that $(h(a), h(b)) = (m, n)$ (see Figure 1). Therefore, $I_i \models \neg E^*(a^*, b^*)$, as desired.

Now that we are able to define the equivalence relation E by a formula in the language of semigroups, we may use this as the base step in an induction

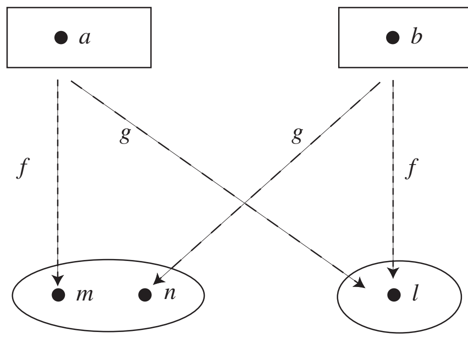

Figure 4.1: Non-equivalent elements

to show that for any formula ψ in the language of equivalence structures, there a corresponding formula ψ^* in the language of inverse semigroups such that for any tuple $\bar{a}$ of elements in $\mathcal{M}_i$,

$$\mathcal{M}_i \models \psi(\bar{a}) \Leftrightarrow I_i \models \psi^*(\bar{a}^*).$$

This holds in particular when ψ is a sentence. Therefore, if $I_0 \equiv I_1$ we may conclude that $\mathcal{M}_0 \equiv \mathcal{M}_1$, and 2 holds.

To see 1, we assume $\lambda : I_0 \to I_1$ is an isomorphism. By Proposition 3.2.1, there is an $f : M_0 \to M_1$ such that (λ, f) is an isomorphism of two-sorted structures from $I_1^* \to I_2^*$. We have

$$\mathcal{M}_0 \models E(a,b) \Leftrightarrow I_0^* \models E^*(a^*, b^*) \Leftrightarrow$$

$$I_1^* \models E^*(\lambda(a^*), \lambda(b^*)) \Leftrightarrow \mathcal{M}_1 \models E(f(a), f(b)),$$

since we must have $f(a)^* = \lambda(a^*)$.

It remains to prove 3. If two equivalence structures are elementarily equivalent, then for any natural number k, they must each contain the same cardinality of equivalence classes of size k. So we need only show that $\mathcal{M}_0$ and $\mathcal{M}_1$ contain the same cardinality of countable infinite equivalence classes. To do this, we utilize the fact that we can define the finite subsets of M_i within the semigroup $I_{fin}(M_i)$.

For any integer n, we have a sentence θ_n in the language of semigroups, such that $I_{fin}(\mathcal{M}_i) \models \theta_n$ if and only if $\mathcal{M}_i$ contains exactly n infinite equivalence classes. The sentence θ_n should say that $\mathcal{M}_i$ contains exactly n distinct equivalence classes that are not contained in any finite subset of M. All of this is first-order expressible using the above definitions of formulas for finite subset, containment and for the equivalence relation given above. If $I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1)$ and $\mathcal{M}_0$ contains exactly n infinite equivalence classes, then $I_{fin}(\mathcal{M}_0) \models \theta_n$, and so $I_{fin}(\mathcal{M}_1) \models \theta_n$, and $\mathcal{M}_1$ contains exactly n infinite equivalence classes. On the other hand, if there are infinitely many infinite equivalence classes in $\mathcal{M}_0$, then none of θ_n hold in $I_{fin}(\mathcal{M}_0)$, and hence none hold in $I_{fin}(\mathcal{M}_1)$, and $\mathcal{M}_1$ also has infinitely many infinite equivalence classes. In either case, $\mathcal{M}_0 \cong \mathcal{M}_1$.

■

Remarks 1. The converse of 2 doesn't hold. Indeed, consider countable equivalence structures $\mathcal{M} = \langle M_0, E_0 \rangle$ such that E_0 contains exactly one equivalence class of size n for each $n < \omega$, and no infinite equivalence classes, and $\mathcal{M}_1 = \langle M_1, E_1 \rangle$ such that E_1 contains exactly one equivalence class of size n for each $n < \omega$, and exactly one infinite equivalence class. Then $\mathcal{M}_0 \equiv \mathcal{M}_1$. This is easy to see from the point of view of Ehrenfeucht-Fraïssé games, since in any n-move game, all equivalence classes of size at least n are the same from the point of view of the game. But if $I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1)$, we have $\mathcal{M}_0 \cong \mathcal{M}_1$ by 3, a contradiction.

2. The cardinality criterion cannot be omitted from 3. Moreover, if κ is any uncountable cardinal, 3 is false with ω replaced by κ. Indeed, consider $\mathcal{M}_0 = \langle M_0, E_0 \rangle$ and $\mathcal{M}_1 = \langle \mathcal{M}_1, E_1 \rangle$ such that M_0 and M_1 each have cardinality κ, where E_0 and E_1 each consist of infinitely many infinite equivalence classes, and no finite equivalence classes. Clearly, we may choose such $\mathcal{M}_0$ and $\mathcal{M}_1$ with $\mathcal{M}_0 \ncong \mathcal{M}_1$, say by letting E_0 have exactly one countably infinite equivalence class, with the remaining classes of cardinality κ, and letting E_1 have no countable equivalence classes. We wish to show that $I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1)$. Recall that a family S of partial isomorphisms from a structure $\mathcal{A}$ to a structure $\mathcal{B}$ has the *back-and-forth property* if for all $f \in S$ the following hold:

1. For all $a \in A$, there exists $f' \in S$ such that $f \subseteq f'$ and $a \in dom(f')$.

2. For all $b \in B$, there exists $f' \in S$ such that $f \subseteq f'$ and $b \in ran(f')$

We say that structures $\mathcal{A}$ and $\mathcal{B}$ are *partially isomorphic* if there exists a family S of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$ such that S has the back-and-forth property. It follows from Karp's Theorem [2] that if two structures are partially isomorphic then they are elementarily equivalent.

Let P be the set of finite partial isomorphisms from $\mathcal{M}_0$ to $\mathcal{M}_1$. For $f \in P$, define:

$$G_f = \{\langle h, f \cdot h \cdot f^{-1} \rangle \mid h \in I_{fin}(\mathcal{M}_0) \wedge dom(h) \cup ran(h) \subseteq dom(f)\}.$$

We claim that that for each $f \in P$, $G_f : I_{fin}(\mathcal{M}_0) \to I_{fin}(\mathcal{M}_1)$ is a finite partial isomorphism. Clearly, $f \cdot h \cdot f^{-1} \in I_{fin}(\mathcal{M}_1)$, and G_f has finite domain and is injective. If $g, h \in I_{fin}$, we have

$$G_f(h) \cdot G_f(g) = (fhf^{-1}) \cdot (fgf^{-1}) = fhgf^{-1} = G_f(hg).$$

Notice that in the second equality, we are using the fact that $ran(g) \subseteq dom(f)$, otherwise it isn't necessarily the case that $f^{-1}fg = g$, since we are dealing with partial functions.

Now, define a set S of partial isomorphisms from $I_{fin}(\mathcal{M}_0)$ to $I_{fin}(\mathcal{M}_1)$ by:

$$S = \{G_f \mid f \in P\}.$$

We will prove that $I_{fin}(\mathcal{M}_0)$ and $I_{fin}(\mathcal{M}_1)$ are partially isomorphic by showing that S has the back-and-forth property. Let $f \in P$ and $h \notin G_f$. Then $dom(h) \cup ran(h) \nsubseteq dom(f)$. We wish to find an extension $f' \supset f$ such that $f' : \mathcal{M}_0 \to \mathcal{M}_1$ is a finite partial isomorphism and $dom(h) \cup ran(h) \subseteq dom(f)$. This is clearly possible, since we are adding only finitely many elements that are not already in $dom(f)$, and there are infinitely many equivalence classes of E_1, all of which are infinite. Now, if $f \subseteq f'$, then $G_f \subseteq G_{f'}$ since $dom(f) \subseteq dom(f')$, and if $h \in dom(G_f)$, then clearly $G_f(h) = G_{f'}(h)$. Hence $G_{f'}$ is the desired extension of G_f. Observe that if $h \in I_{fin}(\mathcal{M}_1)$,

$$h \in ran(G_f) \Leftrightarrow \exists g \in I_{fin}(\mathcal{M}_0) \text{ with}$$

$$dom(g) \cup range(g) \subseteq dom(f), h = fgf^{-1} \Leftrightarrow dom(h) \cup ran(h) \subseteq ran(f),$$

It then follows symmetrically that we may extend an arbitrary G_f to $G_{f'}$ such that $h \in ran(G_{f'})$.

4.2 Computable Equivalence Structures

We next consider the semigroups of partial computable isomorphisms of computable equivalence structures, and obtain a "computable analogue" to the results of the previous section. The idea is that from the first-order theory of semigroup of partial computable isomorphisms of a computable equivalence structure, we may recover the original structure up to computable isomorphism. Even better, we may distinguish each computable isomorphism type of computable equivalence structures with a single sentence in the language of semigroups. Recall that for a computable structure $\mathcal{M}$, $I_c(\mathcal{M})$ is defined to be the semigroup of all partial computable partial automorphisms of $\mathcal{M}$.

Theorem 4.2.1 *[5]*

Let $\mathcal{M}$ be a nontrivial computable equivalence structure. Then there is a first-order sentence θ in the language of semigroups such that for any computable equivalence structure $\mathcal{N}$ such that $I_c(\mathcal{N}) \models \theta$, we have $\mathcal{M} \cong_c \mathcal{N}$.

We need several lemmas.

Lemma 4.2.2 *Let $\mathcal{M} = \langle M; E\rangle$ be a computable equivalence structure. Define the formula* $\mathtt{Fin}(f)$ *in the language of semigroups as follows:*

$$\mathtt{Fin}(f) \Leftrightarrow_{def} \neg\exists g(dom(g) \subseteq dom(f) \wedge ran(g) \subset dom(g))$$

Then for any $f \in I_c(\mathcal{M})$, $I_c(\mathcal{M}) \models \mathtt{Fin}(f)$ *if and only if* $dom(f)$ *is finite.*

Proof.

If $f \in I_c(\mathcal{M})$ and $I_c(\mathcal{M}) \models \neg\mathtt{Fin}(f)$, then $dom(f)$ has a subset that is in one-to-one correspondence with a proper subset of itself, and hence is infinite.

Now assume that $dom(f)$ is infinite. We will consider the following cases:

Case 1. *There exists an* $x \in dom(f)$ *such that* $dom(f) \cap (x/_E)$ *is infinite.*

In this case, $A =_{def} dom(f) \cap (x/_E)$ is an infinite computably enumerable set, since it is the intersection of two computably enumerable sets. Any injective function with domain A and range contained in A is a partial automorphism, so there exists $g \in I_c(\mathcal{M})$ such that $dom(g) = A$, $ran(g) \subset A$, and $I_c(\mathcal{M}) \models \neg\mathtt{Fin}(f)$.

Case 2. *The set* $dom(f)$ *has nonempty intersection with infinitely many equivalence classes.*

In this case, there is an infinite computably enumerable set A consisting of exactly one element from every equivalence class which has nonempty intersection with $dom(f)$. Again, it is the case that any injective function with domain A and range contained in A is a partial automorphism, so it follows that $I_c(\mathcal{M}) \models \neg\mathtt{Fin}(f)$. ■

It follows that the two-sorted structure $I^{\diamondsuit}(\mathcal{M}) = \langle I_c(\mathcal{M}); M, \mathtt{Fin}, \in\rangle$, where $\mathtt{Fin}$ is the set of all finite subsets of M and $\in$ is the membership relation on $\mathcal{M} \times \mathtt{Fin}$, is first-order definable in $I_c(\mathcal{M})$ without parameters. This means that each formula in the language of $I^{\diamondsuit}(\mathcal{M})$ expressing a first-order property of elements of $I_c(\mathcal{M})$ can be transformed into a first-order formula expressing the same property in the language of $I_c(\mathcal{M})$. Indeed, we can identify finite sets with elements $f \in I_c(\mathcal{M})$ satisfying the formula `Fin`. Any such f_0 and f_1 are identified with the same finite set if and only if

$$I_c(\mathcal{M}) \models \forall x\,(x \in dom(f_0) \leftrightarrow x \in dom(f_1)).$$

Finally, x belongs to the set identified with f if and only if $x \in dom(f)$.

Lemma 4.2.3 *There exists a first-order formula* $\mathtt{Nat}\,(v)$ *in the language of semigroups such that for any* $p \in I_c(\mathcal{M})$*,* $I_c(\mathcal{M}) \models \mathtt{Nat}\,(p)$ *if and only if there exists an injective computable function* $f : \omega \to M$ *with the following properties:*

1. $dom(p) = \{f(i) \mid i < \omega\}$;

2. $\forall i\,(p(f(i)) = f(i+1))$;

3. $\forall i\,\forall j\,(i \neq j \to \langle f(i), f(j)\rangle \notin E)$;

4. $\forall x \exists i(\langle f(i), x\rangle \in E)$.

Proof.

The formula $\mathtt{Nat}\,(v)$ will be the conjunction of the following conditions, each of which is expressible by a first-order formula in the language of semigroups by the above considerations:

1. The set $dom(v) \setminus range(v)$ contains exactly one element.

2. Let a_0 be the unique element in the set $dom(v) \setminus range(v)$. Then for all x, there exists an $x_1 \in dom(v)$ such that $\langle x, x_1\rangle \in E$ and a finite set F such that $a_0 \in F$ and for all $t \in F \setminus \{x_1\}$, it is the case that $v(t) \in F$.

3. Any two elements in $dom(v)$ are not E–equivalent.

Clearly, if p satisfies the conditions 1-4 given in the lemma, then

$$I_c(\mathcal{M}) \models \texttt{Nat}\,(p).$$

Now assume that $I_c(\mathcal{M}) \models \texttt{Nat}\,(p)$. Clearly $dom(p)$ contains exactly one element of every equivalence class. Let $x \in dom(p)$, and let F be a finite set as in condition 2 in the definition of $\texttt{Nat}$. Consider the sequence:

$$a_0, p(a_0), p^2(a_0), p^3(a_0), \ldots.$$

If there is no $m \in \omega$ such that $p^m(a_0) = x$, then all the elements in the sequence belong to F by its definition. All of the elements of the sequence are distinct, since if $p^k(a_0) = p^l(a_0)$, for some $k < l < \omega$ then $a_0 = p^{l-k}(a_0)$, and $a_0 \in range(p)$, which is a contradiction. It follows that F contains an infinite subset, which is impossible. We define $f(i) = p^i(a_0)$, and clearly conditions 1 and 2 from the statement of the lemma must also hold. ■

We begin the proof of Theorem 4.2.1.

Proof.

We will consider two cases.

Case 1. *The equivalence relation E has finitely many equivalence classes.*

Note that the we can express the condition of case 1 holding in the language of semigroups by the sentence $\neg\exists v \texttt{Nat}\, v$. Given our above interpretations of finite sets and of the equivalence relation, we can also express the property "the equivalence class of x is infinite" by the following first-order formula:

$$\neg\exists F \in \mathtt{Fin}\forall y(E^*(x,y) \to y \in F)$$

For each $n < \omega$, there exists a first-order formula in the language of equivalence relations that is satisfied by x exactly when the equivalence class of x consists of n elements. Of course, this formula may be readily translated into the language of semigroups. We can now distinguish the cardinality of the equivalence class of any element with a formula in $I_c(\mathcal{M})$.

Suppose the equivalence structure consists of exactly m equivalence classes with cardinalities $k_0, \ldots, k_{m-1}$, where $k_i \in \omega \cup \{\omega\}$. This can be expressed in $I_c(\mathcal{M})$ by the following sentence:

$$\exists x_0, \ldots, x_{m-1}\Big[\bigwedge_{i<j<m} \langle x_i, x_j\rangle \notin E \wedge \forall x \left(\bigvee_{i<m} \langle x, x_i\rangle \in E\right) \wedge \\ \bigwedge_{i<m}(x_i/_E \textit{ contains } k_i \textit{ elements})\Big].$$

We claim this sentence is the desired θ in the statement of the theorem. If $\mathcal{N}$ is a computable equivalence structure such that the sentence is true in $I_c(\mathcal{N})$, then $\mathcal{N}$ consists of m equivalence classes with the same cardinalities as those of $\mathcal{M}$. An easy construction yields a computable isomorphism from $\mathcal{M}$ to $\mathcal{N}$. For each structure, fix a (finite) set of representatives of each equivalence class. The construction is as follows: At stage $s+1$, we will have a finite partial isomorphism f_s from $\mathcal{M}$ to $\mathcal{N}$. Let $x \in \mathcal{M}$ be the least element such that $x \notin \mathtt{dom}\,(f_s)$. Determine which of the representatives x is equivalent to, then find the least $y \in \mathcal{N}$ such that $y \notin \mathtt{ran}\,(f_s)$ and y is equivalent to the corresponding representative in $\mathcal{N}$. Such a y always exists, since the corresponding representatives are in equivalence classes of the same size. Let $f_{s+1} = f_s \cup \{(x,y)\}$. We claim $f =$

$\bigcup_s f_s$ is the desired computable isomorphism. The function f is computable by construction. It is injective and surjective since the least appropriate elements are chosen at each stage, and it clearly preserves the equivalence structure.

Case 2. *The equivalence relation E has infinitely many equivalence classes.*

The case will be broken into two subcases. In either subcase, we will need to define an isomorphic copy of the standard model of arithmetic within $I_c(\mathcal{M})$. Note that this case is distinguished by the formula $\exists v \mathtt{Nat}\,(v)$. Let $p \in I_c(\mathcal{M})$ be any element such that $I_c(\mathcal{M}) \models \mathtt{Nat}\,(p)$. We note that the action of p on its domain is the same as that of the successor function on the natural numbers up to relabeling. With this in mind, we will use p as a parameter in our definition of operations and relations of arithmetic.

We may elementarily define the zero element 0_p as the unique element in the set $\mathtt{dom}\,(p) \setminus \mathtt{ran}\,(p)$, which exists by definition of the formula $\mathtt{Nat}$. We may define the successor function on the set $\mathtt{dom}\,(p)$ as $s_p(x) = p(x)$. Of course, once we have a zero element and successor function defined, it is natural to identify the element $p^n(0_p)$ with the integer n. We may then define an ordering on the elements of $\mathtt{dom}\,(p)$, which corresponds to the usual ordering on the natural numbers as:

$$p^m(0_p) <_p p^n(0_p) \Leftrightarrow_{def} m < n,$$

which may be written in the language of semigroups as follows:

$$\begin{aligned} x <_p y \quad \Leftrightarrow_{def} \quad & x \neq y \wedge \exists S \in \mathtt{Fin}(0_p \in S \wedge x \in S \wedge y \notin S \wedge \\ & \forall t \in S \setminus \{x\}(p(t) \in S)) \end{aligned}$$

The formula utilizes the fact that we have defined the finite sets, and says that there is a finite set of elements of $\mathtt{dom}\,(p)$, of which 0_p and x are members and y is not, and which is closed under the application of p, except for the application of p to x. Such a set must be a finite sequence of the form $0_p, p(0_p), \ldots, p^n(0_p)$, where $p^n(0_p) = x$, since x is the only element that "escapes" from the set. Since y is nowhere in the sequence, we must have $x <_p y$ as desired.

We define the operations $+_p$ and $\times_p$ which correspond to usual addition and multiplication as follows:

$$
\begin{aligned}
x +_p y = z \quad \Leftrightarrow_{\text{def}} \quad & \exists f \big[\mathtt{dom}\,(f) \supseteq \{t \mid 0_p \leqslant_p t \leqslant_p y\} \wedge f(0_p) = x \wedge \\
& \forall t <_p y (s_p(f(t)) = f(s_p(t))) \wedge f(y) = z\big] \\
x \times_p y = z \quad \Leftrightarrow_{\text{def}} \quad & \exists f \big[\mathtt{dom}\,(f) \supseteq \{t \mid 0_p \leqslant_p t \leqslant_p y\} \wedge \\
& f(0_p) = 0_p \wedge \forall t <_p y ((f(s_p(t))) = f(t) +_p x) \wedge \\
& f(y) = z\big].
\end{aligned}
$$

We now consider the first of the two subcases.

Subcase 1. *The set of cardinalities of classes of E is finite.*

If the condition of subcase 1 holds, the equivalence structure is said to have *bounded character*. This subcase can be distinguished by a first-order sentence α in the language of semigroups that says there exists a finite set F, such that for any $x \in M$, there exists $y \in F$ and $p \in I_c(\mathcal{M})$ which is a bijection from $x/_E$ onto $y/_E$. We will then have $I_c(\mathcal{M}) \models \alpha$ if and only if the equivalence structure $\mathcal{M}$ has bounded character.

In this subcase, we let

$$K = \{k_0 < k_1 < \ldots < k_{m-1}\}$$

be the set of all possible cardinalities of classes of E. We do not exclude the possibility that $k_{m-1} = \omega$. For each $i < m$, let $\psi_i(v)$ be a formula saying that the cardinality of $v/_E$ equals k_i. Now E is a computable relation and p is a partial computable function, so for any fixed i, the set

$$\{n \mid \textit{the cardinality of } p^n(0_p)/_E \textit{ equals } k_i\}$$

is arithmetical, and therefore definable by a first-order formula $\varphi_i(n)$ in the language of arithmetic, with a single free variable n. We may view φ_i as a formula in the language of semigroups with parameter p, using our definitions of 0_p, $<_p$, $+_p$ and $\times_p$, and identifying elements of $\mathtt{dom}\,(p)$ with the appropriate integers as defined above, so that $\varphi_i(n)$ will hold only when $n \in \mathtt{dom}\,(p)$. We define the formula $\beta(p)$ as

$$\forall t \in \mathtt{dom}\,(p) \bigvee_{i<m} \big(\varphi_i(t) \wedge \psi_i(t)\big).$$

A word of explanation about this formula is warranted. The idea is that the φ_i code the desired size of the equivalence class of each element of $\mathtt{dom}\,(p)$ (our "integers"), and then $\psi_i(t)$ holds if each equivalence class does indeed have the correct size. Note that by the definition of the φ_i, for any given t, $I_c(\mathcal{M}) \models \varphi_i(t)$ for exactly one i. Also note that we are using the boundedness of character to be able to take a *finite* disjunction over the sizes of equivalence classes. We claim that the desired θ from the statement of the theorem is

$$\theta \Leftrightarrow_{def} \alpha \wedge \exists p(\mathtt{Nat}\,(p) \wedge \beta(p)),$$

that is, $I_c(\mathcal{N}) \models \theta$ if and only if $\mathcal{M} \cong_c \mathcal{N}$. If $I_c(\mathcal{N}) \models \theta$, then we have a $q \in I_c(\mathcal{N})$ such that $I_c(\mathcal{N}) \models \mathtt{Nat}\,(q) \wedge \beta(q)$, and such that for all $n \in \omega$, the cardinality of $p^n(0_p)/_E$ in $\mathcal{M}$ is the same as that of $q^n(0_q)/_E$ in $\mathcal{N}$.

We can easily use these p and q to construct a computable isomorphism f between $\mathcal{M}$ and $\mathcal{N}$. We construct the isomorphism in stages. At stage $s+1$, we inherit a finite partial isomorphism f_s from $\mathcal{M}$ to $\mathcal{N}$. Let $x \in \mathcal{M}$ be the least element not in the domain f_s. Find n such that $\mathcal{M} \models E(x, p^n(0_p))$. Certainly such an n exists, and can be found computably since E and p are computable. We may now computably find the least $y \in \mathcal{N}$ such that

$$y \notin \mathtt{ran}\,(f_s) \wedge \mathcal{N} \models E(y, q^n(0_q)).$$

It will always be possible to find such a y, since the size of the equivalence class of $p^n(0_p)$ must be the same as that of $q^n(0_q)$. We set $f_{s+1} = f_s \cup \{(x, y)\}$, and let $f = \bigcup\limits_s f_s$. Again, because the corresponding equivalence classes have the same size, and because we always choose the least appropriate y, f will be surjective. It is clear that f is injective, and preserves satisfaction of E, hence f is the desired computable isomorphism.

Now, if we assume $f : \mathcal{M} \to \mathcal{N}$ is a computable isomorphism, then clearly $I_c(\mathcal{N}) \models \alpha$, and $q =_{def} fpf^{-1}$ is the desired witness for the second conjunct of θ. Hence $I_c(\mathcal{N}) \models \theta$, and the first subcase is complete.

Subcase 2. *The set of cardinalities of classes of E is infinite.*

If the condition of subcase (2) holds, we say that the equivalence structure has *unbounded character.*

In this case, we will need to define a first-order formula $\mathtt{Card}\,(x, y, p)$ in the language of semigroups, which for a parameter p satisfying $I_c(\mathcal{M}) \models \mathtt{Nat}\,(p)$ expresses the following property of x and y:

> "*$x, y \in \mathtt{dom}\,(p)$ and there exists an $n < \omega$ such that $y = p^n(0_p)$ and the cardinality of $x/_E$ equals n*".

We will first proceed to show how the existence of such a formula implies the result.

First note that for any $p \in I_c(\mathcal{M})$ such that $I_c(\mathcal{M}) \models \mathtt{Nat}\,(p)$ the relation $C(m, n)$, defined as:

$$\begin{aligned} C(m,n) \quad =_{\text{def}} \quad & \{\langle m,n\rangle \mid (n = 0 \wedge p^m(0_p)/_E \textit{ is infinite }) \vee \\ & (n \neq 0 \wedge p^m(0_p)/_E \textit{ contains exactly } n \textit{ elements})\} \end{aligned}$$

is arithmetical.

Now the following sentence γ, which says the cardinalities of each class in the sequence $0_p, p(0_p), p^2(0_p), \ldots$ are actually correctly described by the arithmetical relation $C(m, n)$ can be written in the language of semigroups:

$$\begin{gathered} \exists p \left[\mathtt{Nat}\,(p) \wedge \forall m \in \mathtt{dom}\,(p)\, \exists n \in \mathtt{dom}\,(p)\, [C'(n,m) \wedge \right. \\ \left. ((n = 0_p \wedge (m/_E \textit{ is infinite})) \vee (n \neq 0_p \wedge (\mathtt{Card}\,(m,n,p))))]\right], \end{gathered} \tag{4.1}$$

where C' is obtained from C by replacing all the occurrences of 0, s, $<$, $+$, and $\times$ with 0_p, s_p, $<_p$, $+_p$, and $\times_p$ respectively. The sentence $\neg\alpha \wedge \gamma$ is the desired sentence characterizing $\mathcal{M}$ up to computable isomorphism. Indeed, if $\mathcal{N}$ is a computable equivalence structure, and $I_c(\mathcal{N}) \models \neg\alpha \wedge \gamma$, then we have a $q \in I_c(\mathcal{N})$ such that $I_c(\mathcal{N}) \models \psi(q)$, where for all $n \in \omega$, the equivalence class of $p^n(0_p)$ has the same cardinality as that of $q^n(0_q)$. As in the previous subcase, it follows that there is a computable isomorphism $f : \mathcal{M} \to \mathcal{N}$. Conversely, if there is a computable isomorphism $f : \mathcal{M} \to \mathcal{N}$, then clearly $I_c(\mathcal{N}) \models \neg\alpha$, and $f \cdot p \cdot f^{-1}$ serves as a witness to the existential formula 4.1.

We now concern ourselves with the proof of the existence of the formula $\texttt{Card}\,(m, n, p)$. We will show that the property that $m/_E$ has at least $k+1$ elements, where $n = p^k(0_p)$, is first-order expressible in the language of semigroups. This will clearly suffice to show the existence of $\texttt{Card}\,(m, n, p)$.

The idea is to say that there is a one-to-one correspondence between the set $m/_E$ (or some subset thereof) and the set $A_n =_{def} \{z \mid z <_p n\}$. Unfortunately, the existence of such a correspondence cannot be defined by means of a partial automorphism from one set to the other since the sets behave differently with respect to the equivalence relation (the elements of $m/_E$ are pairwise equivalent while no pair of elements from A_n are equivalent). We will introduce "special" partial automorphisms q and d, as shown below, to deal with this difficulty. Suppose, for example, we wanted to express the fact that the equivalence class of the element 0_p has at least four elements. This would certainly hold if we could find partial automorphisms q and d as in Figure 4.2.

We may if we like assume that d is undefined outside of the shown region. In general, as long we are able to such find a configuration for a given n, there

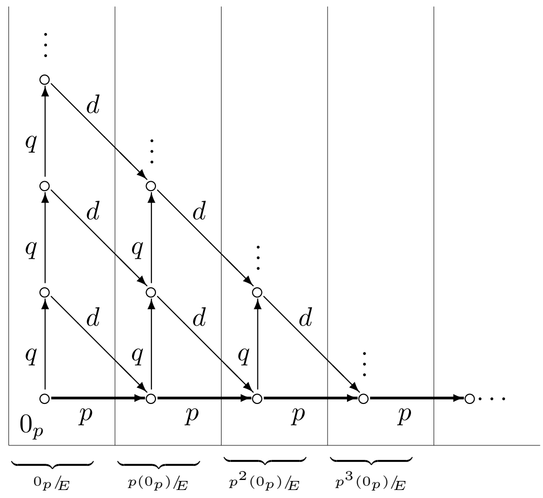

Figure 4.2: Action of automorphisms q and d.

will be an injective function f from a subset of $0_p/_E$ (namely $0_p/_E \cap \texttt{dom}\,(d)$) to the set $\{p^i(0_p) \mid i < n\}$, defined as $f(q^i(0_p)) = d^i(q^i(0_p))$ for all $i < n$. Of course, as noted before, f will not actually be a partial automorphism, but as long as we can guarantee that such an injection exists, it need not be a partial automorphism.

We now specify the properties that the q and d must satisfy. The partial automorphism q has the property that given an element $x \in \texttt{dom}\,(q)$, the equivalence class of x can be generated by repeated applications of q, i.e.:

$$x/_E = \{y \mid \exists i < \omega(y = q^i(x))\}.$$

The partial automorphism d, on the other hand, maps elements diagonally from one equivalence class to another. That is, $d(q^r(p^s(0_p)))$, if defined, will be equal to $q^{r-1}(p^{s+1}(0_p))$.

We would now like to be able to express for arbitrary $y \in \texttt{dom}\,(p)$, and $n \in \omega$ the fact that $y/_E$ has at least n elements by saying in a first-order way that such q and d exist to give us a configuration analogous to the diagram. There are two obstacles. Firstly, for this particular p, we can only express the property of an equivalence class having at least n elements for the particular class $0_p/_E$. Secondly, even for 0_p there may not be enough elements in some $p^i(0_p)/_E$ to have such a configuration. However, for arbitrary $a \in \texttt{dom}\,(p)$ and $n \in \omega$ we can certainly find p' such that $I_c(\mathcal{M}) \models \texttt{Nat}\,(p')$ and $0_{p'} = y$, and we use the assumption of subcase 2 to ensure that the equivalence classes of $0_{p'}, \ldots p'^{n-1}(0_{p'})$ all have sufficiently many elements to guarantee the existence of d' that plays the role of d above. Indeed, because we are only worried about the action of p' on a finite number of representatives, we may find such a p' that differs from p on only finitely many elements, and is therefore also computable.

We need only describe the necessary properties for such p, q and d with first-order formulas. All that is required of p is that it satisfy the formula $\texttt{Nat}\,(v)$ defined before. This is the first condition given below. Conditions (b)–(d) give the necessary first-order properties for q to ensure that it generates each equivalence class:

a) $\texttt{Nat}\,(p)$;

b) $\forall x\,(\langle q(x), x\rangle \in E)$;

c) $\forall x\,(q(x)\!\downarrow \rightarrow q(x) \neq x)$;

d) $\forall a \in \texttt{dom}\,(p)\,\forall y \in a/_E[y \neq a \rightarrow \exists F \in \texttt{Fin}\,(a, y \in F \wedge$

$\forall t \in F \setminus \{y\}(q(y)\downarrow \in F))]$.

We already know from our discussion of the formula $\mathtt{Nat}\,(v)$ that condition (a) guarantees that p generates a complete set of representatives as $\{p^i(0_p) \mid i \in \omega\}$, and a similar argument yields that every equivalence class of E has the form

$$\{q^k(p^i(0_p)) \mid k < \omega \wedge q^k(p^i(0_p))\downarrow\},$$

for an appropriate $i < \omega$. Now we describe the behavior of d, whose action draws diagonals on Figure 4.2, up to the diagonal whose lower right end is the element $y \in \mathtt{dom}\,(p)$. This action can be specified by the following conditions:

e) $y \in \mathtt{ran}\,(d)$;

f) for all $x \in \mathtt{dom}\,(p)$, the condition $0_p \leqslant_p x <_p y$ implies $d(q(x))\downarrow$ and $d(q(x)) = p(x)$;

g) for all t, if $q(d(t)) \in \mathtt{dom}\,(d)$, then $d(q(t)) = q(d(t))$.

Conditions (f) and (g) ensure that d interacts with p and q in the desired way, and condition (e) guarantees that the domain of d includes at least $n+1$ elements where $y = p^n(0_p)$.

The conjunction of the formulas corresponding conditions (a)–(g) gives a formula $\Theta(p,q,d,y)$. When this formula is satisfied by the arguments p,q,d,y, then the number of elements in the class 0_p is greater than the natural number corresponding to y. On the other hand, if the cardinality of the class $a/_E$ is greater than n then there exist p,q,d,y such that $a = 0_p$, $\Theta(p,q,d,y)$, and $y = p^{n-1}(0_p)$.

Finally, we can formulate with a first-order formula the property that the $x/_E$ has more than n elements where $y = p^n(0_p)$ as:

> *there exist partial computable automorphisms* γ, p', q, d *and a* $y' \in \texttt{dom}\,(p')$ *such that* $x = 0_{p'}$, $\Theta(p', q, d, y')$, $\texttt{dom}\,(p) \subseteq \texttt{dom}\,(\gamma)$, $\texttt{dom}\,(p') \subseteq \texttt{dom}\,(\gamma^{-1})$, $\gamma p \gamma^{-1} = p'$, $\gamma^{-1} p' \gamma = p$, *and* $\gamma(y) = y'$.

Here the partial automorphism γ is such that $\gamma(p^n(0_p)) = p'^n(0_{p'})$. Clearly, γ is computable and a partial automorphism. The condition $\gamma(y') = y$ then guarantees that $y' = p'^n(0_{p'})$, and therefore $\Theta(p', q, d, y')$ holding guarantees that there are $n+1$ elements in the equivalence class of x. ■

4.3 Partial Orders

We next give result with a similar flavor to Theorem 4.1.1, but with weaker conclusions. Again, we consider structures for a language with a single binary relation symbol, but this time we denote the symbol $<$, and insist that its interpretation in the structures we consider be a strict partial order. (We work with strict partial orders for convenience, but the results all hold for partial orderings with only minor modifications to the proofs.) We also define for a partially-ordered structure $\mathcal{M}$, a partially-ordered structure $\mathcal{M}^{rev}$ with the same universe, but with its *reverse order* by:

$$\text{for all } a, b \in M,\ \mathcal{M}^{rev} \models a < b \Leftrightarrow \mathcal{M} \models b < a$$

Theorem 4.3.1 *[5]*

Let $\mathcal{M}_0 = \langle M_0, <_0 \rangle$ and $\mathcal{M}_1 = \langle M_1, <_1 \rangle$ be strictly partially-ordered structures for a language with a single binary relation symbol $<$. For $i = 0, 1$, let I_i be an inverse semigroup such that $I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i)$. Then

$$I_0 \equiv I_1 \Rightarrow (\mathcal{M}_0 \equiv \mathcal{M}_1 \vee \mathcal{M}_0 \equiv \mathcal{M}_1^{rev}).$$

In particular,

$$I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1) \Rightarrow (\mathcal{M}_0 \equiv \mathcal{M}_1 \vee \mathcal{M}_0 \equiv \mathcal{M}_1^{rev})$$

Proof.

We would like, in a manner similar to equivalence relations to interpret the partial-order into the language of semigroups. The first step in our attempt to do this is to define pairs of comparable atoms in the language of semigroups:

$$C(x, y) \Leftrightarrow_{def} \neg \exists f[(f(x), f(y)) = (y, x)].$$

The idea is to use a pair of comparable elements to "set" the order. We will have a formula in four free variables, which we read as"x is a-b less than y":

$$x <_{a,b} y \Leftrightarrow_{def} C(a, b) \wedge \exists f[(f(a), f(b)) = (x, y)].$$

So, we will be able to talk about the partial order in the language of semigroups, but only up to our choice of a and b. That is, if $\check{a} < \check{b}$, then

$$I_i \models x <_{a,b} y \Leftrightarrow \mathcal{M}_i \models x < y.$$

On the other hand if $\check{a} > \check{b}$, then

$$I_{fin}(\mathcal{M}) \models x <_{a,b} y \Leftrightarrow \mathcal{M}^{rev} \models x < y.$$

These observations will serve as the base case of the induction below.

Given a formula ϕ in the language of partial orders, we would like to define an equivalent formula in the language of semigroups, as we did for equivalence structures. This isn't actually possible because the order can only be defined modulo reversal. However we may proceed as follows. Given $\phi(\bar{x})$, we let $\tilde{\phi}(\bar{x}, a, b)$ (renaming variables if necessary) be the formula in the language of semigroups that we get by replacing every occurrence of $<$ in $\phi(\bar{x})$ with $<_{a,b}$. We then define $\phi^*(\bar{x})$ to be $\exists ab\tilde{\phi}(\bar{x}, a, b)$. If $I_{fin}(\mathcal{M}) \models \phi^*(\bar{c}^*)$ for $\bar{c} \in M$, with witnesses to the existential quantifiers such that $\check{a} < \check{b}$, then an easy induction shows that $\mathcal{M} \models \phi(\bar{c})$. If, however, the witnesses are such that $\check{a} > \check{b}$, then $\mathcal{M}^{rev} \models \phi(\bar{c})$. In particular, if ϕ is a sentence, we have

$$I_{fin}(\mathcal{M}) \models \phi^* \Leftrightarrow \mathcal{M} \models \phi \vee \mathcal{M}^{rev} \models \phi.$$

Suppose the conclusion of the theorem doesn't hold. Then there are sentences ϕ and ψ, such that $\mathcal{M}_0 \models \phi \wedge \psi$, $\mathcal{M}_1 \models \neg\phi$ and $\mathcal{M}_1^{rev} \models \neg\psi$. Now, $I_{fin}(\mathcal{M}_0) \models (\phi \wedge \psi)^*$, and therefore $I_{fin}(\mathcal{M}_1) \models (\phi \wedge \psi)^*$. So, either $\mathcal{M}_1 \models \phi \wedge \psi$ or $\mathcal{M}_1^{rev} \models \phi \wedge \psi$, a contradiction. ■

We wish to apply the above results to Boolean algebras. We usually do not think of Boolean algebras as structures in the language of strict partial orders, but rather in the language $\langle \cap, \cup, ^-, 0, 1\rangle$. We noted above that any Boolean algebra has a natural partial order associated with it. Conversely, certain structures in the language of strict partial orders may be thought of as Boolean algebras, in the sense that the usual Boolean algebra operations are definable from the strict partial order.

Definition 4.3.2 *A Boolean Algebra in the language of strict partial orders is a structure $\mathcal{B}$ for the language $\langle < \rangle$ with the following properties:*

1. *B has a largest element 1, and a smallest element 0;*
2. *For all $a, b \in B$, there is an element $a \cup b \in B$, such that $a \cup b$ is the least upper bound of a and b with respect to $\leq$.*
3. *For all $a, b \in B$, there is an element $a \cap b \in B$ such that $a \cap b$ is the greatest lower bound of a and b with respect to $\leq$.*
4. *For all $a \in B$, there is an element $\bar{a} \in B$ such that $a \cup \bar{a} = 1$, and $a \cap \bar{a} = 0$.*

Essentially, $\mathcal{B}$ has the ordinary operations of a Boolean algebra, only these are not named in the language, but rather defined in terms of a partial order on the structure. It is easily checked that if $\mathcal{B}$ is a Boolean algebra in the language of strict partial orders, then the structure $\langle B; \cap, \cup, ^{-}, 1, 0 \rangle$ is in fact a Boolean algebra, and furthermore

$$a < b \Leftrightarrow a \neq b \wedge a \cup b = a.$$

Also, the structure $\mathcal{B}^{rev}$ is a Boolean algebra in the language of strict partial orders, and in fact $\langle B; \cup, \cap, ^{-}, 0, 1 \rangle$ is the corresponding Boolean algebra. It then follows that if $\mathcal{B}$ is a Boolean algebra in the language of strict partial orders, then $\mathcal{B} \cong \mathcal{B}^{rev}$ by the map $f : \mathcal{B} \to \mathcal{B}^{rev}$ defined by $f(a) = \bar{a}$ for $a \in \mathcal{B}$. The following corollary to the previous theorem is immediate:

Corollary 4.3.3 *If $\mathcal{M}_0$ is a partially-ordered structure such that $\mathcal{M}_0 \equiv \mathcal{M}_0^{rev}$ (and in particular, if $\mathcal{M}_0$ is a Boolean algebra in the language of partial orders),*

$\mathcal{M}_1$ is a partially-ordered structure, and I_0 and I_1 are as in the previous theorem, then

$$I_0 \equiv I_1 \Rightarrow \mathcal{M}_0 \equiv \mathcal{M}_1.$$

Remark. It is not true that for partial orders

$$\mathcal{M}_0 \equiv \mathcal{M}_1 \Rightarrow I_{fin}(\mathcal{M}_0) \equiv I_{fin}(\mathcal{M}_1).$$

Indeed, suppose that $\mathcal{M}_0$ is a partial order of type ω (the natural numbers), and $\mathcal{M}_1$ is a partial order of type $\omega + \zeta$ (the natural numbers followed by a copy of the integers). Then, using Ehrenfeucht-Fraïssé games, we may easily show $\mathcal{M}_0 \equiv \mathcal{M}_1$. However, we may describe a first-order sentence ϕ in the language of semigroups which says that there are comparable elements a and b, such that there is no idempotent whose domain contains all elements between a and b. Then $I_{fin}(\mathcal{M}_0) \models \neg\phi$ and $I_{fin}(\mathcal{M}_1) \models \phi$.

The next result tells us that we can recover the isomorphism type of a partial order up to reversal from the isomorphism type of any inverse semigroup of its partial automorphisms containing all the finite ones.

Theorem 4.3.4 *[5]*

Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be partially-ordered structures and I_0 and I_1 inverse semigroups such that:

$$I_{fin}(\mathcal{M}_i) \subseteq I_i \subseteq I(\mathcal{M}_i) \text{ for } i = 0, 1.$$

Then,

$$I_0 \cong I_1 \Rightarrow (\mathcal{M}_0 \cong \mathcal{M}_1 \wedge \mathcal{M}_0 \cong \mathcal{M}_1^{rev}).$$

Proof.

Let $\lambda : I_0 \to I_1$ be an isomorphism. By Proposition 3.2.1, there is an isomorphism of two-sorted models $(\lambda, f) : I_0^* \to I_1^*$. Fix comparable elements $a < b$, $a, b \in M_0$, and note that $f(a)$ and $f(b)$ must also be comparable. Assume $f(a) < f(b)$. Then,

$$\mathcal{M}_0 \models c < d \Leftrightarrow I_0 \models c <_{a,b} d \Leftrightarrow I_1 \models f(c) <_{f(a),f(b)} f(d) \Leftrightarrow \mathcal{M}_1 \models f(c) < f(d),$$

and $\mathcal{M}_0 \cong \mathcal{M}_1$. However, if $f(a) > f(b)$, we end up with

$$\mathcal{M}_0 \models c < d \Leftrightarrow \mathcal{M}_1 \models f(c) > f(d),$$

and $\mathcal{M}_0 \cong \mathcal{M}_1^{rev}$. ■

Once again, we have an immediate corollary:

Corollary 4.3.5 *If $\mathcal{M}_0$ is a partial order such that $\mathcal{M}_0 \cong \mathcal{M}_0^{rev}$ (in particular, if $\mathcal{M}_0$ is a Boolean algebra in the language of partial orders), and if I_0 and I_1 are as in Theorem 4.3.4, then*

$$I_0 \cong I_1 \Rightarrow \mathcal{M}_0 \cong \mathcal{M}_1.$$

Chapter 5

Semigroups of Partial Computable Automorphisms of Boolean Algebras

5.1 Defining the Order on Boolean Algebras

In this chapter we consider semigroups of partial computable automorphisms of Boolean algebras in the usual language of $\langle \cup, \cap, ^{-}, 0, 1 \rangle$. We recall some terminology. An element a in a Boolean algebra $\mathcal{B}$ is called an atom if

$$\forall b, c(a = b \cup c \Rightarrow (b = a \vee c = a)),$$

that is, a has no nontrivial splitting. We call a Boolean algebra $\mathcal{B}$ *atomless* if no element of $\mathcal{B}$ is an atom. Recall that in any Boolean algebra, a natural partial order $\subseteq$ is first-order definable as $c \subseteq d \Leftrightarrow_{def} c \cap d = c$. An equivalent definition of atom can be given in terms of this partial order: An element $a \in \mathcal{B}$ is an atom if and only if

$$\forall b(b \subseteq a \Rightarrow (b = 0 \vee b = a)).$$

For any $a \in \mathcal{B}$, we may define $\hat{a}$, the *restriction of* $\mathcal{B}$ *to* a, as the Boolean algebra with universe $\{b \mid b \subseteq a\}$, where $\cup$, $\cap$ and 0 are inherited from $\mathcal{B}$, $1 = a$, and $^-$ is complementation in $\mathcal{B}$ relative to a (The complement of an element $b \in \mathcal{B}$ relative to a is by definition $\bar{b} \cap a$). Finally, we say $a \in \mathcal{B}$ is an *atomless element* of $\mathcal{B}$ is $\hat{a}$ is an atomless Boolean algebra.

We would like to be able to define the pairs of comparable elements with respect to the order $\subseteq$ with a formula in the language of semigroups as with partially-ordered structures. The same formula works as in that case, although it is a little more difficult to show that this is the case.

Proposition 5.1.1 *Suppose $\mathcal{B}$ is a Boolean algebra in the language $\langle \cup, \cap, ^-, 0, 1 \rangle$. There is a formula $C(x, y)$ in the language of semigroups, such that if I is an inverse semigroup such that $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$, then*

$$(I \models C(a^*, b^*)) \Leftrightarrow (\mathcal{B} \models a \subset b \vee \mathcal{B} \models b \subset a).$$

Proof.

Set

$$C(x,y) \Leftrightarrow_{def} \neg\exists f[(f(x), f(y)) = (y,x)].$$

The direction ($\Leftarrow$) is clear, since it is impossible that both $a \subset b$ and $b \subset a$. Now assume $I \models C(a^*, b^*)$. We may assume that $a \notin \{0,1\} \wedge b \notin \{0,1\}$. The only obstacle to the existence of a partial automorphism that transposes a and b is that a binary operation $\cup$ or $\cap$ is not preserved. This can happen only if, say, $a \cup b = b$, or, say, $a \cap b = b$. But the former equation is equivalent (for $a \neq b$) to $a \subset b$, and the latter to $b \subset a$. We must have $a \neq b$, and thus, the implication ($\Rightarrow$) also holds. ■

We now wish to define the order on a Boolean algebra with a first-order formula in the language of semigroups with parameters. We will need to use a pair of comparable elements as parameters. For any partial automorphism f of a Boolean algebra $\mathcal{B}$, we have that

$$0_{\mathcal{B}} \in \mathtt{dom}\,(f) \Rightarrow f(0_{\mathcal{B}}) = 0_{\mathcal{B}},$$

and

$$1_{\mathcal{B}} \in \mathtt{dom}\,(f) \Rightarrow f(1_{\mathcal{B}}) = 1_{\mathcal{B}}.$$

Therefore, the elements $0_{\mathcal{B}}$ and $1_{\mathcal{B}}$ must be handled separately, and (identifying $0_{\mathcal{B}}$ with $0^*_{\mathcal{B}}$ and $1_{\mathcal{B}}$ with $1^*_{\mathcal{B}}$) will also be taken to be parameters of our formula.

Proposition 5.1.2 *Let $\mathcal{B}$ be a Boolean Algebra, and I be an inverse semigroup such that $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$. Let $L(x,y)$ be the following formula in the* *language of semigroups with distinct parameters $c, d, 0_{\mathcal{B}}, 1_{\mathcal{B}}$ where $\check{c} \subset \check{d}$:*

$$[\exists f((f(c), f(d) = (x, y)) \vee (x = 0_{\mathcal{B}}) \vee (y = 1_{\mathcal{B}})] \wedge x \neq y.$$

Then,

$$I \models L(a^*, b^*) \Leftrightarrow a \subset b \ \blacksquare$$

There is no way to distinguish the element $0_{\mathcal{B}}$ from the element $1_{\mathcal{B}}$ in the language of semigroups, so these elements must be retained as parameters in the formula L. However, since for any partial automorphism $f \in I(\mathcal{B})$ we have both $f \cup \{(0_{\mathcal{B}}, 0_{\mathcal{B}})\} \in I(\mathcal{B})$ and $f \cup \{(1_{\mathcal{B}}, 1_{\mathcal{B}})\} \in I(\mathcal{B})$, we have the following fact, which will be useful later:

Lemma 5.1.3 *Let $\mathcal{B}$ be a Boolean algebra and I be any semigroup such that $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$. Then there is an automorphism $\lambda : I \to I$ such that $\lambda(0_{\mathcal{B}}) = 1_{\mathcal{B}}$.* $\blacksquare$

Recall that a structure $\mathcal{M}$ is said to be computable if its universe is a computable set (we may assume the universe is the natural numbers) and the relations and operations of the structure are uniformly computable. Recall that two structures are computably isomorphic if there is a isomorphism from one structure to the other that is a computable function. If $\mathcal{B}$ is a computable Boolean algebra with an atomless element, then from the isomorphism type of $I_c(\mathcal{B})$, we are able to recover the original Boolean algebra, up to computable isomorphism:

(The following result is due to Lipacheva [11]. It existed in an internal Russian publication unbeknownst to the author and is proved here independently).

Theorem 5.1.4 *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable Boolean algebras in the language $\langle \cup, \cap, ^-, 0, 1\rangle$, such that $\mathcal{B}_0$ contains an atomless element. Then*

$$I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1) \Rightarrow \mathcal{B}_0 \cong_c \mathcal{B}_1$$

5.2 Tree Representation of Boolean Algebras

A very useful technique in proving Theorem 5.1.4 is representation of Boolean algebras as trees. This construction is given in [7]. For the remainder of this chapter, we let T refer to $2^{<\omega}$, the full binary tree. Nodes in T are finite binary strings. The symbol $^\frown$ denotes concatenation. The empty string is denoted by $<>$. There is a natural partial order $\subseteq$ on $2^{<\omega}$ given by set containment. For nodes $\sigma, \tau \in T$, the notation $\sigma \mid \tau$ signifies that σ and τ are incomparable.

For any countable Boolean algebra $\mathcal{B}$, we may find an associated subtree $S \subset T$, and a function $f : S \to \mathcal{B}$ (which we may think of as a labeling of nodes of S), such that S and f have the following properties:

1. $f(<>) = 1$ (i.e., the root of the tree is sent to the greatest element of the Boolean algebra);

2. For any node $\sigma \in S$, σ is either a terminal node (in which case $f(\sigma)$ is an atom of the Boolean algebra $\mathcal{B}$), or $\sigma^\frown 0$ and $\sigma^\frown 1$ are in S (in this case we say σ *splits*);

3. If $\sigma \in S$ splits, then $f(\sigma) = f(\sigma^\frown 0) \cup f(\sigma^\frown 1)$, and $f(\sigma^\frown 0) \cap f(\sigma^\frown 1) = 0$;

4. For any $a \in \mathcal{B}$, where $a \neq 0_{\mathcal{B}}$, $a = \rho(\sigma_1) \cup ... \cup \rho(\sigma_n)$ for some finite collection $\{\sigma_i\} \subset T$.

Given any countable boolean algebra $\mathcal{B}$, and an enumeration $\{b_i \mid i \in \omega\}$ of the elements of the Boolean algebra, we may construct S and f with the desired properties in stages. At each stage e, we will have a finite subtree $S_e \subset S$ such that $S = \bigcup_e S_e$, and a function $f_e : S_e \to \mathcal{B}$ such that f_e, wherever it is defined, agrees with f. At stage $e = -1$, we set $S_{-1} = \{<>\}$, and let $f(<>) = 1$. At stage e, $e \in \omega$, we consider the element b_e, and extend each leaf of S_{e-1} that is split by b_e. More precisely, for each leaf σ of S_{e-1} such that $b_e \cap f_{e-1}(\sigma) \neq f_{e-1}(\sigma)$ and $b_e \cap f_{e-1}(\sigma) \neq 0$, we will add each $\sigma^\frown 0$ and $\sigma^\frown 1$ to S_e and define f_e on these nodes such that $f_e(\sigma^\frown 0) = f_{e-1}(\sigma) \cap b_e$ and $f_e(\sigma^\frown 1) = f_{e-1}(\sigma) \cap \bar{b}_e$. On S_{e-1}, f_e will agree with f_{e-1}. We let $S = \bigcup_e S_e$ and $f = \bigcup_e f_e$.

We claim the construction gives the desired properties for S and f. Clearly 1 and 3 hold by construction. Note that 3 implies that if $\sigma, \tau \in S$ and $\sigma \mid \tau$ then $f(\sigma) \cap f(\tau) = 0$. To see 2, note that if $b_i \in \mathcal{B}$ is not an atom, then there is $j \in \omega$ such that $b_j \subset b_i$. If $i < j$, and $\sigma \in S_{j-1}$ with $f_{e-1}(\sigma) = b_i$, then σ is not a leaf of S_j, and therefore not a leaf of S. If $j < i$, then b_i will have nonempty intersection with the labels of at least two of the leaves of S_{i-1}, and will therefore will not be the label of any leaf of S. Clearly, 4 holds, because if $b_i \in \mathcal{B}$, then b_i is the union of labels of some nodes of S_i, which itself is a finite subset of S.

For any binary tree $S \subset T$, we may "reverse" the construction and find a

Boolean algebra represented by the tree. The elements of the Boolean algebra will be equivalence classes of finite sets of nodes of the tree. Two finite sets of nodes A and B will be equivalent if A is obtainable from B by finitely many additions or subtractions of nodes τ, where $\tau \subset \sigma \in B$, or replacements of $\sigma^\frown 0$ and $\sigma^\frown 1 \in B$ with σ, or vice versa. For such a tree S, we will denote this Boolean algebra by $B(S)$. The operations are most easily defined given $a, b \in B(S)$ by letting $a \cup b$ be the union of any pair of representatives of a and b, and letting $\bar{a}$ be the set of all $\sigma \in S$ with $lh(\sigma) < n$, where n is the length of the longest node in a, and such that there is no $\tau \in a$ with $\tau \nmid \sigma$.

We make the following observations about the above constructions:

1. Tree representations of Boolean algebras are not in general unique, i.e., if a different enumeration of the elements of a Boolean algebra is given, a different tree may result.

2. Tree representations of Boolean algebras may be naturally identified with subsets of the Boolean algebras. If Boolean algebra $\mathcal{B}$ has representation S, via function f, then $f(S) \subset \mathcal{B}$. We may identify S with $f(S)$, and now view $S \subset \mathcal{B}$.

3. The unique (up to isomorphism) countable atomless Boolean algebra has unique tree representation T (recall we use T for the full binary tree in this chapter). The notation B(T) of course represents that Boolean algebra.

5.3 Partial Computable Tree Representations of Computable Boolean Algebras

The content of the lemmas of this section may be regarded as the "computable version" of the above construction and remarks, adapted to the various contexts needed for the proof of the theorem.

Lemma 5.3.1 *Let a be an element of a computable Boolean algebra $\mathcal{B}$. Let $\hat{a}$ be the Boolean algebra that is the restriction of $\mathcal{B}$ to a. Then there exists a subtree $S \subseteq T$ and a partial computable function $f : S \to \hat{a}$ such that the following hold:*

1. $f(<>) = a$;
2. *For any node $\sigma \in S$, σ is either a terminal node (in which case $f(\sigma)$ is an atom of the Boolean algebra $\mathcal{B}$), or $\sigma^\frown 0$ and $\sigma^\frown 1$ and in S;*
3. *For any $\sigma \in T$, $f(\sigma) = f(\sigma^\frown 0) \cup f(\sigma^\frown 1)$ and $f(\sigma^\frown 0) \cap f(\sigma^\frown 1) = 0$;*
4. *Any $b \in \hat{a}, b \neq 0$ is the union of a finite number of elements in the range of f.*

   The pair $\langle S, f \rangle$ will called a partial computable tree representation of $\hat{a}$. We have $S = T$ if and only if $\hat{a}$ is atomless.

Proof.

We note that $\hat{a}$ is a computable subalgebra of $\mathcal{B}$, and that the construction of f described above is partial computable when applied to a computable Boolean algebra. As noted in remark 3 of the previous section, the tree associated with $\hat{a}$ will be the full binary tree T exactly when a is atomless. ■

The following result is proved in [7].

Lemma 5.3.2 *If $\langle S, f_0\rangle$ and $\langle S, f_1\rangle$ are partial computable tree representations of computable Boolean algebras $\mathcal{B}_0$ and $\mathcal{B}_1$, respectively, then there exists a unique computable isomorphism $g : \mathcal{B}_0 \to \mathcal{B}_1$ such that $g(f_0(\sigma)) = f_1(\sigma)$ for all $\sigma \in S$.*

An immediate consequence of the previous lemma is there is a unique computable atomless Boolean algebra up to computable isomorphism. We may identify the full binary tree T with a (computable) subset of $B(T)$ in a natural way. Moreover, if $\mathcal{B}_0$ and $\mathcal{B}_1$ are computable atomless Boolean algebras, viewing T as a subset of each via any partial computable tree representations, we may find a computable isomorphism $f : \mathcal{B}_0 \to \mathcal{B}_1$ such that f restricted to T is the identity on T. We may also extend successor-preserving mappings from $T \to T$ to partial computable isomorphisms of $B(T)$ uniquely:

Lemma 5.3.3 *Let $a \neq 1_{\mathcal{B}}$ be an atomless element of a Boolean algebra $\mathcal{B}$, and let f be a function that satisfies the conclusion of the Lemma 5.3.1. If $g : T \to T$ is a computable, successor-preserving function, then fgf^{-1} may be uniquely extended to a partial computable automorphism $\hat{g}$ of $\mathcal{B}$ so that $\mathtt{dom}\,(\hat{g}) = \hat{a}$ and $\mathtt{ran}\,(\hat{g}) \subseteq \hat{a}$.*

Proof.

Note that since g is successor-preserving, the pair $\langle T, fg\rangle$ is a partial computable tree representation of the Boolean algebra $\hat{b}$, where $b = fg(<>)$. By Lemma 5.3.2, there is a unique computable isomorphism $\hat{g} : \hat{a} \to \hat{b}$ such that

$\hat{g}f = fg$. Now, for $x \in \texttt{ran}\,(f)$, we have $\hat{g}(x) = fgf^{-1}(x)$. Clearly $\texttt{dom}\,(\hat{g}) = \hat{a}$, and $\texttt{ran}\,(\hat{g}) \subseteq \hat{a}$. The function $\hat{g}$ preserves the operations $\cap_{\mathcal{B}}$ and $\cup_{\mathcal{B}}$, since the corresponding operations on $\hat{a}$ and $\hat{b}$ are just the restriction of these. We also have $\hat{g}(0_{\mathcal{B}}) = 0_{\mathcal{B}}$. For all $x, y \in \hat{a}$, $x \neq \bar{y}$ holds, so complementation is trivially preserved. The Boolean algebra $\hat{b}$ is generated by elements of the form $fgf^{-1}(x), x \in \hat{a}$, so the restriction of the partial computable automorphism of $\mathcal{B}$ to $\hat{a}$ satisfying the conclusions of the lemma must be a computable isomorphism from $\hat{a}$ onto $\hat{b}$. The uniqueness of $\hat{g}$ follows immediately. ■

The next lemma says that any computable Boolean algebra may be embedded into the computable atomless Boolean algebra in such a way that (viewing $T \subset B(T)$) we may generate the range of the embedding from its intersection with T, which is a subtree of T.

Lemma 5.3.4 *Let $\mathcal{A}$ be a computable Boolean algebra, and $B(T)$ be a computable atomless Boolean algebra. View $T \subset B(T)$ in the natural way. There exists a computable embedding $f : \mathcal{A} \to B(T)$ where $S = (f(A) \cap T)$ is a subtree of T such that $B(S) = f(A)$, where $B(S)$ is the subalgebra of $B(T)$ whose elements are finite unions of elements of S.*

Proof.

The technique to construct f is similar to the construction of tree representations. Essentially, we will find a subtree $S \subset T$ such that S together with the restriction of f^{-1} to S form a partial computable tree representation of $\mathcal{A}$. We begin by constructing a partial function $\rho : \mathcal{A} \to T$ in stages such that $\rho(A) = S$. At each stage s, we will have $\rho_s \subset \rho$, $D_s = dom(\rho_s)$, $T_s = ran(\rho_s)$.

The tree T_s will be a subtree of T at each stage, and we will denote the set of leaves of T_s by L_s. Let $(a_i)_{i\in\omega}$ be an enumeration of $\mathcal{A}$ without repetitions, where $a_0 = 1^{\mathcal{A}}$. The construction proceeds as follows:

Stage 0. Set $\rho_0(a_0) =<>$.

Stage s+1 For each $b \in \rho^{-1}(L_s)$ split nontrivially by a_{s+1}, set $\rho_{s+1}(b \cap a_{s+1}) = \rho(b)^\frown 0$ and $\rho_{s+1}(b \backslash a_{s+1}) = \rho(b)^\frown 1$. For all other $b \in D_s$, $\rho_{s+1}(b) = \rho_s(b)$.

Let $\rho = \bigcup_s \rho_s$. Clearly, ρ is injective, so we may speak of ρ^{-1}.

As in the tree representation construction, any $a \in \mathcal{A}$ can be represented as a finite union of elements in $dom(\rho)$. In order to compute $f(a)$, we find a finite collection of elements $\{b_i\}$ of $dom(\rho)$ such that $a = \bigcup_i b_i$ (by running the construction until the stage where a is enumerated, say), and set $f(a) = \bigcup_i \rho(b_i)$. It immediately follows that $B(S) = f(\mathcal{A})$.

We claim that $S = \rho(A)$, verifying that S is a subtree of T. Clearly, $\rho(A) \subset S$. If $c \in S$, then $c \in \mathtt{ran}\,(f)$, and $c = \bigcup_i b_i$, for some finite collection $\{b_i\}$ of elements of $\mathtt{ran}\,(\rho)$. Now, $c \in T$, and $b_0 \subseteq c$. Then $c \in \mathtt{ran}\,(\rho)$, since $\mathtt{ran}\,(\rho)$ is a subtree of T. ■

We now show that if we use the above lemma to embed computable Boolean algebras into the atomless computable Boolean algebra and they embed "in the same way" (i.e., the subtree of T generated by range of each embedding is the same), then the Boolean algebras will actually have to be computably isomorphic.

Lemma 5.3.5 *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable Boolean algebras. Let $a_0 \in \mathcal{B}_0$ and*

$a_1 \in \mathcal{B}_1$. Let $f_i : \hat{a}_i \to B(T)$, $i = 0, 1$ be computable embeddings such that:

1. $\mathtt{ran}\,(f_0) \cap T = \mathtt{ran}\,(f_1) \cap T$;
2. *Every element in* $\mathtt{ran}\,(f_i)$ *may be written as a finite union of elements from* $\mathtt{ran}\,(f_i) \cap T$.

Then there is a computable isomorphism $g : \hat{a}_0 \to \hat{a}_1$.

Proof.

Denote the restriction of f_i^{-1} to $\mathtt{ran}\,(f_0) \cap T$ as $\hat{f}_i$. Note that $\langle ran(f_i) \cap T, \hat{f}_i \rangle$ is a computable tree representation of $\hat{a}_i$, for $i = 0, 1$. The result then follows immediately from Lemma 5.3.2. ■

5.4 Defining the Embeddings

Our goal is now to be able to describe, in the language of semigroups, exactly what subtree of T is generated by a particular embedding. This allows us to conclude that if different Boolean algebras have pieces that embed into $B(T)$ in the same way, that this will be reflected in the infinitary theory of the semigroups of partial computable isomorphisms of those Boolean algebras. The next lemma shows how this is possible.

Lemma 5.4.1 *Let $a \neq 1$ be an atomless element in a computable Boolean algebra $\mathcal{B}$. Let us view $T \subset \hat{a}$. There exist partial computable automorphisms ϕ and ψ of $\mathcal{B}$ such that the following hold:*

1. *$dom(\phi) = dom(\psi) = \hat{a}$;*

2. $ran(\phi) \subset \hat{a}$ *and* $ran(\psi) \subset \hat{a}$*;*

3. $\{\psi^m\phi^n(a) \mid m, n \in \omega\} = T$*;*

4. The relation $\psi^m\phi^n(a) = \sigma$ *is computable in* m, n *and* σ*.*

Proof.

We will define computable successor-preserving embeddings $\tilde{\phi}$ and $\tilde{\psi}$ from $T \to T$ in such a way that their unique extensions by Lemma 5.3.3 to partial computable automorphisms of $\mathcal{B}$ with domain $\hat{a}$, denoted by ϕ and ψ respectively, will clearly have the desired properties.

Define $\tilde{\phi} : T \to T$ by $\tilde{\phi}(\sigma) = 0^\frown\sigma$ (See Figure 5.1). Now $\tilde{\phi}$ is clearly computable, injective and successor-preserving, and therefore may be uniquely extended to the desired ϕ.

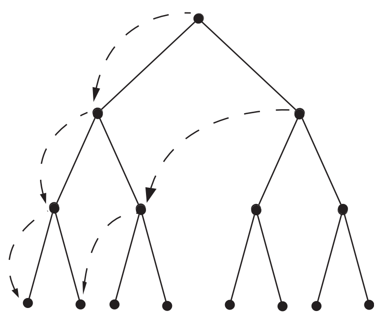

Figure 5.1: The function $\tilde{\phi}$

The embedding ϕ allows us to move down the tree to whichever level we choose. On the other hand, ψ will permute nodes on any given level of the tree, so that we may get to the desired node once we have gotten down to the correct level using ϕ (See Figure 5.2). We define $\tilde{\psi}$ inductively by level in the tree as follows. Let $\tilde{\psi}(<>) = <>$. Assume that $\tilde{\psi}$ has been defined for all nodes of length less than or equal to s, and that $\tilde{\psi}$ applied to all nodes of length s is a permutation consisting of a single cycle. Let σ be the string consisting of $s+1$ zeroes, and let $pr(\tau)$ denote the immediate predecessor of a node τ. We keep a list l_{s+1} of nodes of length $s+1$, representing the single cycle of the permutation. We set $l_{s+1} = \{\sigma\}$ to begin. As long as there is a node of length $s+1$ that hasn't been added to l_{s+1}, we extend l_{s+1} by adding $\psi(pr(\tau))^\frown 0$ (where τ is the last element in the list) if that element hasn't appeared in l_{s+1}, and adding $\psi(pr(\tau))^\frown 1$ otherwise. Essentially, our cycle consists of two "copies" of the cycle on the nodes of length s, first going through the nodes that end in 0, then the nodes that end in 1, then returning to σ.

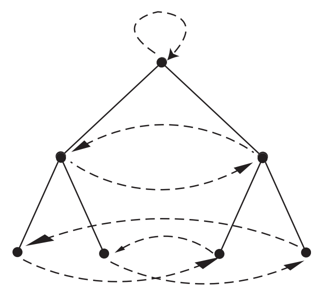

Figure 5.2: The function $\tilde{\psi}$

Extend $\tilde{\phi}$ and $\tilde{\psi}$ by Lemma 5.3.3 to ϕ and ψ respectively, which are partial computable automorphisms of $\mathcal{B}$. Properties 1 and 2 hold by the properties of the extension in Lemma 5.3.3. Any node in $\sigma \in T$ may be written as $\sigma = \psi^m \phi^n(a)$, by letting $n = lh(\sigma)$, to get to the correct level of T, and then applying ψ as many times as needed to reach σ, so 3 holds. The functions ϕ and ψ are computable, so 4 holds. ■

We are now ready to give the proof of the theorem. The idea is that if a Boolean algebra $\mathcal{B}$ has an atomless element a, then we may split the Boolean algebra into two pieces ($\hat{b}$ and $\hat{\hat{b}}$ for some $b \in \mathcal{B}$) each of which embed into $\hat{a}$. (We must use two pieces because the top element $1_{\mathcal{B}}$ is distinguished by a constant, and therefore cannot be mapped to $a \neq 1_{\mathcal{B}}$.) We can then find an infinitary formula in the language of inverse semigroups which will specify exactly how these pieces embed into $\hat{a}$. If two Boolean algebras have isomorphic semigroups of partial computable isomorphisms, then they must each split into two pieces that embed into the computable atomless Boolean algebra in the same way. From this, we may reconstruct an isomorphism between the two Boolean algebras.

Proof of Theorem 5.1.4.

We now proceed with the proof of the theorem. Let $b \in B_0$ be any element other than 0 or 1, let $a \in B$ be an atomless element, and let ϕ and ψ be the partial computable isomorphisms from Lemma 5.4.1. By Lemma 5.3.1, we identify $\hat{a}$ with $B(T)$, so, by Lemma 5.3.4, there are computable embeddings $f_0 : \hat{b} \to \hat{a}$ and $f_1 : \hat{\hat{b}} \to \hat{a}$ such that $ran(f_i) \cap T$ is a subtree of T that generates $ran(f_i)$ for $i = 0, 1$.

Next, we describe an infinitary formula in the language of semigroups describing the properties of ϕ, ψ, f_0, and f_1, which will be parameters in our infinitary formula. Thinking of a Boolean algebra as a partially ordered set, we fix two comparable elements $c_0, c_1 \in B_0$ with $c_0 \subset c_1 \wedge c_0 \neq 0 \wedge c_1 \neq 1$ so that we may define the order on B_0 in the language of semigroups with parameters $c_0, c_1, 0_{\mathcal{B}_0}, 1_{\mathcal{B}_1}$ (which will also be parameters of our infinitary formula) as in Proposition 5.1.2. The remaining parameters of our infinitary formula will be a and b. Let γ_1 be a formula asserting that a is atomless (using the fact that we have defined the order on B_0 in $I_c(\mathcal{B})$). Let γ_2 assert that the domains of f_0 and f_1 are $\hat{b}$ and $\hat{\hat{b}}$, respectively, and that the range of each is contained in $\hat{a}$. Let γ_3 be as follows:

$$\bigwedge(\phi^m\psi^n(a) \in ran(f_0) \Rightarrow \phi^{m'}\psi^{n'}(a) \in ran(f_0)),$$

where the infinite conjunction is over all m, n, m', and n' such that

$$\phi^m\psi^n(a) \subseteq \phi^{m'}\psi^{n'}(a).$$

Let γ_4 be an analogous sentence for f_1. These sentences say that $ran(f_0) \cap T$ and $ran(f_1) \cap T$ are subtrees of T. Let γ_5 be

$$(\forall x \in ran(f_0))[\bigvee_{\bar{m},\bar{n}} x = \psi^{m_1}\phi^{n_1}(a) \cup ... \cup \psi^{m_k}\phi^{n_k}(a)],$$

where $\bar{m}$ and $\bar{n}$ run over all finite tuples of natural numbers of equal length. Let γ_6 be an analogous sentence for f_1. We write formulas γ_7 and γ_8, which specify $ran(f_0) \cap T$ and $ran(f_1) \cap T$, respectively, simply by enumerating in an infinitary conjunction the elements of T which appear in the range of f_0 and f_1, using ψ and ϕ. Finally, we need formulas that assert that ψ and ϕ have the properties that we require of them, i.e., that every element in a generating tree of $\hat{a}$ may be realized by application of ψ and ϕ to a, and interact in the required way. We'll have a formula γ_9 that says that the domain of each ψ and ϕ is equal to $\hat{a}$

and that the range of each is contained in $\hat{a}$, and via and infinitary disjunction, that every element in $\hat{a}$ can be written as a finite union of elements of the form $\psi^m\phi^n(a)$ (we must enumerate all possible finite unions of elements of this form). Finally, we must ensure that the desired tree structure is obtained by elements of the form $\psi^m\phi^n$. We do this via an infinitary formula γ_{10} in which for each pair of elements of the form $a_1 = \psi^m\phi^n(a)$ and $a_2 = \psi^{m'}\phi^{n'}(a)$, whichever of $a_1 \subset a_2$, $a_2 \subset a_1$, $a_1 = a_2$,or $a_1 \cap a_2 = 0$ holds is enumerated as a conjunct.

We let γ be the conjunction of all of the γ_i, and γ' be γ with all parameters replaced by their isomorphic images in $I_c(\mathcal{B}_1)$ (denoted by $f_0', f_1', \phi', \psi', a', b', c_0', c_1', 0'$ and $1'$).

Now, $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$ and therefore, $I_c(\mathcal{B}_1) \models \gamma'$. Assume that $c_0' \subset c_1'$. We may assume that $0' = 0_{\mathcal{B}_1}$ and $1' = 1_{\mathcal{B}_1}$ by 5.1.3. Therefore the order defined in the language of semigroups by $C(x, y)$ with the parameters $c_0', c_1', 0', 1'$ agrees with the order $\subseteq$ on $\mathcal{B}_1$. Now, by Lemma 5.3.5, we have that $\hat{b} \cong_c \hat{b'}$ and $\hat{\hat{b}} \cong_c \hat{\hat{b'}}$, and

$$\mathcal{B}_0 \cong_c \hat{b} \times \hat{\hat{b}} \cong_c \hat{b'} \times \hat{\hat{b'}} \cong_c \mathcal{B}_1.$$

If $c_1' \subset c_0'$, we have $\mathcal{B}_0 \cong_c \mathcal{B}_1^{-1}$. However, $\mathcal{B}_1 \cong_c \mathcal{B}_1^{-1}$ simply by mapping $x \to \bar{x}$.

■

Chapter 6

Relatively Complemented Distributive Lattices

6.1 Definitions and Basic Results

A *relatively complemented distributive lattice* (RCDL) is a generalization of a Boolean algebra. Like Boolean algebras, we may consider RCDLs to be structures in either of two languages, the language of strict partial orders, or the language of Boolean algebras with the constant symbol 1 omitted. A strict partial ordering $\mathcal{B} = \langle B, < \rangle$ with a smallest element is called a relatively complemented distributive lattice if it is a distributive lattice under the ordering $\leq$ and for all $a \leq b$ in $\mathcal{B}$ there exists a unique relative complement of a in b, i.e., an element a' such that $sup\{a, a'\} = b$ and $inf\{a, a'\} = 0$. We may also consider

$\mathcal{B}$ to be a structure in the language $\langle \cap, \cup, \setminus, 0 \rangle$, where 0 is the least element of $\mathcal{B}$, $x \cap y = inf\{x, y\}$, $x \cup y = sup\{x, y\}$, and $x \setminus y$ is the relative complement of $x \cap y$ in x. We have the following proposition:

Proposition 6.1.1 *1. If $\mathcal{B}$ is an RCDL in the language $\langle < \rangle$, and $a \in \mathcal{B}$, the structure $\langle \hat{a}, < \rangle$, where $\hat{a} = \{x \in B \mid x \leq a\}$ is a Boolean algebra. In particular, if $\mathcal{B}$ has a greatest element, then $\mathcal{B}$ is a Boolean algebra.*

2. *If $\mathcal{B}$ is an RCDL in the language $\langle \cap, \cup, \setminus, 0 \rangle$, and $a \in \mathcal{B}$, then $\langle \hat{a}; \cap, \cup, ^{-}, 0, 1 \rangle$, where $\hat{a}$ is as above, 1 is interpreted as a, and $\hat{b} = a \setminus b$, is a Boolean algebra.*

Proof.

1. Follows immediately from definition of Boolean algebra in the language of strict partial orders.

2. The underlying partially-ordered structure of an RCDL is a distributive lattice, so associativity, commutativity, distributivity and adsorption hold in the structure $\langle B; \cap, \cup, \setminus, 0 \rangle$, and therefore in the restriction $\langle \hat{a}; \cap, \cup, ^{-}, 0, 1 \rangle$. Complements are taken relative to the greatest element a, so complementation also holds.

$\blacksquare$

Note that changing the language in which we consider a RCDL changes semigroup of partial automorphisms of the structure. Indeed, if $\mathcal{B}$ is a RCDL (in the language of strict partial orders) and a, b, c are pairwise distinct elements such that $a \cup b = c$ and a', b', c' are pairwise distinct elements such that $a' \cup b' \neq c'$ but $c' \geq a', b'$ and a' and b' are incomparable, then there is a partial automorphism f of $\mathcal{B}$ such that $f = \{(a, a'), (b, b'), (c, c')\}$. However, if $\mathcal{B}'$ is the corresponding structure of $\mathcal{B}$ in the language $\langle \cap, \cup, \setminus, 0 \rangle$, then clearly f is not a partial automorphism of $\mathcal{B}'$, since it does not preserve satisfaction of $\cup$. Therefore, when proving results about partial automorphism semigroups of RCDLs, we must consider these cases separately. The following is essentially a corollary to Theorem 4.3.1 (about strict partial orders).

Corollary 6.1.2 *If $\mathcal{B}_0$ and $\mathcal{B}_1$ are RCDLs in the language $\langle < \rangle$, and $I_i, i = 0, 1$ are inverse semigroups such that*

$$I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i), i = 0, 1,$$

then

$$I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1.$$

Proof.

By Theorem 4.3.1, $\mathcal{B}_0 \equiv \mathcal{B}_1$, or $\mathcal{B}_0 \equiv \mathcal{B}_1^{rev}$. But $\mathcal{B}_1^{rev}$ has a greatest element, a property which is first-order expressible, so if $\mathcal{B}_0 \equiv \mathcal{B}_1^{rev}$, then $\mathcal{B}_0$ also has a greatest element. By Proposition 6.1.1, $\mathcal{B}_0$ is a Boolean algebra and therefore by, Corollary 4.3.3, $\mathcal{B}_0 \equiv \mathcal{B}_1$. ■

We have a similar corollary to Theorem 4.3.4.

Corollary 6.1.3 *If $\mathcal{B}_0$ and $\mathcal{B}_1$ are RCDLs in the language $\langle < \rangle$ and I_i are inverse semigroups such that*

$$I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i), i = 0, 1,$$

then

$$I_0 \cong I_1 \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1.$$

Proof. By Theorem 4.3.4, $\mathcal{B}_0 \cong \mathcal{B}_1$ or $\mathcal{B}_0 \cong \mathcal{B}_1^{rev}$. In the latter case, we may conclude that both $\mathcal{B}_0$ and $\mathcal{B}_1$ have greatest elements, and are therefore Boolean algebras. The result then follows from Corollary 4.3.5. ■

Similar results hold for RCDLs in the language $\langle \cap, \cup, \setminus, 0 \rangle$.

Theorem 6.1.4 *[5]*

Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be RCDLs in the language $\langle \cap, \cup, \setminus, 0 \rangle$ and I_i inverse semigroups such that

$$I_{fin}(\mathcal{B}_i) \subseteq I_i \subseteq I(\mathcal{B}_i), i = 0, 1.$$

Then

1. $I_0 \equiv I_1 \Rightarrow \mathcal{B}_0 \equiv \mathcal{B}_1$
2. $I_0 \cong I_1 \Rightarrow \mathcal{B}_0 \cong \mathcal{B}_1$

Proof.

For any RCDL $\mathcal{B}$ with more than two elements, the element $0_{\mathcal{B}}$ is the unique element that is not moved by any partial automorphism, so we can give a first-order definition of the element $0_{\mathcal{B}_i}$ in $I_{\mathcal{B}_i}$ as follows:

$$(x \neq \mathbf{\Lambda}) \wedge \forall p(p(x) = x \vee p(x) = \mathbf{\Lambda}).$$

We can then define the set of pairs of distinct comparable elements of the set $B \setminus \{0\}$ in the structure I_i^* by the first-order formula $\texttt{Comp}\,(u, v)$

$$(u, v \notin \{0, \mathbf{\Lambda}\}) \wedge \neg\exists p(p(u) = v \wedge p(v) = u).$$

We verify that the formula holds exactly for pairs of nonequal nonzero comparable elements. Suppose a and b are nonequal comparable elements, say $a \subset b$. Then $a \cup b = b$ must hold. Now if $I_i^* \models \texttt{Comp}\,(a, b)$, we must have $b \cup a = a$, and $a = b$, a contradiction. On the other hand, suppose that $I_i^* \models \texttt{Comp}\,(a, b)$ for some $a, b \in I_i^*$. Then there is no $p \in I_i$ such that $p(a) = b$ and $p(b) = a$, so clearly $a \neq b$. The only obstacle to the existence of such a p is preservation of some binary operation with only a and b involved. If the operatoin in question is $\cup$, we must have $\mathcal{B}_i \models a \cup b = b$ (modulo reversal of a and b) and $\mathcal{B}_i \models b \cup a \neq a$. The first condition is equivalent to $a \subseteq b$, however. A similar argument works to show that if $\cap$ isn't preserved, the elements must be comparable. It now suffices to show that $\setminus$ is preserved by reversal of two nonzero nonequal elements a, b. This is so since for all such pairs a, b, $B_i \models \neg(a \backslash b = b) \wedge \neg(a \backslash a = b) \wedge \neg(a \backslash b = a)$, since any conjunct could only hold if $b = 0_{\mathcal{B}_i}$.

The next step is to define the pairs $\langle a, b \rangle$ such that $a < b$. We use the following condition, which may easily be expressed as a first-order formula in the language of semigroups:

$$\texttt{Comp}\,(a, b) \wedge$$

$$\exists c \neq 0\, (c <_{a,b} b \wedge (\ a \textit{ and } c \textit{ have no } <_{a,b}\textit{–lower bound in } B \setminus \{0\})).$$

Indeed, if $a < b$, then certainly $\texttt{Comp}\,(a, b)$ holds, and $c = b \setminus a$ is a witness to the second conjunct. On the other hand, if it is not the case that $a < b$ and the formula holds, then because $\texttt{Comp}\,(a, b)$ holds, we must have $b < a$, but for any c, $a \cup c$ is a nonzero "lower bound" for a and c in the reverse order, a contradiction.

This yields the fact that the usual order on B_i is now definable by a first-order formula in the language of semigroups. We can assert that $x < y$ by saying that either there is some pair (a, b) satisfying the above formula where $x <_{a,b} y$, or that $x = 0$ and $y \neq 0$. The standard RCDL operations are then definable from the order as discussed above.

The considerations above together with Proposition 3.2.1 mean that there exists a mapping $^\sharp$ from the language $\sigma = \langle \cap, \cup, \setminus, 0 \rangle$ into the language of semigroups such that for all RCDLs $\mathcal{B}$ and all inverse semigroups I such that $I_{fin}(\mathcal{B}) \subseteq I \subseteq I(\mathcal{B})$ and for all sentences φ of the language σ the following holds:

$$\mathcal{B} \models \varphi \;\Leftrightarrow\; I \models \varphi^\sharp,$$

which implies 1. Furthermore, in a manner similar to Theorem 4.3.4 it follows that the isomorphism type of I uniquely determines the isomorphism type of $\mathcal{B}$, which implies 2. ■

6.2 Computable Relatively Complemented Distributive Lattices

We have a result about computable RCDLs analogous to our result about computable Boolean algebras. In order to formulate it, we first need some results about the representation of RCDLs, which are similar to the results about representation of Boolean algebras.

Proposition 6.2.1 *Let $\mathcal{B}$ be an arbitrary countable RCDL. The RCDL $\mathcal{S}$ gen-*

erated by a finite family $\{a_0, \dots a_n\} \subseteq B$ is finite and consists of all finite unions of elements of the form (which we will refer to as intersection form):

$$a_0^{\varepsilon_0} \cap a_1^{\varepsilon_1} \cap \dots \cap a_{n-1}^{\varepsilon_{n-1}}, \tag{6.1}$$

where $\varepsilon_i \in \{0, 1\}$, $i < n$ and

$$x^\varepsilon = \begin{cases} x & \text{if } \varepsilon = 1 \\ \left[\bigcup_{i<n} a_i\right] \setminus x & \text{if } \varepsilon = 0. \end{cases}$$

(We shall say that an element that is written as the union of elements in intersection form is in union form).

Furthermore, the nonzero elements that are in intersection form are the atoms of the algebra generated by $\{a_0, \dots a_n\}$.

Proof.

We proceed by induction on the RCDL operations.

We first show that a_i is expressible in union form for $i < n$. Take a_0, for example. It can be written as $a_0 = \bigcup\limits_{\sigma \in 2^{[n-1]}} a_0 \cap a_1^{\sigma(0)} \cap \dots \cap a_{n-1}^{\sigma(n-2)}$, where σ ranges over all binary sequences of length $n - 1$.

Clearly, if $b \in \mathcal{S}$ and $c \in \mathcal{S}$ can be written as a union of elements in intersection, then so can $b \cup c$. To show the induction over $\cap$, suppose that $b = \bigcup\limits_{i \in F} b_i$ and $c = \bigcup\limits_{j \in G} c_j$ where each F and G are finite sets, and b_i and c_j are in intersection form. Then, by distributivity $b \cap c = \bigcup\limits_{i \in F, j \in G} b_i \cap c_i$. It remains only to show that $b_i \cap c_j$ can be written in intersection form for all i and j. This is clearly so because either $b_i \cap c_j = b_i$ or $b_i \cap c_j = 0_{\mathcal{B}}$, since $a_i^0 \cap a_i^1 = 0_{\mathcal{B}}$ for all i. This also shows that the nonzero elements of intersection form are the atoms of $\mathcal{S}$.

To show that relative complements are preserved, i.e. that $b \setminus c$ can be written as a union of such elements, we may assume that $b \supseteq c$, since $b \setminus c = b \setminus b \cap c$. But if $b \supseteq c$, then each of the summands of c must also appear in b. Then $b \setminus c$ can be written simply as b with those summands also in c removed, which is still in union form. ■

Proposition 6.2.2 *1. Up to isomorphism, there is a unique countably infinite atomless RCDL with no greatest element.*

2. Up to computable isomorphism, there is a unique infinite computable atomless RCDL with no greatest element

Proof.

We prove the existence of a countable atomless RCDL with no greatest element. Consider the algebra of all subsets of the rational numbers $\mathbb{Q}$ with the RCDL operations and constant $\cap, \cup, \setminus, \varnothing$ interpreted as set intersection, union, difference and the empty set respectively. We verify that its subalgebra $\mathcal{C}$ generated by all intervals of the form $[a, b)$ is a countable atomless RCDL with no greatest element. Since $\mathcal{C}$ is generated as finite combinations of elements from a countable set, it is countable. The nonzero elements are finite unions of intervals of the form $[a, b)$ each of which properly contains an interval of the same form, so no element is an atom. Clearly, $\mathcal{C}$ is relatively complemented and has no greatest element. Furthermore, $\mathcal{C}$ may be taken to be computable, since each element may be thought of a set of rational numbers (consisting of the endpoints of the intervals). We may then code the rationals as natural numbers, and code each element of $\mathcal{C}$ as a finite set of natural numbers in some standard computable coding of finite sets. From the code of any two such sets representing a and b, we may computably find the codes for the sets representing $a \cap b$, $a \cup b$ and $a \setminus b$.

Next, we show that any two countable atomless RCDLs $\mathcal{A}$ and $\mathcal{B}$ without greatest elements are isomorphic. We fix numberings of their universes as:

$$A = \{a_0, a_1, \ldots\},$$

$$B = \{b_0, b_1, \ldots\}.$$

We show through a back-and-forth argument that $\mathcal{A} \cong \mathcal{B}$. Suppose that we have an isomorphism f' between two finitely generated subalgebras $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$. Let $a \in A$. We show how to extend f' to an isomorphism f between two finitely generated extensions of $\mathcal{A}'$ and $\mathcal{B}'$ so that $a \in dom(f)$.

On the elements $x \in dom(f')$, we let $f'(x) = f(x)$. Let $c_0, \ldots, c_{k-1}$ be the list of all atoms of $\mathcal{A}'$. Then the atoms of the RCDL generated by the set $A' \cup \{a\}$ are exactly the nonzero elements among those of the form $c_i \cap a$, $c_i \setminus a$, $i < k$, and $a \setminus \left(\bigcup_{i<k} c_i\right)$. If $a \setminus \left(\bigcup_{i<k} c_i\right) \neq 0$, find an element $b \in \mathcal{B}$ greater than $\bigcup_{i<k} f(c_i)$ and let $f(a \setminus \left(\bigcup_{i<k} c_i\right)) = b \setminus \left(\bigcup_{i<k} f(c_i)\right)$. Then for each $i < k$, if $c_i \cap a \neq 0$ and $c_i \setminus a \neq 0$, find an element b strictly between 0 and $f(c_i)$ and let $f(c_i \cap a) = b$, $f(c_i \setminus a) = f'(c_i) \setminus b$. Thus, f is defined on all atoms of the RCDL generated by the set $\mathcal{A}' \cup \{a\}$. Next extend the mapping to an isomorphism f from this algebra into $\mathcal{B}$. Clearly, $f' \subseteq f$.

A symmetrical argument shows that for an arbitrary element $b \in B$, one can extend f' to an isomorphism f between two finitely generated extensions of $\mathcal{A}'$ and $\mathcal{B}'$ such that $b \in ran(f)$.

The back-and-forth argument precedes as follows. We define a sequence of isomorphisms $f_n, n \in \omega$, from finitely generated subalgebras of $\mathcal{A}$ to finitely generated subalgebras of $\mathcal{B}$. Let $f_0 = \varnothing$. For $n = 2m+1, m \in \omega$, we let f_n be an extension of f_{n-1} as above to include the first a_i such that $a_i \notin dom(f_{n-1})$. For $n = 2m+2, m \in \omega$, we let f_n be an extension of f_{n-1} so as to include the first b_i such that $b_i \notin ran(f_{n-1})$. Then $f = \bigcup\limits_{n \in \omega} f_n$ will be an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Note that in the case where $\mathcal{A}$ and $\mathcal{B}$ are computable then the above construction is also computable, and hence the isomorphism f is also computable.

■

In order to prove the main result of this section, we will need a representation of RCDLs similar to the tree representation of Boolean algebras given above. Since our structures have no top element, we cannot use trees as our representative structures, for these "trees" would have no root node. Instead we use a "rootless" analogue of trees, as follows.

Let $2^{<\mathbb{Z}}$ be the countable set consisting of all functions f from sets of kind $\mathbb{Z} \upharpoonright m = \{x \in \mathbb{Z} \mid x < m\}$ into the set $\{0, 1\}$ such that the set $\{x \mid f(x) \neq 0\}$ is finite. We fix a computable coding of the set $2^{<\mathbb{Z}}$ such that given the code of $f \in 2^{<\mathbb{Z}}$ one can effectively compute the maximal element of $\mathtt{dom}(f)$, and the index for the finite set $\{x \mid f(x) \neq 0\}$. We identify elements of $2^{<\mathbb{Z}}$ with their codes.

For a function $f \in 2^{<\mathbb{Z}}$, if $k = \sup(\texttt{dom}\,(f))$, we let

$$\begin{aligned} f^{-} &= f \setminus \{\langle k, f(k)\rangle\}, \\ f\breve{\ }0 &= f \cup \{\langle k+1, 0\rangle\}, \\ f\breve{\ }1 &= f \cup \{\langle k+1, 1\rangle\}. \end{aligned}$$

The element f^{-} is called the *predecessor* of f. The elements $f\breve{\ }0$ and $f\breve{\ }1$ are called *successors* of f. The elements of the set $2^{<\mathbb{Z}}$ may be thought of as elements of an infinite 2-branching tree with no root. Thus, each $f \in 2^{<\mathbb{Z}}$ splits into $f\breve{\ }0$ and $f\breve{\ }1$, and f^{-} precedes f. The set $2^{<\mathbb{Z}}$ is naturally ordered by inclusion.

We denote by $\mathcal{A}$ the unique (up to computable isomorphism) computable nontrivial atomless RCDL with no greatest element.

Proposition 6.2.3 *There exists an injective computable mapping θ from the set $2^{<\mathbb{Z}}$ into $\mathcal{A}$ such that the following hold:*

1. *Each element of $\mathcal{A}$ is the union of a finite subset of $ran(\theta)$.*

2. *For all $f, g \in 2^{<\mathbb{Z}}$, the following holds:*

$$f \subseteq g \Leftrightarrow \theta(f) \geq \theta(g)$$

3. *For all $f \in 2^{<\mathbb{Z}}$, the following holds:*

$$\theta(f\breve{\ }0) \cup \theta(f\breve{\ }1) = \theta(f) \text{ and } \theta(f\breve{\ }0) \cap \theta(f\breve{\ }1) = 0.$$

4. *There is an element $a \in \mathcal{A}$ and computable automorphisms ϕ and ψ of $\mathcal{A}$ such that for every $b \in ran(\theta)$ there exist $k \in \mathbb{Z}$ and $l, m \in \omega$ such that $b = \phi^k \psi^l \phi^m(a)$.*

Proof.

We assume that the universe A of $\mathcal{A}$ is the natural numbers, and that the natural number 0 represents the least element 0 of $\mathcal{A}$. To avoid confusion, we will write a_i instead of i when speaking about elements of $\mathcal{A}$. We will define the mapping θ in stages.

We first define $f_0 \in 2^{<\mathbb{Z}}$ as follows:

$$f_0 = \{(z, 0) \mid z \in \mathbb{Z} \wedge z \leqslant 0\}.$$

Let $\theta_0(f_0) = a_1$.

Assume that the finite mapping θ_i is defined and satisfies the following conditions:

a. The set $\mathtt{dom}\,(\theta_i)$ contains a least element f^* under inclusion, and $\mathtt{ran}\,(f^*) = \{0\}$.

b. For all f, if $f \in \mathtt{dom}\,(\theta_i) \setminus \{f^*\}$ then $f^- \in \mathtt{dom}\,(\theta_i)$.

c. For all $f \in \mathtt{dom}\,(\theta_i)$, $f\breve{\ }0 \in \mathtt{dom}\,(\theta_i) \Leftrightarrow f\breve{\ }1 \in \mathtt{dom}\,(\theta_i)$.

d. $0 \notin \mathtt{ran}\,(\theta_i)$.

e. For all f, if $f\breve{\ }0, f\breve{\ }1 \in \mathtt{dom}\,(\theta_i)$ then $\theta_i(f\breve{\ }0) \cup \theta_i(f\breve{\ }1) = \theta_i(f)$ and $\theta_i(f\breve{\ }0) \cap \theta_i(f\breve{\ }1) = 0$.

Now, if $f, g \in \texttt{dom}\,(\theta_i)$ are incomparable sequences such that each differs from f^*, we have $f^* \subset f$ and $f^* \subset g$. Let h be maximal such that $h \subset f$ and $h \subset g$, say $h\breve{\ }0 \subseteq f$ and $h\breve{\ }1 \subseteq g$. We have $\theta_i(h\breve{\ }0) \cap \theta_i(h\breve{\ }1) = 0$, $\theta_i(f) \leq \theta_i(h\breve{\ }0)$ and $\theta_i(g) \leq \theta_i(h\breve{\ }1)$. Thus, $\theta_i(f) \cap \theta_i(g) = 0$.

It follows that if $f \in \texttt{dom}\,(\theta_i)$ has no successors in $\texttt{dom}\,(\theta_i)$, then θ_i is not split by any element of $\texttt{ran}\,(\theta_i)$, and therefore is an atom in the algebra generated by $\texttt{ran}\,(\theta_i)$. By the splitting conditions (c) and (e), every element in said algebra is a finite union of images of elements without successors, which are exactly the atoms of the algebra.

For each pair of elements $a_{i+2} \cap \theta_i(f)$ and $a_{i+2} \setminus \theta_i(f)$ such that neither element is equal to 0 and f has no successors in $\texttt{dom}\,(\theta_i)$, let $\theta_{i+1}(f\breve{\ }0) = a_{i+2} \cap \theta_i(f)$ and $\theta_{i+1}(f\breve{\ }1) = a_{i+2} \setminus \theta_i(f)$. If $a_{i+2} \cup \theta_i(f^*) > \theta_i(f^*)$, we let $\theta_{i+1}((f^*)^-) = \theta_i(f^*) \cup a_{i+2}$ and $\theta_{i+1}((f^*)^-\breve{\ }1) = a_{i+2} \setminus \theta_i(f^*)$. We also let $\theta_{i+1}(f) = \theta_i(f)$, for all $f \in \texttt{dom}\,(\theta_i)$.

Let $\theta = \bigcup_{i<\omega} \theta_i$. Since $\mathcal{A}$ is atomless and has no greatest element, $dom(\theta) = 2^{<\mathbb{Z}}$. Conditions 1–3 follow immediately from the construction.

We now define the automorphisms ϕ and ψ.

The automorphism ϕ shifts the generators of $\mathcal{A}$. It is defined on generators $\theta(f)$ as follows:

$$\phi(\theta(f)) = \theta(\lambda(f)),$$

where

$$\lambda(f)(i) = \begin{cases} f(i-1), & \text{if } (i-1) \in \texttt{dom}\,(f) \\ \text{undefined}, & \text{otherwise} \end{cases}$$

We want to define the automorphism ψ so that it will permute generators less than $\theta(f_0)$ in such a way that each $\theta(f_0{}^\smile\varepsilon_1{}^\smile\ldots{}^\smile\varepsilon_k)$, $\varepsilon_1,\ldots,\varepsilon_k \in \{0,1\}$ can be obtained as $\psi^j(\theta(f_0{}^\smile\underbrace{0^\smile\cdots{}^\smile 0}_{k\text{ times}}))$, for some $j \in \omega$. To do so, first we define an automorphism τ on the structure $\langle 2^{<\mathbb{Z}}; \subseteq \rangle$ and then we put $\psi(\theta(f)) = \theta(\tau(f))$. We put $\tau(f) = f$, for all $f \in 2^{<\mathbb{Z}}$ such that $f_0 \nsubseteq f$. In particular, we let $\tau(f_0) = f_0$. The set F of elements f such that $f_0 \subseteq f$ may naturally be identified with the full binary tree T. On F, the automorphism τ will act the same way as the automorphism ψ on T in Lemma 5.4.1.

The precise definition of τ is as follows. Assume now that we have already defined all values of τ on the set

$$F_k = \{f_0{}^\smile\varepsilon_1{}^\smile\ldots{}^\smile\varepsilon_k \mid \varepsilon_1,\ldots,\varepsilon_k \in \{0,1\}\}$$

for some $k < \omega$ so that τ forms a single cycle on the set F_k, i.e., τ acts on F_k as follows:

$$g_0 \xrightarrow{\tau} g_1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} g_{2^k} \xrightarrow{\tau} g_0$$

Then τ will act on F_{k+1} as follows:

$$g_0{}^\smile 0 \xrightarrow{\tau} g_1{}^\smile 0 \xrightarrow{\tau} \cdots \xrightarrow{\tau} g_{2^k}{}^\smile 0 \xrightarrow{\tau} g_0{}^\smile 1 \xrightarrow{\tau}$$

$$\xrightarrow{\tau} g_1{}^\smile 1 \xrightarrow{\tau} \cdots \xrightarrow{\tau} g_{2^k}{}^\smile 1 \xrightarrow{\tau} g_0{}^\smile 0.$$

Now we let $\psi(\theta(f)) = \theta(\tau(f))$.

Computability of automorphisms ϕ and ψ follows immediately from the construction. For $f \in 2^{<\mathbb{Z}}$, if $\mathtt{ran}(f) \neq 0$, we define $lh(f) = i - j + 1$ where $i = sup\{x \mid x \in \mathtt{dom}(f)\}$ and $j = inf\{x \mid f(x) \neq 0\}$. If $\mathtt{ran}(f) = 0$, $lh(f) = 0$, by definition. We may naturally identify f with the finite sequence of length $lh(f)$ that agrees with f between i and j. Any $f \in 2^{<\mathbb{Z}}$ may be obtained by applications of λ and τ and λ^{-1} to f_0 as follows. First apply $\lambda^{lh(f)}$ to (f_0). Then apply τ the necessary number of times l so that the finite sequence associated with f agrees with $\tau^l \lambda^{lh(f)}(f_0)$ restricted to the integers greater than 0. Now, $\tau^l \lambda^{lh(f)}(f_0)$ needs only be shifted to agree with f. This can be done with applications of ϕ or ϕ^{-1}. It follows that 4 holds. ■

Finally, we need the following proposition:

Proposition 6.2.4 *Let $\mathcal{B}$ be a computable RCDL. There exists a computable embedding from $\mathcal{B}$ into $\mathcal{A}$, the computable atomless RCDL with no greatest element.*

Proof.

Let the $B = \{b_i \mid i \in \omega\}$ be the universe of B. The construction proceeds in stages. At each stage, we will have a finite function f_s which is an isomorphism from a finitely generated subalgebra of $\mathcal{B}$ to such a subalgebra of $\mathcal{A}$.

Stage 0. Let $f_0 = \{(0_{\mathcal{B}}, 0_{\mathcal{A}})\}$, i.e. the function that maps the 0 element of $\mathcal{B}$ to the 0 of $\mathcal{A}$, and is defined nowhere else.

Stage s+1. Let f_s be defined with domain a finite subalgebra of $\mathcal{B}$. For $x \in \mathtt{dom}\,(f_s)$, we set $f_{s+1}(x) = f_s(x)$. For each atom a of the subalgebra with universe $dom(f_s)$, we perform the following steps:

1. If both $a \cap b_s$ and $a \setminus b_s$ are nonempty, we set $f_{s+1}(a \cap b_s)$=c where $c \in \mathcal{A}$ is the first element enumerated in $\mathcal{A}$ such that $0_{\mathcal{A}} < c < f(a)$, and then

set $f_{s+1}(a \setminus b_s) = f_s(a) \setminus c$.

2. If $b_s \setminus u$ is nonempty, where $u = \bigcup_{x \in ran(f_s)} x$, we set $f_{s+1}(b_s \setminus u) = d$, where d is the first element enumerated in $\mathcal{A}$ such that $d \cap f_{s+1}(x) = 0_{\mathcal{A}}$ for each x such that $f_{s+1}(x)$ is so far defined.

3. Naturally extend f_{s+1} so that is defined on the subalgebra generated by the elements on which it was defined in the previous two steps.

Clearly, $f_s, s \in \omega$ is an increasing sequence, so we may set $f = \bigcup_s f_s$. We have $b_s \in \mathtt{dom}\,(f_s)$, since it is clearly the union of elements on which f_s is defined in steps 1 and 2 above. Hence $dom(f) = B$. Clearly f is computable, since we need only run the (computable) construction up to step s to determine $f(b_s)$.

■

We are finally ready to prove the aforementioned theorem. The condition here is that rather than containing an atomless element, the RCDLs in question must contain a computable copy of $\mathcal{A}$, the unique computable atomless RCDL without a greatest element.

Theorem 6.2.5 *Let $\mathcal{B}_0$ and $\mathcal{B}_1$ be computable RCDLs in the language $\langle \cap, \cup, \setminus, 0 \rangle$. Suppose that there is a computable embedding from $\mathcal{A}$ into $\mathcal{B}_0$ and that $I_c(\mathcal{B}_0) \cong I_c(\mathcal{B}_1)$. Then $\mathcal{B}_0 \cong_c \mathcal{B}_1$*

Proof.

In the proof of Theorem 6.1.4, it was established that if $\mathcal{B}$ is an RCDL in the language $\langle \cap, \cup, \setminus, 0 \rangle$, and I is any inverse semigroup such that $I_{fin}(\mathcal{B}) \subseteq I \subseteq I_c(\mathcal{B})$, then the associated ordering $<$ of the RCDL is definable in the model

I^*, along with the relations $\langle \cap, \cup, \setminus, 0\rangle$. This in true in particular of the models $I_c^*(\mathcal{B}_i), i = 0, 1$. By the assumptions of the theorem, we therefore have:

$$\begin{gathered}\langle I_c(\mathcal{B}_0), B_0, ap, \cdot, ^{-1}, <, \cap, \cup, \setminus, 0, \rangle \cong \\ \langle I_c(\mathcal{B}_1), B_1, ap, \cdot, ^{-1}, <, \cap, \cup, \setminus, 0, \rangle.\end{gathered} \tag{6.2}$$

We let β be a computable embedding of $\mathcal{A}$ into $\mathcal{B}_0$, and, by Proposition 6.2.4, γ a computable embedding of $\mathcal{B}_0$ into $\mathcal{A}$. The composition $\xi = \beta \cdot \gamma$ is a self-embedding of $\mathcal{B}_0$. Let $\varphi, \psi \in I_c(\mathcal{A})$ and an $a \in \mathcal{A}$ be as in Proposition 6.2.3.

Take an arbitrary $b \in B_0$. By Proposition 6.2.3, the element $\gamma(b)$ can be represented as a union

$$\gamma(b) = \bigcup_{i=1}^{n-1} \varphi^{k_i} \psi^{l_i} \varphi^{m_i}(a),$$

for appropriate $n, m_i, l_i \in \omega$, $k_i \in \mathbb{Z}$, $i = 1, \ldots, n$. It follows that the element $\xi(b) = \beta \cdot \gamma(b)$ can be represented as

$$\xi(b) = \bigcup_{i=1}^{n-1} (\beta\varphi\beta^{-1})^{k_i} (\beta\psi\beta^{-1})^{l_i} (\beta\varphi\beta^{-1})^{m_i} \beta(a)$$

Denote $\Phi = \beta\varphi\beta^{-1}$ and $\Psi = \beta\psi\beta^{-1}$, $a' = \beta(a)$. Note that $\xi, \Phi, \Psi \in I_c(\mathcal{B}_0)$ and the following condition is satisfied:

> *for all $b \in \mathcal{B}_0$, there exist $n, l_i, m_i < \omega$ and integers k_i, $i < n$ such that*
>
> $$\xi(b) = \bigcup_{i=1}^{n-1} \Phi^{k_i} \Psi^{l_i} \Phi^{m_i}(a'). \tag{6.3}$$

Denote the isomorphic images of ξ, Φ, Ψ, b, and a' with respect to the isomorphism (6.2) by ξ_1, Φ_1, Ψ_1, b_1, and a_1 respectively.

Then we have

$$\xi_1(b_1) = \bigcup_{i=1}^{n-1} \Phi_1^{k_i} \Psi_1^{l_i} \Phi_1^{m_i}(a_1). \tag{6.4}$$

This gives us the following algorithm to compute the isomorphism between $\mathcal{B}_0$ and $\mathcal{B}_1$.

Given $b \in \mathcal{B}_0$, use exhaustive search over all $n, k_i, l_i, m_i,\ i < n$ to find a decomposition for $\xi(b)$ of kind (6.3) and then find the isomorphic image b_1 of b as the unique element of $\mathcal{B}_1$ satisfying (6.4). ■

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