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TURING DEGREES OF ISOMORPHISM TYPES OF ALGEBRAIC OBJECTS

WESLEY CALVERT, VALENTINA HARIZANOV AND ALEXANDRA SHLAPENTOKH

Abstract

The Turing degree spectrum of a countable structure $\mathcal{A}$ is the set of all Turing degrees of isomorphic copies of $\mathcal{A}$. The Turing degree of the isomorphism type of $\mathcal{A}$, if it exists, is the least Turing degree in its degree spectrum. We show that there are elements with isomorphism types of arbitrary Turing degrees in each of the following classes: countable fields, rings, and torsion-free Abelian groups of any finite rank. We also show that there are structures in each of these classes the isomorphism types of which do not have Turing degrees. The case of torsion-free Abelian groups of finite rank settles a question left open by Knight, Downey and Jockusch [Downey, *Complexity, logic, and recursion theory*, Lecture Notes in Pure and Applied Mathematics 187 (ed. A. Sorbi; Marcel Dekker, New York, 1997) 157–205].

1. *Introduction*

One of the main goals of computable algebra is to understand how algebraic properties of structures interact with their computability theoretic properties. While in algebra and model theory isomorphic structures are often identified, in computable model theory, they can have very different algorithmic properties. Here, we study Turing degrees of isomorphism types of algebraic structures from some well-known classes. This is a natural way, introduced by Jockusch and Richter (see [**26**]), of expressing the algorithmic complexity of the structure. We consider only countable structures for computable (usually finite) languages. The universe A of an infinite countable structure $\mathcal{A}$ can be identified with the set ω of all natural numbers. Furthermore, we often use the same symbol for the structure and its universe. (For the definition of a language and a structure, see [**21**, p. 8], and for a definition of a computable language, see [**22**, p. 509].)

Let $\{\mathcal{A}_j, j \in \omega\}$ be a sequence of structures contained in a structure $\mathcal{B}$. Then, by $\prod_{j\in\omega} \mathcal{A}_j$, we mean the smallest substructure of $\mathcal{B}$ containing $\mathcal{A}_j$ for all j. This is not the usual definition of a product, but in the examples in this paper, it corresponds to the usual weak direct product, and, more importantly, it expresses a rather intuitive way of putting structures together. More specifically, we will be looking at products of number fields and function fields and at products of rings contained in a number field. In the case of fields, we fix an algebraic closure of $\mathbb{Q}$, a rational function field over a finite field of characteristic $p > 0$ or over $\mathbb{Q}$, as required, and we can set $\mathcal{B}$ to be this algebraic closure. In the case of subrings of a number field, the number field itself is a natural choice for $\mathcal{B}$.

We say that a set X is *Turing reducible* to (computable in) a set Y, in symbols $X \leqslant_T Y$, if X can be computed by an algorithm with Y in its oracle. Turing reducibility is the more basic notion, in terms of which Turing degree is defined. We say that the sets X and Y are Turing equivalent, or have the same Turing degree, if $X \leqslant_T Y$ and $Y \leqslant_T X$. We use $\equiv_T$ for Turing

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equivalence. We also write $\deg(X) = \deg(Y)$ or $Y \in \deg(X)$ instead of $X \equiv_T Y$. (Detailed information about Turing degrees and their structure can be found in [**27**, **29**].)

When measuring complexity of structures, we identify them with their atomic diagrams. The atomic diagram of a structure $\mathcal{A}$ is the set of all quantifier-free sentences in the language of $\mathcal{A}$ expanded by adding a constant symbol for every $a \in A$, which are true in $\mathcal{A}$. The *Turing degree* of $\mathcal{A}$, $\deg(\mathcal{A})$, is the Turing degree of the atomic diagram of $\mathcal{A}$. Hence, $\mathcal{A}$ is *computable* (recursive) if and only if $\deg(\mathcal{A}) = \mathbf{0}$. (Some authors call a structure computable if it is only isomorphic to a computable one.) We also say that a set or a procedure is computable (effective), relative to $\mathcal{B}$, sometimes said computable in $\mathcal{B}$, if it is computable relative to the atomic diagram of $\mathcal{B}$.

The *Turing degree spectrum* of a countable structure $\mathcal{A}$ is

$$\mathrm{DgSp}(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

A countable structure $\mathcal{A}$ is *automorphically trivial* if there is a finite subset X of the domain A such that every permutation of A which restricts to the identity on X is an automorphism of $\mathcal{A}$. If a structure $\mathcal{A}$ is automorphically trivial, then all isomorphic copies of $\mathcal{A}$ have the same Turing degree. It was shown in [**14**] that if the language is finite, then that degree must be $\mathbf{0}$. On the other hand, Knight [**19**] proved that for an automorphically non-trivial structure $\mathcal{A}$, $\mathrm{DgSp}(\mathcal{A})$ is closed upwards, that is, if $\mathbf{b} \in \mathrm{DgSp}(\mathcal{A})$ and $\mathbf{d} > \mathbf{b}$, then $\mathbf{d} \in \mathrm{DgSp}(\mathcal{A})$. Hirschfeldt *et al.* [**15**] established that for every automorphically non-trivial structure $\mathcal{A}$, there is a symmetric irreflexive graph, a partial order, a lattice, a ring, an integral domain of arbitrary characteristic, a commutative semigroup, or a 2-step nilpotent group, the degree spectrum of which coincides with $\mathrm{DgSp}(\mathcal{A})$.

Since the Turing degree of a structure is not invariant under isomorphisms, Jockusch and Richter introduced the following complexity measures of the isomorphism type of a structure.

DEFINITION 1.1. (i) The *Turing degree of the isomorphism type* of $\mathcal{A}$, if it exists, is the least Turing degree in $\mathrm{DgSp}(\mathcal{A})$.

(ii) Let α be a computable ordinal. The *αth jump degree* of a structure $\mathcal{A}$ is, if it exists, the least Turing degree in

$$\{\deg(\mathcal{B})^{(\alpha)} : \mathcal{B} \cong \mathcal{A}\}.$$

Obviously, the notion of the 0th jump degree of $\mathcal{A}$ coincides with the notion of the degree of the isomorphism type of $\mathcal{A}$. (A general discussion of the jump operator can be found in [**27**, Section 13.1; **29**, Chapter III].)

Richter [**26**] proved that if $\mathcal{A}$ is a structure without a computable copy and satisfies the effective extendability condition explained below, then the isomorphism type of $\mathcal{A}$ has no degree. Richter's result uses a minimal pair construction. Distinct non-zero Turing degrees $\mathbf{a}$ and $\mathbf{b}$ form a *minimal pair* if

$$(\mathbf{c} \leqslant \mathbf{a} \wedge \mathbf{c} \leqslant \mathbf{b}) \Rightarrow \mathbf{c} = \mathbf{0}.$$

(See [**29**] for the minimal pair method.) A structure $\mathcal{A}$ satisfies the effective extendability condition if for every finite structure $\mathcal{M}$ isomorphic to a substructure of $\mathcal{A}$, and every embedding σ of $\mathcal{M}$ into $\mathcal{A}$, there is an algorithm that determines whether a given finite structure $\mathcal{N}$ extending $\mathcal{M}$ can be embedded into $\mathcal{A}$ by an embedding extending σ. Richter [**26**] also showed that every linear order and every tree, as a partially ordered set, satisfies the effective extendability condition. Recently, Khisamiev [**18**] proved that every Abelian p-group, where p is a prime number, satisfies the effective extendability condition. Hence, the isomorphism type of a countable linear order, a tree, or an Abelian p-group, which is not isomorphic to a computable one, does not have a degree.

Recently, Csima [**7**] proved that if $\mathcal{A}$ is a prime model of a complete decidable theory with no computable prime model, then the isomorphism type of $\mathcal{A}$ does not have a Turing degree, while for every $n \geqslant 1$, the structure $\mathcal{A}$ has the nth jump degree $\mathbf{0}^{(n)}$. (See [**12**] for more information on computability of prime models.)

If $\mathcal{A}$ is a non-standard model of the Peano arithmetic, then the isomorphism type of $\mathcal{A}$ has no degree. On the other hand, Knight [**19**] showed that for any Turing degree $\mathbf{d}$, there is a non-standard model of Peano arithmetic with the first jump degree $\mathbf{d}'$. Knight also established that the only possible first jump degree for a linear order is $\mathbf{0}'$.

Ash, Jockusch, and Knight [**2**], and Downey and Knight [**9**] proved that for every computable ordinal $\alpha \geqslant 1$, and every Turing degree $\mathbf{d} \geqslant \mathbf{0}^{(\alpha)}$, there is a linear order $\mathcal{L}$ the αth jump degree of which is $\mathbf{d}$, and such that $\mathcal{L}$ does not have βth jump degree for any $\beta < \alpha$. Jockusch and Soare [**17**] proved that for a Turing degree $\mathbf{d}$ and $n \in \omega$, if a Boolean algebra $\mathcal{B}$ has the nth jump degree $\mathbf{d}$, then $\mathbf{d} = \mathbf{0}^{(n)}$. They used a method by Feiner to show also that if $\mathbf{d} \geqslant \mathbf{0}^{(\omega)}$, then there is a Boolean algebra with ωth jump degree $\mathbf{d}$ (see [**17**]). Oates [**23**] proved that for every computable ordinal $\alpha \geqslant 1$, and every Turing degree $\mathbf{d} > \mathbf{0}^{(\alpha)}$, there is an Abelian group $\mathcal{G}$ the αth jump degree of which is $\mathbf{d}$, and $\mathcal{G}$ does not have βth jump degree for any $\beta < \alpha$.

For additional background information on computability (recursion) theory, see [**27**, **29**]. For the computable model theory, see [**8**, **13**, **12**, **24**]. In the sections that follow, we will use some facts from algebra and number theory. The relevant algebraic material can be found in [**1**, **5**, **10**, **11**, **16**, **20**, **25**, **28**].

2. *Turing degrees of the isomorphism types of structures*

We would like to further investigate Turing degrees of the isomorphism types of Abelian groups, rings, and fields. Our general theorem will be a modification of the following result by Richter.

THEOREM 2.1 [**26**]. *Let T be a theory in a finite language L such that there is a computable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of* finite *structures for L, which are pairwise non-embeddable. Assume that for every set $X \subseteq \omega$, there is a countable model $\mathcal{A}_X$ of T such that*

$$\mathcal{A}_X \leqslant_T X,$$

and for every $j \in \omega$

$$\mathcal{A}_j \text{ is embeddable in } \mathcal{A}_X \Leftrightarrow j \in X.$$

Then, for every Turing degree $\mathbf{d}$, there is a countable model of T whose isomorphism type has degree $\mathbf{d}$.

Richter used Theorem 2.1 to show that for every Turing degree $\mathbf{d}$, there is an Abelian torsion group the isomorphism type of which has degree $\mathbf{d}$. The following generalization of Theorem 2.1 allows infinite structures in the computable sequence of structures.

THEOREM 2.2. *Let $\mathcal{C}$ be a class of countable structures in a finite language L, closed under isomorphisms. Assume that there is a computable sequence $\{\mathcal{A}_i, i \in \omega\}$ of computable (possibly infinite) structures in $\mathcal{C}$ satisfying the following conditions.*

- *There exists a finitely generated structure $\mathcal{A} \in \mathcal{C}$ such that for all $i \in \omega$, $\mathcal{A} \subset \mathcal{A}_i$.*
- *For any $X \subseteq \omega$, there is a structure $\mathcal{A}_X$ in $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{A}_X$ and*

$$\mathcal{A}_X \leqslant_T X,$$

and for every $i \in \omega$, there exists an embedding σ such that

$$\sigma : \mathcal{A}_i \hookrightarrow \mathcal{A}_X, \quad \sigma_{|\mathcal{A}} = \mathrm{id},$$

if and only if $i \in X$.

- *Suppose that any $\mathcal{A}_X$ is isomorphic to some structure $\mathcal{B}$ under isomorphism $\tau : \mathcal{A}_X \leftrightarrow \mathcal{B}$. Consider a pair of structures $\mathcal{A}_i, \mathcal{A}_j$ such that exactly one of them embeds in $\mathcal{B}$ via σ with $(\tau^{-1} \circ \sigma)_{|\mathcal{A}} = \mathrm{id}$. Then, there is a uniformly effective procedure with oracle $\mathcal{B}$ for deciding which of the two structures embeds in $\mathcal{B}$.*

Then, for every Turing degree $\mathbf{d}$*, there is a structure in* $\mathcal{C}$ *with isomorphism type of degree* $\mathbf{d}$.

Proof. Let $D \subseteq \omega$ be such that $\deg(D) = \mathbf{d}$. As usual, let

$$D \oplus \overline{D} =_{\mathrm{def}} \{2n : n \in D\} \cup \{2n+1 : n \notin D\}.$$

We will show that $\mathcal{A}_{D\oplus\overline{D}}$ is a structure in $\mathcal{C}$, the isomorphism type of Turing degree $\mathbf{d}$. Clearly, by assumption,

$$\deg(\mathcal{A}_{D\oplus\overline{D}}) \leqslant \deg(D \oplus \overline{D}) = \mathbf{d}.$$

Now, let a structure $\mathcal{B}$ be such that there exists an isomorphism $\tau : \mathcal{A}_{D\oplus\overline{D}} \leftrightarrow \mathcal{B}$. Then, by the definition of $D \oplus \overline{D}$ and an assumption of the theorem, for every $j \in \omega$,

$$j \in D \Leftrightarrow (\exists\sigma)\left[\sigma : \mathcal{A}_{2j} \hookrightarrow \mathcal{B} \wedge (\tau^{-1} \circ \sigma)_{|\mathcal{A}} = \mathrm{id}\right]$$

or, equivalently,

$$j \notin D \Leftrightarrow (\exists\sigma)\left[\sigma : \mathcal{A}_{2j+1} \hookrightarrow \mathcal{B} \wedge (\tau^{-1} \circ \sigma)_{|\mathcal{A}} = \mathrm{id}\right].$$

Thus, by an assumption of the theorem, we conclude that $\mathbf{d} \leqslant \deg(\mathcal{B})$. Hence, $\deg(\mathcal{A}_{D\oplus\overline{D}}) = \mathbf{d}$, and the degree of the isomorphism type of $\mathcal{A}_{D\oplus\overline{D}}$ is $\mathbf{d}$. ☐

Moreover, if the structure $\mathcal{A}_{D\oplus\overline{D}}$ in the proof of Theorem 2.2 is automorphically non-trivial, then the degree spectrum of $\mathcal{A}_{D\oplus\overline{D}}$ is the cone above $\mathbf{d}$.

3. *Fields the isomorphism types of which have arbitrary Turing degrees*

In this section, we show that certain sequences of computable linearly disjoint fields satisfy conditions of Theorem 2.2. First, we state two definitions and prove two lemmas to describe sequences we will consider and their role in the construction of fields with isomorphism types of arbitrary Turing degrees.

Definition 3.1. Let F be any field. Let $\{L_i, i \in \omega\}$ be a sequence of finite extensions of F such that for any $i \in \omega$, L_i and $\prod_{j\in\omega\setminus\{i\}} L_j$ are linearly disjoint over F, or, in other words,

$$[L_i : F] = [L : \prod_{j\in\omega\setminus\{i\}} L_j] > 1,$$

where $L = \prod_{j\in\omega} L_j$. Then, we call the sequence $\{L_i, i \in \omega\}$ *totally linearly disjoint* over F.

Definition 3.2. Let F be any field. Let $\{L_i, i \in \omega\}$ be a sequence of algebraic extensions of F, and let $L = \prod_{i\in\omega} L_i$. Suppose further that for any embedding $\sigma : L \hookrightarrow \tilde{F}$, where $\tilde{F}$ is the algebraic closure of F, such that $\sigma_{|F} = \mathrm{id}$, we have for all i, either $\sigma(L_i) = L_i$ or $\sigma(L_i) \not\subset L$. Then, we call the sequence $\{L_i, i \in \omega\}$ *stable with respect to* F. If $F = \mathbb{Q}$ or F is a finite field, then we say that the sequence $\{L_i, i \in \omega\}$ is *stable*.

Lemma 3.3. *Let K be a computable finitely generated field. Then there exists a computable sequence $\mathcal{P} = \{f_i(T) \in K(T)\}$ of monic polynomials irreducible over K such that the following* *conditions are satisfied for a field sequence* $\{M_i\}$, *where* $M_i = K(\alpha_i)$ *and* α_i *is a root of* $f_i(T)$ *in some fixed algebraic closure of* K*:*

- $\{M_i, i \in \omega\}$ *is totally linearly disjoint over* K*;*
- $\{M_i, i \in \omega\}$ *is stable with respect to* K*.*

Proof. First of all, observe that it would be enough to show that there exists a computable sequence $\mathcal{P}$ of polynomials such that the corresponding fields M_i are Galois extensions of K and the sequence $\{M_i\}$ is totally linearly disjoint over K. We will produce an explicit description of a required polynomial sequence.

Let C_0 be the algebraic closure of $\mathbb{Q}$ or a finite field in K. (It is possible that $C_0 = K$.) Since K is finitely generated, C_0 is either a number field or a finite field. In the case of a number field, do the following. Let q be the largest rational prime ramified in the extension $C_0/\mathbb{Q}$. (Such a prime always exists since only finitely many primes ramify in a finite extension of number fields; see, for example, [**20**, Chapter I].) Let $q < p_1 < p_2 < \cdots$ be any ascending computable sequence of rational primes. Let ξ_i be the p_ith primitive root of unity. Let $M_i = K(\xi_i)$. Thus, each M_i is clearly Galois over K. It remains to show that the linear disjointedness condition is satisfied. We will first show that the sequence $\{C_i\}$, where $C_i = C_0(\xi_i)$, is totally linearly disjoint over C_0. For $j \in \omega$, let $D_j = \prod_{i \neq j} C_i$. It is enough to show that ξ_j is of degree $p_j - 1$ over D_j. Observe that in any number field contained in D_j, p_j is not ramified. (This follows from the general properties of cyclotomics; see, for example, [**20**, Chapter IV].) On the other hand, in any number field containing C_j, p_j is ramified with ramification degree at least $p_j - 1$. So suppose that ξ_j is of degree smaller than $p_j - 1$ over D_j. Then, for some number field $N_j \subset D_j$, $[N_j(\xi_j) : N_j] < p_j - 1$. Therefore, the ramification degree of p_j in this extension is smaller than $p_j - 1$, which is impossible.

We now show that the sequence $\{M_i\}$ is also totally linearly disjoint. First of all, we observe that by [**1**, Theorem 13, Chapter XV], for $j \in \omega$, $\prod_{i \neq j} M_i = KD_j$. Secondly, by [**1**, Theorem 11], the degree of ξ_j over KD_j is equal to the degree of ξ_j over D_j. The only remaining point is to note that we have the explicit formula for the pth cyclotomic polynomial: $X^{p-1} + X^{p-2} + \cdots + 1$. Thus, the polynomial sequence corresponding to the field sequence $\{M_i\}$ is certainly computable.

Our next task is to address the case of positive characteristic. Let $q > 0$ be the characteristic of C_0, and let q^r be the size of C_0. Let $r < p_1 < \cdots < p_n < \cdots$ be any ascending computable sequence of rational primes. Let α_i be any root of the polynomial $f_i(X) = (X^{q^{p_i}} - X)/(X^q - X)$. In other words, α_i generates the extension of degree p_i over the finite field of q elements. Let $C_i = C_0(\alpha_i)$ and $M_i = K(\alpha_i)$. Observe that C_i/C_0 and M_i/K are cyclic extensions. From this point on, we can pretty much proceed as in the case of characteristic 0. The only difference will stem from the fact that instead of using ramification to establish total linear disjointedness of the sequence, we can use the fact that a finite field has a unique extension of any fixed degree (within a fixed algebraic closure). □

We are now ready to prove the second technical result of this section.

LEMMA 3.4. *Let* K *be a finitely generated computable field. Let* $\mathcal{P} = \{f_i(T) \in K(T)\}$ *be a computable sequence of monic polynomials irreducible over* K*. Let* α_i *be a root of* f_i*, and let* $M_i = K(\alpha_i)$*. Assume further that the sequence* $\{M_i, i \in \omega\}$ *is totally linearly disjoint over* K *and is stable with respect to* K*. Let* $\mathcal{A} = K$*. Let* $\mathcal{A}_i = M_i$*, and for any* $X \subset \omega$*, let* $\mathcal{A}_X = M_X$*, where*

$$M_X = \prod_{i \in X} M_i.$$

Then, the conditions of Theorem 2.2 are satisfied.

Proof. First of all, as shown in [**24**], every computable field with a splitting algorithm has a computable algebraic closure. Let $\tilde{K}$ be a computable algebraic closure of K, which as a finitely generated computable field has a splitting algorithm (see [**10**]) and assume that $\alpha_i \in \tilde{K}$ for all i. Next, let $X \subset \omega$.

(1) We show that M_i is embeddable into $M_X = \prod_{i \in X} M_i$ under any embedding σ keeping K fixed if and only if $i \in X$. Let $\sigma : M_i \hookrightarrow M_X$ be such an embedding. Given our assumptions on the linear disjointedness of the elements of the sequence, σ can be extended to $\tilde{\sigma} : M \hookrightarrow \tilde{K}$, where $M = \prod_{j \in \omega} M_j$. Thus, by our assumptions, we conclude that $\sigma(M_i) = M_i$, and therefore, $M_i \subset M_X$. However, by our assumptions again, this is possible if and only if $i \in X$.

(2) Next, we show that $M_X \leqslant_T X$. Let α_i be a root of f_i. Given our assumptions, M is a computable field, and there exists a computable function which for a given element of M will produce its coordinates with respect to the basis

$$\Omega = \left\{ \prod_{i \in I} \alpha_i^{a_i} : I \subset \omega,\ \ |I| < \infty,\ \ 0 \leqslant a_i < \deg(f_i) \right\}.$$

Thus, given $\beta \in M$ and X, we can determine, computably in X, whether $\beta \in M_X$. Consequently, $M_X \leqslant_T X$. (Since the reverse reducibility is obvious, we can actually show that $M_X \equiv_T X$.)

(3) Finally, let $\tau : M_X \leftrightarrow \mathcal{B}$ be an isomorphism, and let $i, j \in \omega$, $i \neq j$, be such that for exactly one of $k = i$ and $k = j$, there exists $\sigma : M_k \hookrightarrow \mathcal{B}$ satisfying the condition that $\tau^{-1} \circ \sigma$ is an identity on K. Since K is finitely generated, we can compute with oracle $\mathcal{B}$ the τ-images of f_i and f_j. Next, we note that there exists an embedding $\sigma : M_i \hookrightarrow \mathcal{B}$ such that $\tau^{-1} \circ \sigma$ is an identity on K if and only if $\tau(f_i)$ has a root in $\mathcal{B}$. Indeed, suppose that there exists $\gamma \in \mathcal{B}$ such that γ is a root of $\tau(f_i)$. Then, $\beta = \tau^{-1}(\gamma) \in M_X$ and f_i has a root $\beta \in M_X$. We claim that

$$K(\beta) = K(\alpha_i) = M_i.$$

Suppose otherwise. Then consider $\lambda : K(\alpha_i) \to K(\beta)$ keeping K fixed and its extension $\tilde{\lambda}$ to M. Since $\tilde{\lambda}$ fixes K, by our assumption on $\{M_i\}$, we conclude that either $K(\beta) = K(\alpha_i)$, or $K(\beta) \not\subset M$ and, in particular, $K(\beta) \not\subset M_X$. Thus, $i \in X$ and $M_i \subset M_X$, and we can set $\sigma(M_i) = \tau(M_i)$. Conversely, if the required σ exists, then $\tau^{-1} \circ \sigma : M_i \to M_X$, and thus $i \in X$ and $\tau(\alpha_i)$ will satisfy $\tau(f_i)$. Therefore, we just need to check systematically all elements of $\mathcal{B}$ until we find a root for $\tau(f_i)$ or $\tau(f_j)$. $\square$

From Lemma 3.3 and Lemma 3.4, we immediately obtain the main theorem of this section.

THEOREM 3.5. *Let K be a computable finitely generated field, and let* $\mathbf{d}$ *be a Turing degree. Then, there is an algebraic extension of K the isomorphism type of which has degree* $\mathbf{d}$.

Below, we present some other sequences which can be used to satisfy the requirements of Lemma 3.4. These constructions have somewhat different flavour from the construction in Lemma 3.3.

EXAMPLE 3.6. Let $K = \mathbb{Q}$, and let $\{p_i, i \in \omega\}$ be the listing of rational primes. Let $f_i(T) = T^2 - p_i$. It is clear that, in this case, the sequence $\{M_i, i \in \omega\}$, where $M_i = (\sqrt{p_i})$, is stable and totally linearly disjoint over $\mathbb{Q}$, and thus all the requirements of Lemma 3.4 are satisfied.

EXAMPLE 3.7. Let $K = \mathbb{Q}(x)$, where x is not algebraic over $\mathbb{Q}$. Let $\{p_i, i \in \omega\}$ be the listing of rational primes. Let $M_i = \mathbb{Q}(x, \sqrt[p_i]{x^2+1})$. Then, the sequence $\{M_i, i \in \omega\}$ is linearly disjoint over $\mathbb{Q}(x)$ and is stable with respect to $\mathbb{Q}(x)$. Hence, Lemma 3.4 applies again.

EXAMPLE 3.8. Let $\mathbb{G}_p$ be a finite field. Let x be transcendental over $\mathbb{G}_p$. Let $\{p_i, i \in \omega\}$ be a listing of rational primes as before. Let α_i be of degree p_i over $\mathbb{G}_p$. Let $M_i = \mathbb{G}_p(\alpha_i, x)$. Then, $\{M_i, i \in \omega\}$ is totally linearly disjoint over $\mathbb{G}_p(x)$ and is stable with respect to $\mathbb{G}_p(x)$.

EXAMPLE 3.9. Let $K = \mathbb{Q}(x)$ or $K = \mathbb{F}_p(x)$, where x is not algebraic over $\mathbb{Q}$ or $\mathbb{F}_p$, respectively. Let $M_i = \mathbb{Q}(\sqrt{x^2 + i})$ or $M = \mathbb{F}_p(\sqrt{x^2 + i})$. Then, $\{M_i, i \in \omega\}$ is totally linearly disjoint over $\mathbb{Q}(x)$ and $\mathbb{F}_p(x)$, respectively, and is stable with respect to $\mathbb{Q}(x)$ and $\mathbb{F}_p(x)$, respectively.

4. *Rings the isomorphism type of which has arbitrary Turing degree*

In this section, we consider sequences of integrally closed subrings of product formula fields: number fields (finite extensions of $\mathbb{Q}$), and finite extensions of rational function fields. In the case of a function field, we will let the constant field be an arbitrary computable field with a splitting algorithm. (For a discussion of fields with splitting algorithms, see [**10**, Sections 17.1, 17.2].)

Let K be a product formula field. In the case of a function field, let C be the constant field, and let $x \in K$ be a non-constant element. If characteristic p is such that $p > 0$, then assume that x is not a pth power in K. Under our assumptions, $K/\mathbb{Q}$ in the case of a number field, or $K/C(x)$ in the case of a function field is a finite and separable extension. Let $R = \mathbb{Z}$ in the case of a number field, and let $R = C[x]$ in the case of a function field. Then all prime ideals of R correspond to prime numbers or irreducible monic polynomials. In the case of $\mathbb{Z}$, these ideals also represent all non-Archimedean valuations of $\mathbb{Q}$, while in the case of $C(x)$, a valuation corresponding to the degree of polynomials does not correspond to a prime ideal of $C[x]$. (It does, however, correspond to a prime ideal of $C[1/x]$.) Now, let O_K be the integral closure of $\mathbb{Z}$ or $C[x]$ in K. Then O_K is called the ring of algebraic integers or integral functions, depending on the choice of K. Now, the prime ideals of R do not necessarily remain prime in O_K, but every prime ideal of R will have finitely many factors in O_K. The set of all prime ideals of O_K (together with the factors of the degree valuation in the case of function fields) will form what is called the set of primes of K. If $x \in (O_K)^*$ and $\mathfrak{p}$ is a prime ideal in O_K, then we define $\operatorname{ord}_{\mathfrak{p}} z$ to be the largest non-negative number such that $z \in (\mathfrak{p})^n$. If $w \in K$, we write $w = z_1/z_2$ for some $z_1, z_2 \in (O_K)^*$, and let $\operatorname{ord}_{\mathfrak{p}} w = \operatorname{ord}_{\mathfrak{p}} z_1 - \operatorname{ord}_{\mathfrak{p}} z_2$. Finally, we set $\operatorname{ord}_{\mathfrak{p}} 0 = \infty$. For more material on valuations and primes of number fields and function fields, the reader is referred to [**5**, **10**, **16**, **25**].

Using the Strong Approximation Theorem (see [**10**, p. 21; **25**, p. 268]), it can be shown that any integrally closed subring of K, the fraction field of which is K, is of the form

$$O_{K,\mathcal{W}} = \{z \in K : (\forall \mathfrak{p} \notin \mathcal{W})[\operatorname{ord}_{\mathfrak{p}} z \geqslant 0]\},$$

where $\mathcal{W}$ is an arbitrary set of (non-Archimedean) primes of K. In case $\mathcal{W}$ is finite, this ring is called a *ring of $\mathcal{W}$-integers*. Unfortunately, there is no universally accepted name for these rings when $\mathcal{W}$ is infinite. Since R is a principal ideal domain, O_K is a free R-module. The basis of O_K as a free R-module is called an *integral basis* of K over R. Therefore, if we represent elements of K using their coordinates with respect to an integral basis, then O_K is computable under such representation.

In what follows, we use an effective listing of primes of a product formula field and basically identify the ith prime on the list with a natural number i (for the purpose of satisfying the requirements of Theorem 2.2). To carry out this plan, we need some way to represent the primes and to make sure that the order at a given prime is computable. There are several ways to do this. In the next lemma, we describe one of them.

LEMMA 4.1. *Let K be a product formula field. Consider the pairs of K-elements representing the primes of K, as described below. Then, given an element $x \in K$ presented by its coordinates with respect to some integral basis of K over its rational subfield, there is an effective procedure to determine which primes of K occur in the divisor of x, as well as the order of x at each of the primes occurring in its divisor.*

Proof. Let $\mathfrak{p}$ be a prime of K. First, by the Weak Approximation Theorem, there exists $t_{\mathfrak{p}} \in O_K$ such that $\text{ord}_{\mathfrak{p}} t_{\mathfrak{p}} = 1$ and for any $\mathfrak{q} \neq \mathfrak{p}$, which is conjugate to $\mathfrak{p}$ over the rational field, $\text{ord}_{\mathfrak{q}} t_{\mathfrak{p}} = 0$. We will identify each prime of K with a pair $(t_{\mathfrak{p}}, p)$, where p is the prime number or the monic irreducible polynomial below $\mathfrak{p}$ in the rational field.

Since we assume that the constant field is computable and has a splitting algorithm, we can produce a computable sequence of all primes of the rational field. Then, for all but finitely many primes p of the rational field, using [**28**, Lemma 4.1, p. 135; **10**, Lemma 17.5, p. 232; **20**, Proposition 25, p. 27] we have an effective procedure for determining the number and relative degree of all K-factors of p. Next, by a systematic search of O_K, using the rational norms of elements of O_K, we can locate $t_{\mathfrak{p}}$ for each factor of a given rational prime p. Finally, for an arbitrary element x of K, by looking at its rational norm, we can determine a finite superset of the primes occurring in its divisor. Then, using the pairs for these potentially occurring primes, we can determine the actual divisor of x. □

REMARK 4.2. For future use, we also note here that for any K-prime $\mathfrak{p}$, by the Strong Approximation Theorem, there exists an element $z_{\mathfrak{p}} \in K$ such that $\mathfrak{p}$ is the only prime of K with $\text{ord}_{\mathfrak{p}} z_{\mathfrak{p}} < 0$. Furthermore, there exist $a_{\mathfrak{p}}, b_{\mathfrak{p}} \in O_K$ such that $z_{\mathfrak{p}} = a_{\mathfrak{p}}/b_{\mathfrak{p}}$. Having constructed sequence $\{(p, t_{\mathfrak{p}})\}$ for each prime of K, we can effectively locate $z_{\mathfrak{p}}$, $a_{\mathfrak{p}}$, and $b_{\mathfrak{p}}$.

We are now ready to prove the technical lemma leading to the main result of this section.

LEMMA 4.3. *Let K be a computable product formula field with a finite degree separable rational subfield F (so either $F = \mathbb{Q}$ or $F = C(x)$ for some $x \in K \setminus C$). In the case of a function field, assume that the constant field C is computable, finitely generated, and has a splitting algorithm. Let $\{p_i, i \in \omega\}$ be an effective listing of primes of F (that is, prime numbers or monic irreducible polynomials in x over C). For each p_i, choose one K-factor p_i, and let $\{(p_i, t_{\mathfrak{p}_i})\}$ be the listing of primes of K corresponding to $\{p_i, i \in \omega\}$, where $t_{\mathfrak{p}_i}$ has the least code in the set*

$$\{t_{\mathfrak{q}} : q \text{ is a } K\text{-factor of } p\}.$$

Let $\mathcal{A} = R$ (that is, $\mathcal{A} = \mathbb{Z}$ if K is a number field, and $\mathcal{A} = C[x]$ if K is a function field). Let $\mathcal{W} = \emptyset$ if K is a number field, and let $\mathcal{W}$ be the (finite) set of all K-poles of x if K is a function field. For any $X \subseteq \omega$, let

$$\mathcal{W}_X = \mathcal{W} \cup \{\mathfrak{p}_i : i \in X\}.$$

Then, for any Turing degree $\mathbf{d}$, there is a set X consisting of primes of K such that $O_{K,\mathcal{W}_X}$ has degree $\mathbf{d}$.

Proof. Let

$$\mathcal{A}_X = O_{K,\mathcal{W}_X},$$

and let $\mathcal{A}_i = \mathcal{A}_{\{i\}}$. Then, we will show that the sequence $\{\mathcal{A}_i, i \in \omega\}$ satisfies the conditions of Theorem 2.2.

In the case of K being a function field, we observe that for all $i \in \omega$, $\mathfrak{p}_i$ is not a pole of x in K. This is true since each $\mathfrak{p}_i$ is a factor of a $C(x)$-prime corresponding to a polynomial in x. We next verify that all conditions of Theorem 2.2 hold.

(1) We show that $O_{K,\mathcal{W}_X} \leqslant_T X$ for any $X \subseteq \omega$. Assuming that we represent elements of K as n-tuples of their coordinates with respect to some integral basis of K over its rational subfield F, by Lemma 4.1, we can effectively compute the divisors of elements of K. Assuming that the element is in $O_{K,\mathcal{W}_\omega}$, we can next determine for each pair (p_i, p_i) with p_i occurring in the norm, whether $i \in X$, and thus establish whether the element is in $O_{K,\mathcal{W}_X}$. (Conversely, if we have the characteristic function of $O_{K,\mathcal{W}_X}$, to determine whether $i \in X$, then it is enough to determine whether $z_{\mathsf{p}_i} \in O_{K,\mathcal{W}_X}$. Hence, we have $O_{K,\mathcal{W}_X} \equiv_T X$.)

(2) Let $\sigma : O_{K,\mathcal{W}_i} \hookrightarrow O_{K,\mathcal{W}_X}$ for some $i \in \omega$ and $X \subset \omega$ with $\sigma|_{C[x]} = \mathrm{id}$ in the case K is a function field. We show that $i \in X$. First, observe that σ can be extended to an automorphism of K, keeping $C(x)$ fixed. Next, observe that $\sigma(z_{\mathsf{p}_i}) = \sigma(a_{\mathsf{p}_i}/b_{\mathsf{p}_i})$ is an element that has negative order only at a prime $\sigma(\mathsf{p}_i)$. By the definition of $O_{K,\mathcal{W}_X}$, $\sigma(\mathsf{p}_i) \in \mathcal{W}_X \subset \mathcal{W}_\omega$. However, $\mathcal{W}_X$ contains only one factor for each p_i. Therefore

$$\sigma(\mathsf{p}_i) = \mathsf{p}_i \in \mathcal{W}_X \Leftrightarrow i \in X.$$

The case of K being a number field is similar.

(3) Let $\tau : O_{K,\mathcal{W}_X} \leftrightarrow \mathcal{B}$ be an isomorphism of rings. Let $i, j \in \omega$, $i \neq j$, be such that for exactly one of $k = i$ and $k = j$, there exists an embedding $\sigma : O_{K,\mathcal{W}_k} \hookrightarrow \mathcal{B}$ with $\tau^{-1} \circ \sigma|_{C[x]} = \mathrm{id}$, in the case when K is a function field. We show that, relative to $\mathcal{B}$, we can compute which of the rings $O_{K,\mathcal{W}_i}$ and $O_{K,\mathcal{W}_j}$ is embeddable into $\mathcal{B}$ in the prescribed manner. Since C is finitely generated and O_K has a basis over R, in $\mathcal{B}$ we can list $\tau(O_K)$ and effectively find $\tau(a_{\mathsf{p}_i})$, $\tau(b_{\mathsf{p}_i})$, $\tau(a_{\mathsf{p}_j})$, and $\tau(b_{\mathsf{p}_j})$. Finally, we systematically look for the solution in $\mathcal{B}$ of the equations $\tau(b_{\mathsf{p}_i})Z = \tau(a_{\mathsf{p}_i})$, and $\tau(b_{\mathsf{p}_j})W = \tau(a_{\mathsf{p}_j})$. It is clear that, by assumption, exactly one of these equations will have a solution. On the other hand, $\tau(b_{\mathsf{p}_k})Z = \tau(a_{\mathsf{p}_k})$ has a solution in $\mathcal{B}$ if and only if $\sigma : O_{K,\mathcal{W}_k} \hookrightarrow \mathcal{B}$ as specified above exists. □

We now state the main theorem of this section.

THEOREM 4.4. *Let K be any finitely generated infinite computable field. Then for any Turing degree $\mathbf{d}$, there is a ring $R \subset K$, the fraction field of which is K, such that the degree of the isomorphism type of R is $\mathbf{d}$.*

Proof. Any finitely generated, infinite, computable field is either a number field or a function field in several variables over a number field or a finite field. The latter is also a product formula field. More specifically, let K be such a field. First of all, note that we can always select a separating transcendence base for K [**10**, Chapter 17]. Let $x_1, \ldots, x_k$ be the elements of this base. Then, for some number field or finite field C, $K/C(x_1, \ldots, x_k)$ is a finite separable extension. Let $\tilde{C}$ be the algebraic closure of $C(x_1, \ldots, x_{k-1})$ in K. Then $\tilde{C}$ is computable and has a splitting algorithm (see [**10**, Chapter 17] again). Furthermore, K is of transcendence degree 1 over $\tilde{C}$, and thus we can think of K as a product formula field over a constant field, satisfying the requirements of Lemma 4.3. □

In lieu of further examples, we offer a different version of Lemma 4.3. The proof is very similar.

LEMMA 4.5. *Let K, F, and C be as in Lemma 4.3. Let $\{\mathsf{p}_i, i \in \omega\}$ be an effective listing of primes of K, excluding in the function field case the poles of some non-constant element x such that $F = C(x)$. Let $\mathcal{A} = O_K$. Let $\mathcal{W} = \emptyset$, if K is a number field, and let $\mathcal{W}$ be the (finite)* *set of all K-poles of x. For any $X \subseteq \omega$, let*

$$\mathcal{W}_X = \bigcup_{i \in X} \{\mathsf{p} \text{ is a factor of } p_i\} \cup \mathcal{W}.$$

Let $\mathcal{W}_i = \mathcal{W} \cup \{\text{all } K\text{-factors of } p_i\}$. *Let*

$$\mathcal{A}_X = O_{K,\mathcal{W}_X}.$$

Let $\mathcal{A}_i = O_{K,\mathcal{W}_i}$. *Then the sequence* $\{\mathcal{A}_i, i \in \omega\}$ *satisfies the conditions of Theorem* 2.2.

5. *Torsion-free Abelian groups of arbitrary finite rank and of arbitrary Turing degree*

Rank 1, torsion-free, Abelian groups are isomorphic to subgroups of $(\mathbb{Q}, +)$. There is a known classification, due to Baer [**4**], of countable, rank 1, torsion-free, Abelian groups. The account here will generally follow the one in a book by Fuchs [**11**, Volume 2]. Given such a group $\mathcal{G}$, for any prime p, we define a function $h_p : G \to \omega$ by setting $h_p(a)$ equal to the largest natural number k such that there is some $b \in G$ with $p^k b = a$. If no such k exists, then we set $h_p(a) = \infty$. Now define the characteristic of a to be the sequence

$$\chi_{\mathcal{G}}(a) = (h_{p_1}(a), h_{p_2}(a), \ldots),$$

where $(p_i)_{i \in \omega}$ are the prime numbers.

In some torsion-free Abelian groups (for example, $(\mathbb{Q}, +)$), it is the case that all non-zero elements have the same characteristic. In these groups, we would need to look no further for invariants. However, in some others (for example, $(\mathbb{Z}, +)$) the characteristics of the various elements are essentially the same, but not identical. We say that two characteristics are equivalent if they are equal except in a finite number of places, and in all places where they differ, both are finite. An equivalence class of characteristics under this relation is called a type. If $\chi_{\mathcal{G}}(a)$ belongs to a type $\mathbf{t}$, then we set $\mathbf{t}_{\mathcal{G}}(a) = \mathbf{t}$ and say that a is of type $\mathbf{t}$. A group $\mathcal{G}$ in which any two non-zero elements have the same type $\mathbf{t}$ is homogeneous, and we say that $\mathbf{t}(\mathcal{G}) = \mathbf{t}$ is the type of $\mathcal{G}$. In particular, we note that any torsion-free Abelian group of rank 1 is homogeneous.

Proposition 5.1 [**4**]. *If $\mathcal{G}$ and $\mathcal{H}$ are torsion-free Abelian groups of rank* 1, *then $\mathcal{G} \simeq \mathcal{H}$ if and only if $\mathbf{t}(\mathcal{G}) = \mathbf{t}(\mathcal{H})$.*

Knight and Downey [**8**] established that for an arbitrary Turing degree $\mathbf{d}$, there exists a torsion-free Abelian group $G_{\mathbf{d}}$ of rank 1 and of finite type, such that the isomorphism type of $G_{\mathbf{d}}$ has degree $\mathbf{d}$. Downey and Jockusch also showed that even rank 1 groups of finite type can fail to have degree (see [**8**], again). On the other hand, it follows from a computability theoretic result of Coles, Downey and Slaman, using a result by Downey, Jockusch and Solomon, that every torsion-free Abelian group of rank 1 has first jump degree (see [**6**]). Coles, Downey and Slaman proved that for every set $A \subseteq \omega$, there is a Turing degree that is the least degree of the jumps of all sets X for which A is computably enumerable in X. On the other hand, Richter [**26**] constructed a non-computably enumerable set that is computably enumerable in two sets that form a minimal pair. This construction implies that there is a set A such that the set of all X for which A is computably enumerable in X has no member of least Turing degree.

The proof by Knight and Downey of the claim that for an arbitrary Turing degree $\mathbf{d}$, there exists a torsion-free Abelian group of rank 1 (and finite type) with isomorphism type of degree

$\mathbf{d}$ is as follows. Let a set $D \subseteq \omega$ be of degree $\mathbf{d}$. Let $\mathcal{G}$ be a rank 1 group defined by the type sequence $(a_n)_{n\in\omega}$ such that

$$a_n = \begin{cases} 1 & \text{if } n \in D \oplus \overline{D}, \\ 0 & \text{if } n \notin D \oplus \overline{D}. \end{cases}$$

There is an isomorphic copy of $\mathcal{G}$ computable in D. Conversely, let $\mathcal{H} \cong \mathcal{G}$. Then, $\mathcal{H}$ has the type sequence $(a_n)_{n\in\omega}$ (by Proposition 5.1). The type sequence of $\mathcal{H}$ is computably enumerable in $\mathcal{H}$, and hence $D \oplus \overline{D}$ is computably enumerable in $\mathcal{H}$. Thus, $D \leqslant_T \mathcal{H}$.

More generally, the construction from Section 4 can be adapted to give examples of torsion-free Abelian groups of finite rank satisfying conditions of Theorem 2.2 and thus produce torsion-free Abelian groups of finite rank and of arbitrary Turing degree $\mathbf{d}$.

THEOREM 5.2. *Let k be a positive integer, and let $\mathbf{d}$ be a Turing degree. There is a torsion-free Abelian group $G_{\mathbf{d}}$ of rank k such that the isomorphism type of $G_{\mathbf{d}}$ has degree $\mathbf{d}$.*

Proof. Let $\{p_i, i \in \omega\}$ be a listing of rational primes. Let $\mathcal{A} = \mathbb{Z}^k$. For any $X \subseteq \omega$, let $\mathcal{V}_X = \{p_i : i \in X\}$. Let

$$\mathcal{A}_X = (O_{\mathbb{Q},\mathcal{V}_X})^k.$$

For each $i \in \omega$, let $\mathcal{A}_i = (O_{\mathbb{Q},\{p_i\}})^k$. Then, the proof that the sequence $\{\mathcal{A}_i, i \in \omega\}$ satisfies the conditions of Theorem 2.2 is almost identical to the proof of Lemma 4.3. $\square$

6. *Algebraic structures of isomorphism types with no Turing degrees*

We now want to investigate structures the isomorphism types of which have no Turing degrees. The enumeration reducibility will play an important role. First, we define the canonical index u of a finite set $D_u \subset \omega$. Let $D_0 = \emptyset$. For $u > 0$, let $D_u = \{n_0, \ldots, n_{k-1}\}$, where $n_0 < \cdots < n_{k-1}$ and $u = 2^{n_0} + \cdots + 2^{n_{k-1}}$. For sets $X, Y \subseteq \omega$, we say that Y is *enumeration reducible* to X, in symbols $X \leqslant_e Y$, if there is a computably enumerable binary relation E such that

$$(x \in X) \Leftrightarrow (\exists u)[D_u \subseteq Y \wedge E(x, u)].$$

Equivalently, $X \leqslant_e Y$ if and only if for every set S, if Y is computably enumerable relative to S, then X is computably enumerable relative to S. Richter showed that a modification of Theorem 2.1 obtained by replacing Turing reducibility $\mathcal{A}_X \leqslant_T X$ by the enumeration reducibility $\mathcal{A}_X \leqslant_e X$ yields a very different conclusion.

THEOREM 6.1 [**26**]. *Let T be a theory in a finite language L such that there is a computable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of finite structures for L, which are pairwise non-embeddable. Assume that for every set $X \subseteq \omega$, there is a countable model $\mathcal{A}_X$ of T such that*

$$\mathcal{A}_X \leqslant_e X,$$

and for every $j \in \omega$,

$$\mathcal{A}_j \text{ is embeddable in } \mathcal{A}_X \Leftrightarrow j \in X.$$

Then, there is a set X such that the isomorphism type of $\mathcal{A}_X$ does not have a degree.

Richter used Theorem 6.1 to show that there is a torsion Abelian group the isomorphism type of which has no degree. The following result is a generalization of Theorem 6.1, which allows infinite structures in the computable sequence of structures. It should be noted that the hypotheses of this theorem are similar to those of Theorem 2.2.

THEOREM 6.2. *Let $\mathcal{C}$ be a class of countable structures in a finite language L, closed under isomorphisms. Assume that there is a computable sequence $\{\mathcal{A}_i, i \in \omega\}$ of computable (possibly infinite) structures in $\mathcal{C}$ satisfying the following conditions.*

- *There exists a finitely generated structure $\mathcal{A} \in \mathcal{C}$ such that for all $i \in \omega$, $\mathcal{A} \subset \mathcal{A}_i$.*
- *For any $X \subseteq \omega$, there is a structure $\mathcal{A}_X$ in $\mathcal{C}$ such that $\mathcal{A} \subset \mathcal{A}_X$ and*

$$\mathcal{A}_X \leqslant_e X,$$

and for every $i \in \omega$, there exists an embedding σ such that

$$\sigma : \mathcal{A}_i \hookrightarrow \mathcal{A}_X, \quad \sigma_{|\mathcal{A}} = \mathrm{id},$$

if and only if $i \in X$.

- *Suppose that any $\mathcal{A}_X$ is isomorphic to some structure $\mathcal{B}$ under isomorphism $\tau : \mathcal{A}_X \leftrightarrow \mathcal{B}$. Then from any enumeration of $\mathcal{B}$, we can effectively enumerate those i for which $\mathcal{A}_i \hookrightarrow \mathcal{B}$ under an embedding σ with $(\tau^{-1} \circ \sigma)_{|\mathcal{A}} = \mathrm{id}$.*

Then, there is a structure $\mathcal{A}_X$ in $\mathcal{C}$ the isomorphism type of which has no Turing degree.

Proof. As for Theorem 6.1, the proof follows from the following well-known lemma.

LEMMA 6.3. *There is a set $X \subseteq \omega$ such that the set of functions*

$$\{f : \mathrm{ran}(f) = X\}$$

has no least Turing degree.

We omit the proof of Lemma 6.3 here. The idea is to construct a set X, which is not computably enumerable, but the set of enumerations of X contains two enumerations the Turing degrees of which form a minimal pair. A full proof may be found in [**26**].

Let X be as in Lemma 6.3, and, toward contradiction, let $\mathcal{M}$ be a copy of $\mathcal{A}_X$ with least Turing degree. Now, we will argue that $\mathcal{M}$ has a least enumeration. Let ν be an enumeration of $\mathcal{M}$ where $\deg(\nu) = \deg(\mathcal{M})$. Let g be another enumeration of $\mathcal{M}$ with $\nu \not\leqslant_T g$. By padding (that is, by passing to a structure the elements of which are of the form (x, t), where x is enumerated into M at time t, and where the operations are the obvious ones), we can obtain an isomorphic copy of $\mathcal{M}$ with the same Turing degree as g, which is a contradiction. Hence, by the final assumption of the theorem, we can pass effectively from ν to an enumeration $\nu_{\mathcal{M}}$ of X.

Let f be an enumeration of X, and we will show that $\nu_{\mathcal{M}} \leqslant_T f$. By assumption, we can pass effectively from f to an enumeration g of $\mathcal{A}_X$, and, since $\mathcal{A}_X \leqslant_e X$ and $\mathcal{M} \leqslant_T \mathcal{A}_X$ (because here we identify $\mathcal{M}$ with its atomic diagram), we can pass effectively from g to an enumeration $\tilde{g}$ of $\mathcal{M}$. Now $\mathcal{M} \leqslant_T \tilde{g}$, and so we have $\nu_{\mathcal{M}} \leqslant_T \mathcal{M} \leqslant_T f$, as was to be shown. This is a contradiction. □

THEOREM 6.4. *There are countable fields, rings, and torsion-free Abelian groups of arbitrary finite rank, the isomorphism types of which do not have Turing degrees.*

Proof. Since the statements of Theorem 6.4 and Theorem 2.2 differ in the place where $\mathcal{A}_X \leqslant_T X$ is replaced with $\mathcal{A}_X \leqslant_e X$, to prove the analogues of Theorems 3.5, 4.4 and 5.2, we need to show that the respective structures satisfy $\mathcal{A}_X \leqslant_e X$.

We will start with the field case, where all the notation and assumptions are as in Lemma 3.4. We need to show that $M_X \leqslant_e X$. Let $\phi : \omega \to X$ be any listing of X. Then, using ϕ, we can list the sets $\{\alpha_{\phi(i)}\}$ and $\{\alpha_{\phi(i_1)}^{m(\phi(i_1))}\cdots\alpha_{\phi(i_k)}^{m(\phi(i_k))}\}$, where $0 \leqslant m(\phi(i_l)) < \deg(f_{\phi(i_l)})$ and $k \in \omega$. Thus, we will be able to list the basis of M_X over K, and then M_X itself. (As in the case of Turing reducibility, it is not hard to see that $X \leqslant_e M_X$ also, and therefore, we really have enumeration equivalence.)

We now proceed to the ring case. Here, all the notation and assumptions are as in Lemma 4.3. Let ϕ be as above, and note that, given ϕ, we can list the set $\{p_{\phi(i)}, \mathsf{p}_{\phi(i)}\}$. The listing of primes will then allow the listing of $O_{K,\mathcal{W}_X}$, where we would proceed by testing elements of K to see whether the primes in the denominator of their divisors have appeared in the list already. This testing process is effective as discussed in Section 4. (As above, we also have here that $X \leqslant O_{K,\mathcal{W}_X}$.) The case of the Abelian groups from Section 5 is again almost identical to the case of the rings.

For the condition on listing the i such that $\mathcal{A}_i$ is embeddable in a copy of $\mathcal{A}_X$, the earlier proofs suffice. In each case, we proved the hypotheses for Theorem 2.2 by first establishing the hypotheses for this theorem. $\square$

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Wesley Calvert
Department of Mathematics and Statistics
Murray State University
Murray
KY 42071
USA

wesley.calvert@murraystate.edu

Valentina Harizanov
Department of Mathematics
George Washington University
Washington
DC 20052
USA

harizanv@gwu.edu

Alexandra Shlapentokh
Department of Mathematics
East Carolina University
Greenville
NC 27858
USA

shlapentokha@mail.ecu.edu