On Decidable Categoricity and Almost Prime Models*

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Abstract

The complexity of isomophisms for computable and decidable structures plays an important role in computable model theory. Goncharov [26] defined the *degree of decidable categoricity* of a decidable model $\mathcal{M}$ to be the least Turing degree, if it exists, which is capable of computing isomorphisms between arbitrary decidable copies of $\mathcal{M}$. If this degree is $\mathbf{0}$, we say that the structure $\mathcal{M}$ is *decidably categorical*. Goncharov established that every computably enumerable degree is the degree of categoricity of a prime model, and Bazhenov showed that there is a prime model with no degree of categoricity. Here we investigate the degrees of categoricity of various prime models with added constants, also called *almost prime models*. We relate the degree of decidable categoricity of an almost prime model $\mathcal{M}$ to the Turing degree of the set $C(\mathcal{M})$ of complete formulas. We also investigate uniform decidable categoricity, characterizing it by primality of $\mathcal{M}$ and Turing reducibility of $C(\mathcal{M})$ to the theory of $\mathcal{M}$.

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1 Introduction and preliminaries

Computable model theorists are interested in algorithmic isomorphisms of stuctures, because non-effective isomorphisms often do not transfer computability-theoretic properties of structures. In particular, algorithmic categoricity has been extensively studied in computable model theory and dates back to Fröhlich and Shepherson and Mal'cev. This paper deals with the complexity of isomorphisms of decidable structures. We will assume that structures are countable and in computable languages. We say that a structure $\mathcal{A}$ is *computable* (or *constructive*) if its domain A is computable and its relations and functions are uniformly computable. That is, $\mathcal{A}$ is computable if A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A such that the atomic diagram of $\mathcal{A}$ is decidable. A structure $\mathcal{A}$ is *decidable* (or *strongly constructive*) if A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A such that the elementary (also called complete) diagram $\mathrm{Th}((\mathcal{A}, a_i)_{i\in\omega})$ of $\mathcal{A}$ is decidable. Every decidable theory has a decidable model and every type realized in a decidable model is computable. The set of all types of T realized in a decidable model of T is effectively enumerable.

For a Turing degree $\mathbf{d}$, a computable structure $\mathcal{A}$ is called $\mathbf{d}$-*computably categorical* if, for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. The $\mathbf{d}$-*computable dimension* of a computable structure $\mathcal{A}$ is the number of computable isomorphic copies of $\mathcal{A}$, up to $\mathbf{d}$-computable isomorphism. Hence, a computably categorical structure has computable dimension 1. Many structures from natural algebraic classes have computable dimension 1 or ω. Goncharov was the first to produce examples of computable structures of finite computable dimension greater than 1. In [30], he showed that for every finite $n \geq 2$, there is a computable structure of computable dimension n (see also [38, 29].)

More recently, Goncharov initiated the investigation of the degrees of categoricity of computable structures (see [22]), as well as of decidable structures (see [26]). In [22], Fokina, Kalimullin, and R. Miller defined, for a computable structure $\mathcal{C}$, the *degree of categoricity* of $\mathcal{C}$ to be the least Turing degree in $\{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computably categorical}\}$, if it exists. Not every computable structure has a degree of categoricity. For more information and related results on which degrees can be degrees of categoricity of computable structures, both in general and for specific natural algebraic classes, see [44, 15, 1, 21, 7, 6, 12, 24, 5, 3, 14, 10, 9, 20, 13].

Definition 1. For a Turing degree $\mathbf{d}$, a computable structure $\mathcal{A}$ is $\boldsymbol{d}$-*decidably categorical* if for every decidable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a $\boldsymbol{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

In particular, with $\mathbf{d} = \mathbf{0}$, $\mathcal{A}$ is *decidably categorical* if for each decidable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

The following characterization of decidably categorical structures is due to Nurtazin [45].

Theorem 1. *([45]) Let $\mathcal{A}$ be a decidable structure. Then $\mathcal{A}$ is decidably categorical if and only if there is a finite tuple $\overrightarrow{c}$ of elements in A such that $(\mathcal{A}, \overrightarrow{c})$ is a prime model of the theory $Th(\mathcal{A}, \overrightarrow{c})$ and the set of complete formulas of this theory is computable.*

Moreover, Nurtazin proved that if there is no such $\overrightarrow{c}$, then there are infinitely many decidable copies of $\mathcal{A}$, no two of which are computably isomorphic.

The degree of decidable categoricity captures the least degree in the set of all Turing degrees capable of computing isomorphisms between decidable isomorphic copies of structures. More precisely, the *decidable categoricity spectrum* of a decidable structure $\mathcal{A}$ is the following set of Turing degrees:

$$DCatSpec(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-decidably categorical}\}.$$

The *degree of decidable categoricity* of $\mathcal{A}$, if it exists, is the least Turing degree in DCatSpec($\mathcal{A}$). Not every decidable structure has a degree of decidable categoricity. In [26], Goncharov showed that every computably enumerable degree is the degree of decidable categoricity of some decidable prime model. Goncharov also investigated decidable categoricity of models of Ehrenfeucht theories. In [27], he proved that there exists a decidable Ehrenfeucht theory T such that it has a decidable prime model that is decidably categorical, and T has a decidable almost prime model that is not decidably categorical. (A structure $\mathcal{M}$ is *almost prime* if, for some tuple $\overrightarrow{c}$ of elements of $\mathcal{M}$, $(\mathcal{M}, \overrightarrow{c})$ is a prime model of its theory.) Decidable categoricity and its degrees for various classes of structures were further investigated in the papers [28, 35, 37, 36, 33, 32, 34, 4, 41, 31, 11]. In [18], the authors investigated the notion of categoricity for n-decidable structures relative to m-decidable isomorphic copies, for $m, n \geq 0$, where a structure is called *n-decidable* if its Σ^0_n-diagram (equivalently, Π^0_n-diagram) is decidable.

Since prime models will play a significant role in our paper, let us review some relevant model-theoretic concepts and results.

Definition 2. For a complete theory T, a formula $\theta(\overrightarrow{x})$ is called *complete* if for every formula $\psi(\overrightarrow{x})$, either $T \vdash \theta(\overrightarrow{x}) \Rightarrow \psi(\overrightarrow{x})$ or $T \vdash \theta(\overrightarrow{x}) \Rightarrow \neg\psi(\overrightarrow{x})$. Hence the set $\{\beta(\overrightarrow{x}) : T \vdash \theta(\overrightarrow{x}) \Rightarrow \beta(\overrightarrow{x})\}$ forms a complete n-type where $n = \mathrm{lh}(\overrightarrow{x})$. A type that contains a complete formula is called *principal.* It is also said to be *generated* by $\theta(\overrightarrow{x})$.

We will write $C(\mathcal{M})$ for the set of all formulas that are complete for $\mathrm{Th}(\mathcal{M})$.

Proposition 1. *Let T be a decidable theory. Every principal type of T is a computable type, and the set of all principal types of T is a Π^0_1 set.*

Let $\mathcal{M}$ be an arbitrary (possibly uncountable) model. $\mathcal{M}$ is *atomic* if every n–tuple of elements from the domain of $\mathcal{M}$ satisfies a complete formula in the theory of $\mathcal{M}$.

Proposition 2. *Let T be a complete theory in an at-most-countable language.*

(i) A countable model $\mathcal{A}$ of T is prime if and only if $\mathcal{A}$ is atomic.

(ii) T has a prime model if and only if every formula consistent with T is a member of a principal type of T.

Let T be a complete decidable theory with a prime model. Then T has a prime model which is decidable in $\emptyset'$.

Theorem 2 (Goncharov-Nurtazin, Harrington)**.** *Let T be a complete decidable theory. The following are equivalent.*
(i) T has a decidable prime model.
(ii) T has a prime model and the set of all principal types of T is computable.

We will use $\preceq$ to denote elementary extensions of models.

Definition 3. A theory T is *model complete* if for every two models $\mathcal{A}$ and $\mathcal{B}$ of T,

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \preceq \mathcal{B}.$$

Neither one of completeness and model completeness implies the other.

Theorem 3. *The following are equivalent for a theory T in a language L.*
(i) The theory T is model complete.
(ii) If $\mathcal{A}$ and $\mathcal{B}$ are models of T and $\mathcal{A} \subseteq \mathcal{B}$, then every existential sentence of L_A true in $\mathcal{B}_A$ is also true in $\mathcal{A}_A$.
(iii) For every formula $\theta(\overrightarrow{x})$, there is a universal formula $\psi(\overrightarrow{x})$ such that

$$T \vdash [\theta(\overrightarrow{x}) \Leftrightarrow \psi(\overrightarrow{x})].$$

(iv) For every formula $\theta(\overrightarrow{x})$, there is an existential formula $\varphi(\overrightarrow{x})$ such that

$$T \vdash [\theta(\overrightarrow{x}) \Leftrightarrow \varphi(\overrightarrow{x})].$$

Every model complete theory has a $\forall\exists$-axiomatization. If a theory is $\forall\exists$-axiomatizable theory with infinite models and is κ-categorical for some infinite cardinal κ, then it is model complete. For more on these topics, see [42].

For more on computable model theory see [17, 19]. Computability-theoretic notation is as in [46]. We will use $\langle \quad \rangle_k$ to denote a computable bijection of ω^k onto ω such that its inverse functions are also computable, with k omitted when it is clear from the context. By $\deg(S)$ we will denote the Turing degree of S, and by $\leq_T$ Turing reducibility. In Section 2 we study degrees of categoricity of almost prime models and compare them to the degrees of the sets of complete formulas. In Section 2.1 we investigate uniform decidable categoricity and how it can be characterized by relating Turing degrees of complete formulas of a prime model to the Turing degree of the theory. It turns out that relative decidable categoricity is equivalent to uniform decidable categoricity after an expansion by constants.

2 Degrees of categoricity of almost prime models

Let $\overrightarrow{c}$ be a finite tuple of elements from the domain of a structure $\mathcal{A}$ such that $(\mathcal{A}, \overrightarrow{c})$ is a prime model of $\mathrm{Th}(\mathcal{A}, \overrightarrow{c})$. (Recall that an $\mathcal{A}$ having such a tuple is said to be *almost prime*.) In [26], Goncharov established that if this $\mathcal{A}$ is decidable, then the Turing degree $\mathbf{c}$ of the set $C(\mathcal{A}, \overrightarrow{c})$ of complete formulas of $(\mathcal{A}, \overrightarrow{c})$ is a c.e. degree such that $\mathcal{A}$ is $\mathbf{c}$-decidably categorical. In some cases, $\mathbf{c}$ is the degree of decidable categoricity for $\mathcal{A}$, that is, the least degree in the decidable categoricity spectrum. In fact, Goncharov [26] constructed, for each c.e. degree $\mathbf{c}$, a decidable prime model $\mathcal{M}$ such that $\deg(C(\mathcal{M})) = \mathbf{c}$ and $\mathbf{c}$ is the degree of decidable categoricity of $\mathcal{M}$. In [2], Bazhenov proved that there is a decidable prime model that does not have any degree of decidable categoricity. In this paper, we will give the full proof of results, which were presented in [25], about the degrees of categoricity of almost prime models of decidable theories and how they relate to the Turing degrees of the set of complete formulas. This will provide a positive answer to Question 1 in [2] regarding the possibility of having a degree of categoricity that is strictly below the Turing degree of the set of complete formulas with added constants. We then investigate uniformity for decidable categoricity.

In the following theorem c and d are constant symbols.

Theorem 4. *Let S be a c.e. set. Then there is a decidable theory T^S, in a language $\mathcal{L}_0$, and a decidable theory $T^c \supseteq T^S$ in $\mathcal{L}_0 \cup \{c\}$, satisfying all of the following. First, T^S has a decidable prime model $\mathcal{M}^S$ which is decidably categorical. Also, T^c has a decidable prime model $\mathcal{M}^c = (\mathcal{M}_c, c)$ such that deg(S) is both the degree of its complete formulas and the degree of decidable categoricity of $\mathcal{M}^c$. Furthermore, there is a prime model $\mathcal{M}^{c,d} = (\mathcal{M}_{c,d}, c, d)$ in $\mathcal{L}_0 \cup \{c, d\}$ such that $\mathcal{M}^{c,d}$ is decidably categorical and the set of complete formulas of Th$(\mathcal{M}^{c,d})$ is computable. Finally, $\mathcal{M}^S \preceq \mathcal{M}_c \preceq \mathcal{M}_{c,d}$.*

Proof. Let S be an infinite c.e. set. Fix a computable 1-1 enumeration of S: $s(0), s(1), \ldots$ Let $S_t = \{s(0), s(1), \ldots, s(t)\}$. We will first define the theory T^S.

The language $\mathcal{L}_0$ for T^S will consist of:

- unary predicate symbols $P_0, P_1, P_2, \ldots$
- unary predicate symbols $Q_0, Q_1, Q_2, \ldots$
- unary predicate symbols $A_0, A_1, A_2, \ldots$
- binary predicate symbols $F_0, F_1, F_2, \ldots$
- binary predicate symbols $G_0, G_1, G_2, \ldots$

The sets (unary predicates) P_t will be infinite, properly nested with all differences infinite, and all contained in P_0. That is, $P_0 \supset P_1 \supset P_2 \supset \cdots$

The sets (unary predicates) Q_i will be infinite, properly nested with all differences infinite, and all contained in Q_0. That is, $Q_0 \supset Q_1 \supset Q_2 \supset \cdots$

The sets P_0 and Q_0 are disjoint. The sets A_k are infinite, disjoint from P_0, disjoint from Q_0, and pairwise disjoint. The binary F-predicates will be functions from some P's to certain A's, encoding the enumeration of S. The binary

G-predicates will be functions from some Q's to corresponding A's. Corresponding to these rules, we have the following axioms. (There is some redundancy in the axioms, but there are all included for greater clarity.)

1. Infinite nesting axioms

Axioms saying that $P_{t+1} \subseteq P_t$ and $|P_t - P_{t+1}| = \infty$ for every $t \in \omega$:

$\forall x\ (P_{t+1}(x) \Rightarrow P_t(x))$
$\exists^{\geq m}\ x\ (P_t(x) \wedge \neg P_{t+1}(x))$ for $m \geq 1$

Axioms saying that $Q_{i+1} \subseteq Q_i$ and $|Q_i - Q_{i+1}| = \infty$ for every $i \in \omega$:

$\forall x\ (Q_{i+1}(x) \Rightarrow Q_i(x))$
$\exists^{\geq m}\ x\ (Q_i(x) \wedge \neg Q_{i+1}(x))$ for $m \geq 1$

2. Disjointness axioms

Axioms saying that $P_0 \cap Q_0 = \emptyset$; $P_0 \cap A_j = \emptyset$ and $Q_0 \cap A_j = \emptyset$, for every $j \in \omega$; and $A_j \cap A_l = \emptyset$ for all $j, l \in \omega$ such that $j \neq l$.
$\neg\exists x\ (P_0(x) \wedge Q_0(x))$
$\neg\exists x\ (P_0(x) \wedge A_j(x))$
$\neg\exists x\ (Q_0(x) \wedge A_j(x))$
$\neg\exists v\ (A_j(v) \wedge A_l(v))$

3. A-infinity axioms

Axioms saying that $|A_j| = \infty$ for every $j \in \omega$.
$\exists^{\geq m} v\ A_j(v)$ for every $m \geq 1$

The relations $F_0, F_1, \ldots$ will be binary relations connecting stages with elements enumerated into S. That is, F_t will connect elements of P_t, and hence $P_{t+1}, P_{t+2}, \ldots$, with elements in $A_{s(t)}$. More precisely, the relation F_t is an ∞-1 function mapping P_t *onto* $A_{s(t)}$. (Hence its restriction to P_u for $u > t$ is a function mapping P_u to, in fact, onto $A_{s(t)}$.)

The binary relation G_t will be a ∞-1 function mapping Q_t *onto* A_t.

4. F-function and G-function axioms

Hence we have the following axioms for every $t \in \omega$.
$(\forall x)(\forall v)[F_t(x, v) \Rightarrow P_t(x) \wedge A_{s(t)}(v)]$
$(\forall x)(\exists v)[P_t(x) \Rightarrow F_t(x, v)]$
$(\forall x)(\forall v)(\forall u)[(F_t(x, v) \wedge F_t(x, u)) \Rightarrow v = u]$
$(\forall v)[A_{s(t)}(v) \Rightarrow \exists^{\geq m} x\ F_t(x, v)]$ for every $m \geq 1$

$(\forall y)(\forall v)[G_t(y, v) \Rightarrow Q_t(y) \wedge A_t(v)]$
$(\forall y)(\exists v)[Q_t(y) \Rightarrow G_t(y, v)]$
$(\forall y)(\forall v)(\forall u)[(G_t(y, v) \wedge G_t(y, u)) \Rightarrow v = u]$
$(\forall v)[A_t(v) \Rightarrow \exists^{\geq m} y\ G_t(y, v)]$ for every $m \geq 1$

5. **Stage** $(t{+}1)$ **infinite F- and G-induced equivalence classes axioms**

For $t \geq 0$, we define $P_{t+1} \subset P_t$ and $F_{t+1} : P_{t+1} \to A_{s(t+1)}$ as follows. For every tuple of values, $v_0, \ldots, v_t$ from $A_{s(0)}, \ldots, A_{s(t)}$, respectively, and every v_{t+1}

from $A_{s(t+1)}$, there are infinitely many elements x in P_t that do not lie in P_{t+1} such that they have these values under $F_0, \ldots, F_t$, respectively. Additionally, there are infinitely many elements x in P_{t+1} such that they have those same values under $F_0, \ldots, F_t$, respectively, and have the value v_{t+1} under F_{t+1}. For $t \geq 0$, we similarly define $Q_{t+1} \subset Q_t$ and $G_{t+1} : G_{t+1} \to A_{t+1}$. Hence we have the following axioms for every $m \geq 1$.

$$(\forall v_0) \cdots (\forall v_t) \left[\bigwedge_{i=0}^{t} A_{s(i)}(v_i) \Rightarrow (\exists^{\geq m} x)[P_t(x) \wedge \neg P_{t+1}(x) \wedge \bigwedge_{i=0}^{t} F_i(x, v_i)] \right]$$

$$(\forall v_0) \cdots (\forall v_{t+1}) \left[\bigwedge_{i=0}^{t+1} A_{s(i)}(v_i) \Rightarrow (\exists^{\geq m} x)[P_{t+1}(x) \wedge \bigwedge_{i=0}^{t+1} F_i(x, v_i)] \right]$$

$$(\forall v_0) \cdots (\forall v_t) \left[\bigwedge_{i=0}^{t} A_i(v_i) \Rightarrow (\exists^{\geq m} y)[Q_t(y) \wedge \neg Q_{t+1}(y) \wedge \bigwedge_{i=0}^{t} G_i(y, v_i)] \right]$$

$$(\forall v_0) \cdots (\forall v_{t+1}) \left[\bigwedge_{i=0}^{t+1} A_i(v_i) \Rightarrow (\exists^{\geq m} x)[Q_{t+1}(y) \wedge \bigwedge_{i=0}^{t+1} (G_i(y, v_i)] \right]$$

The theory T^S is the deductive closure of these axioms in groups 1–5.

We will define finite language restrictions T_n of theory T^S for $n \in \omega$, setting $r(n) =_{\text{def}} \max\{s(0), \ldots, s(n)\}$. (Hence $r(n) \geq n$.) The axioms for T_n are those axioms above that involve the predicates $P_0, \ldots, P_n$, $Q_0, \ldots, Q_{r(n)}$, $A_0, \ldots, A_{r(n)}$, $F_0, \ldots, F_n$, $G_0, \ldots, G_{r(n)}$, plus the following additional axioms for every $m \geq 2$.

6. Infinity of the complement axioms for T_n, $n \geq 0$

$$\exists^{\geq m}\, x \left(\bigwedge_{i=0}^{r(n)} \neg A_i(x) \wedge \neg P_0(x) \wedge \neg Q_0(x) \right)$$

Lemma 1. *The theory T_n is consistent.*

Proof. We can build a canonical countable model $\mathcal{M}_n$ for T_n, uniformly in n, by choosing three pairwise disjoint infinite sets: V, X and Y. They can be chosen to be computable. The set V is the union of uniformly computable, pairwise disjoint, infinite sets $V = \bigcup_{k \in \omega} V^{[k]}$, where the k-th row set is $V^{[k]} = \{v_{\langle k,0 \rangle}, v_{\langle k,1 \rangle}, v_{\langle k,2 \rangle}, \ldots\}$ with elements listed without repetition. Similarly, we have $X = \bigcup_{k \in \omega} X^{[k]}$ and $Y = \bigcup_{k \in \omega} Y^{[k]}$. Now, uniformly in k, $X^{[k]}$ is a union of uniformly computable, pairwise disjoint, infinite sets indexed by tuples of length $(k+1)$, $X^{[k]} = \{x_{\langle k; j_0, \ldots, j_k \rangle} : j_0, \ldots, j_k \in \omega\}$. Similarly, for each k, uniformly in k, $Y^{[k]}$ is a union of uniformly computable, pairwise disjoint, infinite sets indexed by tuples of length $(k+1)$, $Y^{[k]} = \{y_{\langle k; j_0, \ldots, j_k \rangle} : j_0, \ldots, j_k \in \omega\}$. In both sets, the elements are listed without repetition.

We define $A_i = V^{[i]}$ for $i \leq r(n)$. We define P_0 as the disjoint union: $P_0 = P_n \cup (P_{n-1} - P_n) \cup (P_{n-2} - P_{n-1}) \cup \cdots \cup (P_0 - P_1)$, where $P_n = X^{[n]}$, $P_{n-1} - P_n = X^{[n-1]}, \ldots, P_0 - P_1 = X^{[0]}$. Similarly, we define Q_0 as the disjoint union: $Q_0 = Q_n \cup \bigcup_{1 \le k \le n} (Q_{k-1} - Q_k)$, where $Q_n = Y^{[n]}$, $Q_{k-1} - Q_k = Y^{[k-1]}$ for $k = 1, \dots, n$. The domain of the model $\mathcal{M}_n$ is $P_0 \cup Q_0 \cup \bigcup_{1 \le i \le r(n)} A_i$. For $x = x_{\langle k; j_0, \dots, j_k \rangle}$, we set $F_t(x, v_{\langle s(t), j_t \rangle})$ for $t = 0, \dots, k$. For $y = y_{\langle k; j_0, \dots, j_k \rangle}$, we set $G_t(y, v_{\langle t, j_t \rangle})$ for $t = 0, \dots, k$. □

Lemma 2. *Each theory T_n is complete, and is decidable uniformly in n.*

Proof. We can show that the theory T_n is $\aleph_0$-categorical by showing that it has a unique (up to isomorphism) countable model. Since T_n has only infinite models and is $\aleph_0$-categorical, it is complete by the Łoś-Vaught test. Hence T_n is a complete theory. Since T_n has a computable set of axioms, uniformly in n, it is decidable. □

Lemma 3. *The theory T_n is model complete. Hence every formula is T_n-equivalent to an existential formula, and every computable model of T_n is decidable.*

Proof. Since T_n is $\forall\exists$-axiomatizable with infinite models and is $\aleph_0$-categorical, it is model complete. □

Lemma 4. *The theory T^S is complete, decidable, and model complete.*

Proof. Since all T_n are complete and model complete and $T_n \subseteq T_{n+1} \subseteq T^S$, we have that T^S is a complete theory, and is also model complete. Since T^S has a computable set of axioms, it is decidable. □

Lemma 5. *The theory T^S has a decidable prime model $\mathcal{M}^S$, which is decidably categorical.*

Proof. For the models constructed in the proof of Lemma 1, it follows that for every $n \in \omega$, $\mathcal{M}_n$ is an elementary submodel of $\mathcal{M}_{n+1}$ when restricted to the language of $\mathcal{M}_n$. Let $\mathcal{M}^S = \bigcup_{n \ge 0} \mathcal{M}_n$. By extending finite embeddings, we see that $\mathcal{M}^S$ can be elementarily embedded in any model of T^S. Thus, $\mathcal{M}^S$ is a prime model of its theory T^S. For this model, we have $\bigcap_{i \ge 0} P_i^{\mathcal{M}} = \emptyset$, because every element in $P_0^{\mathcal{M}}$ belongs to some difference $P_{k-1}^{\mathcal{M}} - P_k^{\mathcal{M}}$. Similarly, we have $\bigcap_{j \ge 0} Q_j^{\mathcal{M}} = \emptyset$. The model $\mathcal{M}^S$ is decidable since the models $\mathcal{M}_n$ are constructed to be decidable uniformly in n.

We can show that for every isomorphic decidable copy of $\mathcal{M}^S$, we can construct a computable isomorphism. □

Definition of T^c. The theory T^c is defined by expanding the language $\mathcal{L}_0$ of T^S to $\mathcal{L} = \mathcal{L}_0 \cup \{c\}$ where c is a constant symbol, and by adding the following axioms to the axioms of T^S.

7. Axioms for c
$P_i(c)$ for $i \in \omega$.

Lemma 6. *The theory T^c is complete, decidable, and model complete, and has a decidable prime model.*

Proof. Let $T_{n,c}$ be the expansion of the theory T_n obtained by adding c to the language, with the axioms $P_i(c)$ for $i \leq n$. The theory $T_{n,c}$ is $\aleph_0$-categorical and has only infinite models, hence is complete. It is also decidable uniformly in n, and is model complete. We can modify the construction in the proof of Lemma 1 for the models $\mathcal{M}_n$ to obtain $\mathcal{M}_{n,c}$ by adding an element c such that $P_n(c)$ and setting $F_t(c, v_{\langle s(t),0\rangle})$ for $t = 0, \ldots, n$. It will follow that for every $n \in \omega$, $\mathcal{M}_{n,c}$ is an elementary submodel of $\mathcal{M}_{n+1,c}$ in the appropriate language. Let $\mathcal{M}^c = \bigcup_{n \geq 0} \mathcal{M}_{n,c}$. The model $\mathcal{M}^c$ is decidable since the models $\mathcal{M}_{n,c}$ are constructed to be decidable uniformly in n. The model $\mathcal{M}^c$ is a prime model of its theory T^c such that when restricted to language $\mathcal{L}_0$, it is a model of T^S realizing its non-principal type $\{P_i(x) : i \in \omega\}$ on a unique element. $\square$

Lemma 7. *The set $C(T^c)$ of complete formulas of the theory T^c has Turing degree* $\deg(S)$.

Proof. We will show that if $n \in S$, then the formula $A_n(x)$ is not complete, while if $n \notin S$, then $A_n(x)$ is a complete formula. This is true since, for all $x \neq x'$ from A_n, x and x' are automorphic if and only if $n \notin S$. If $n = s(t)$ for some t, then both formulas $F_t(c, x)$ and $\neg F_t(c, x)$ are consistent with $A_n(x)$. Hence $\deg(S) \leq \deg(C(T^c))$.

To show that $\deg(C(T^c)) \leq \deg(S)$ we will use the fact that every complete formula of T^c is satisfied in $\mathcal{M}^c$ and hence in some $\mathcal{M}_{n,c}$. That is, consider every $(c, \overrightarrow{b})$, where $\overrightarrow{b}$ is a finite tuple of distinct elements from $\mathcal{M}^c$ and all different from c. Using the oracle S, we can find $n \in \omega$ such that all relations involving $c, \overrightarrow{b}$ take place in $\mathcal{M}_{n,c}$. We can now write all atomic and negated atomic formulas satisfied by $c, \overrightarrow{b}$ and their images under F_i and G_i in the language of $\mathcal{M}_{n,c}$, and take their conjunction, thus obtaining $\psi(c, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ where $\overrightarrow{x}$ corresponds to $\overrightarrow{b}$, $\overrightarrow{y}$ to F-images and $\overrightarrow{z}$ to G-images. The existential formula $(\exists \overrightarrow{y})(\exists \overrightarrow{z})\ \psi(c.\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is satisfied by $\overrightarrow{b}$, and if a tuple $c, \overrightarrow{b_1}$ satisfies it, then $\overrightarrow{b_1}$ is automorphic with $\overrightarrow{b}$. Hence $(\exists \overrightarrow{y})(\exists \overrightarrow{z})\ \psi(c, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z})$ is a complete formula. We can now enumerate all formulas implied (under the theory T^c) by these formulas, thus obtaining a $\deg(S)$-computable enumeration of all complete formulas. $\square$

Lemma 8. *The degree of decidable categoricity of $\mathcal{M}^c$ is deg(S).*

Proof. It is not hard to see that $\mathcal{M}^c$ is $\mathbf{s}$-decidably categorical, where $\mathbf{s} = \deg(S)$. To see that $\mathbf{s}$ is the least degree in the decidable categoricity spectrum, we will consider a model $\mathcal{M}^*$ of T^c, built as in Lemma 6 except that we define F_i in this model to satisfy $F_t(c, v_{\langle s(t),\langle t,s(t)\rangle\rangle})$. Now every isomorphism $f : \mathcal{M}^c \to \mathcal{M}^*$ is such that $S \leq_T f$, since $k \in S \Leftrightarrow [f(v_{\langle k,0\rangle}) = v_{\langle k,\langle t,k\rangle\rangle} \wedge s(t) = k]$. □

Let us now define an expansion of the theory T^c by adding a new constant symbol d.

Definition of $T^{c,d}$. The theory $T^{c,d}$ is defined by expanding the language $\mathcal{L}$ to $\mathcal{L}^d = \mathcal{L} \cup \{d\}$, and by adding the following axioms to the axioms of T^c.

8. Axioms for d
$Q_i(d)$ for $i \in \omega$.
$(\exists y)[F_t(c, y) \wedge G_{s(t)}(d, y)]$ for $t \in \omega$.
By the same methods as before, we can establish the following lemma.

Lemma 9. *The theory $T^{c,d}$ is decidable, and has a decidable prime model $\mathcal{M}^{c,d}$.*

Proof. We can build a model of $T^{c,d}$ in which, for all n, we have $P_n(c)$ and $Q_n(d)$, and for every t, we have $F_t(c, v_{\langle s(t),0\rangle})$ and $G_t(d, v_{\langle t,0\rangle})$. □

Moreover, the theory $T^{c,d}$ is model complete, every computable model of $T^{c,d}$ is decidable, and every first-order formula is equivalent under $T^{c,d}$ to an existential formula.

Lemma 10. *The degree of decidable categoricity of $\mathcal{M}^{c,d}$ is $\mathbf{0}$. The set $C(T^{c,d})$ of complete formulas is computable.*

This proves the theorem. □

Hence we have the following corollary.

Theorem 5. *There is a decidable model $\mathcal{M}$ and an element d from its domain such that $(\mathcal{M}, d)$ is a prime model of $Th(\mathcal{M}, d)$ and the degree of decidable categoricity of $(\mathcal{M}, d)$ exists and lies strictly below the Turing degree of the set of complete formulas of $Th(\mathcal{M})$.*

However, the more interesting question of Bazhenov remains open: "Is there a prime model with a degree $\boldsymbol{d}$ of decidable categoricity, but whose set of complete formulas has Turing degree $\boldsymbol{c} > \boldsymbol{d}$?"

We can use a direct sum of theories to construct a new theory T corresponding to any computable list of c.e. Turing degrees.

Theorem 6. *For every computable list of c.e. sets $D_0, D_1, \ldots, D_n, \ldots$, there exists a complete decidable theory T with a computable sequence of decidable almost prime models $\mathcal{M}_0 \preccurlyeq \mathcal{M}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{M}_n \preccurlyeq \cdots$ and a computable sequence of finite lists of elements $\overline{a}_0 = \emptyset, \overline{a}_1, \ldots, \overline{a}_n, \ldots$ such that for every $n \geq 0$ the model $(\mathcal{M}_n, \overline{a}_0, ..., \overline{a}_n)$ is prime and decidable with degree of decidable categoricity equal to $deg(D_n)$ and with $C(\mathcal{M}_n) \equiv_T D_n$. Since $\overline{a}_0$ is the empty tuple, $\mathcal{M}_0$ itself is the prime model of T.*

2.1 Uniformly decidably categorical structures

Uniform computable categoricity has been studied by Ventsov [47], Kudinov [40, 39], and Downey, Hirschfeldt and Khoussainov [16]. A partial summary appears in [43]. For uniformity, we require not that only computable isomorphism between the computable structures must exist, but that there is an algorithm to find it. In this section we will carry out an analogous investigation for decidable structures. For infinite structures we may assume that their domain is ω. We will fix a Gödel numbering of all sentences in the language expanded by the constants for the elements in the domain. Thus, we may view the elementary diagram $E(\mathcal{M})$ of a structure $\mathcal{M}$ as an element of 2^ω. Such a real $E(\mathcal{M})$ will often be used as the oracle for a Turing functional.

Definition 4. A countable structure $\mathcal{M}$ is *uniformly decidably categorical* if there exists a Turing functional Γ such that, for all structures $\mathcal{A}$ and $\mathcal{B}$ with domain ω such that $\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}$, the function $\Gamma^{E(\mathcal{A})\oplus E(\mathcal{B})}$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Suppose that Γ is the functional for a uniformly decidably categorical structure $\mathcal{M}$, and suppose $\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}$. Choose $b_0, \ldots, b_{n-1}$ so that

$$(\forall i < n) \left[\Gamma^{E(\mathcal{A})\oplus E(\mathcal{B})}(i)\downarrow = b_i\right].$$

Fix finite strings $\sigma \subseteq E(\mathcal{A})$ and $\tau \subseteq E(\mathcal{B})$ with $\Gamma^{\sigma\oplus\tau}(i)\downarrow = b_i$ for all $i \leq k$. Now, σ states that a certain finite set $\{\beta_0(0,\ldots,m),\ldots,\beta_k(0,\ldots,m)\}$ of facts all hold in $\mathcal{A}$. (If $\sigma(j) = 0$, then the negation of that formula serves as β_j in our finite set.) We define the formula $\alpha_\sigma(\overrightarrow{x})$ to be

$$(\exists y_n)\cdots(\exists y_m)\left[\delta \wedge \bigwedge_{i\leq k} \beta_i^*(x_0,\ldots,x_{n-1};\ y_n,\ldots,y_m)\right]$$

where we get β_i^* from $\beta_i(0,\ldots,m)$ by replacing each $p < n$ by x_p and each $j \geq n$ by y_j. The δ here is just a formula saying that all of the variables involved are pairwise not equal to each other.

Lemma 11. *Suppose that Γ is the functional for a uniformly decidably categorical structure $\mathcal{M}$, as in Definition 4. Whenever $\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}$ with initial segments $\sigma \subseteq E(\mathcal{A})$ and $\tau \subseteq E(\mathcal{B})$ and*

$$(\forall i < n)\ [\Gamma^{E(\mathcal{A})\oplus E(\mathcal{B})}(i)\downarrow = b_i],$$

with oracle use $\subseteq (\sigma\oplus\tau)$, the formula $\alpha_\sigma(\overrightarrow{x})$ generates (over the theory $T = Th(\mathcal{M})$) the type realized by the tuple $(0, 1, \ldots, n-1)$ in $\mathcal{A}$. (Likewise, the formula defined analogously from τ generates the type realized by $\overrightarrow{b}$ in $\mathcal{B}$.)

Proof. Suppose $\varphi(x)$ were a formula such that $\models_{\mathcal{A}} \varphi(0,\ldots,n-1)$ but $T \not\vdash (\alpha_\sigma \Rightarrow \varphi)$. Then there must be another tuple $\overrightarrow{c} \in \mathcal{A}^{<\omega}$ with $\models_{\mathcal{A}} \alpha_\sigma(\overrightarrow{c}) \wedge \neg\varphi(\overrightarrow{c})$.

Now, we define $\mathcal{C}$ to be the isomorphic image of $\mathcal{A}$ under some permutation h of ω that has $h(i) = c_i$ for each i and fixes all elements $\geq n$ except the elements c_i. The details, which appear immediately below, ensure that $\sigma \subseteq E(\mathcal{C})$, so $\Gamma^{E(\mathcal{C})\oplus E(\mathcal{B})}(i)\downarrow = b_i$ for each i. Now,

$$h^{-1} \circ \left(\Gamma^{E(\mathcal{C})\oplus E(\mathcal{B})}\right)^{-1} \circ \left(\Gamma^{E(\mathcal{A})\oplus E(\mathcal{B})}\right)$$

is an automorphism of $\mathcal{A}$ mapping each i to c_i, contradicting the assumption that $\models_{\mathcal{A}} \varphi(0, \ldots, n-1) \wedge \neg\varphi(\overrightarrow{c})$. Therefore, $\alpha_\sigma(\overrightarrow{x})$ does indeed generate the type of $\overrightarrow{d}$ in $\mathcal{A}$, over T.

Defining the permutation h is easy unless, for some $i \neq j$, we have $i = c_j$. The general procedure, which handles this situation, is as follows. Define $h(i) = c_i$ for all i. If possible, define $h(c_0) = 0$ (so that h simply interchanges 0 and c_0). This is not possible if $c_0 = i_0$ with $0 < i_0 < n$, in which case $h(c_0) = c_{i_0}$, and we define $h(c_{i_0}) = 0$ if possible (so that h permutes 0, c_0, and c_{i_0} cyclically). This is not possible if $c_{i_0} = i_1$ for some $i_1 < n$ not in $\{0, i_0\}$, in which case $h(c_{i_0}) = c_{i_1}$, and we define $h(c_{i_1}) = 0$ if possible. Continue until we have placed 0 in the image of h. If this fails to define h^{-1} on all of $\{0, \ldots, n-1\}$, then for the least j with $h^{-1}(j)$ currently undefined, do this process again, with j in place of 0: define $h(c_j) = j$ if possible, and so on. Since n is a finite number, this will eventually end with a permutation h defined on $\{0, c_0, \ldots, n, c_n\}$. Then we set h to equal the identity on all other elements of ω. All orbits of h are thus finite, and since we define $\mathcal{C}$ to make h an isomorphism, any pair (i, j) in the same cycle must be in the same orbit under automorphisms of $\mathcal{A}$.

□

Recall that by $C(\mathcal{M})$ we denote the set of all formulas that are complete for $\mathrm{Th}(\mathcal{M})$.

Theorem 7. *A countable structure $\mathcal{M}$ is uniformly decidably categorical if and only if $C(\mathcal{M}) \leq_T Th(\mathcal{M})$ and $\mathcal{M}$ is a prime model of $Th(\mathcal{M})$.*

Proof. We write $T = \mathrm{Th}(\mathcal{M})$. The backwards implication is readily seen. If $\mathcal{A}$ and $\mathcal{B}$ are copies of $\mathcal{M}$ with domain ω, we build an isomorphism g from $\mathcal{A}$ onto $\mathcal{B}$ (using oracles for $E(\mathcal{A})$ and $E(\mathcal{B})$) by a back-and-forth construction. The functional Γ will be the procedure by which this g is built. If g is already defined on a finite domain $D = \{a_1, \ldots, a_n\} \subseteq \omega$ within $\mathcal{A}$, then for the smallest $a \notin D$, we search for a formula $\alpha(x_0, \ldots, x_n) \in C(\mathcal{M})$ such that $\models_{\mathcal{A}} \alpha(a, a_1, \ldots, a_n)$. Here we use the $E(\mathcal{A})$ oracle both to decide truth in $\mathcal{A}$ and to decide $\mathrm{Th}(\mathcal{A})$, which of course is equal to $\mathrm{Th}(\mathcal{M})$ and therefore (by assumption) allows us to decide $C(\mathcal{M})$. Since $\mathcal{M}$ is a prime model of T, so is $\mathcal{A}$, and so the tuple $(a, \overrightarrow{a})$ must realize a principal type, hence must satisfy some formula in $C(\mathcal{M})$. Thus eventually our search produces the desired α. When it does, we then use the $E(\mathcal{B})$-oracle to search for some $b \in \omega = \mathrm{dom}(\mathcal{B})$ such that $\models_{\mathcal{B}} \alpha(b, g(a_1), \ldots, g(a_n))$; it must exist because $\mathcal{B} \cong \mathcal{A}$. Finally we define $g(a) = b$. By the completeness of α, this new g must extend to an isomorphism (since by inductive hypothesis, the old g did). At the next stage we do the same with the least element $\notin$ range(g), and continuing this process through ω-many stages computes the isomorphism $g = \Gamma^{E(\mathcal{A})\oplus E(\mathcal{B})}$.

The forwards implication is more difficult. Suppose that there exists some Γ as in Definition 4. We first show that $\mathcal{M}$ is a prime model of T. Given any $x \in \mathcal{M}$, there is some presentation $\mathcal{A}$ of $\mathcal{M}$, with domain ω, for which some isomorphism from $\mathcal{M}$ onto $\mathcal{A}$ maps x to 0. Lemma 11 shows that the formula $\alpha_\sigma(x_0)$ given there (taking σ to be the first half of the use of the computation there) generates the type realized by 0 in $\mathcal{A}$, which is to say, the type realized by x in $\mathcal{M}$. Thus $\mathcal{M}$ is indeed a prime model of T.

To show that $C(\mathcal{M}) \leq_T T$, we will give a T-computable enumeration of $C(\mathcal{M})$. This will suffice, because $C(\mathcal{M})$ is always Π_1 relative to T. To give the enumeration, we start by searching for pairs of strings σ and τ and an n and a finite tuple $\overrightarrow{b} \in \omega^n$ such that $\Gamma^{\sigma\oplus\tau}(i)\!\downarrow= b_i$ for all $i < n$. Whenever we find such pairs and tuples, we translate σ and τ each into formulas. The formula $\alpha_\sigma(x_0, \ldots, x_{n-1})$ is defined exactly as above:

$$(\exists y_n)\cdots(\exists y_m)\left[\delta \wedge \bigwedge_{i\leq k} \beta_i(x_0, \ldots, x_{n-1}, y_n, \ldots, y_m)\right],$$

using as many variables as the largest domain element mentioned in σ. Likewise, $\gamma_\tau(u_0, \ldots, u_{n-1})$ is defined to reflect the information given by τ about the elements $(b_0, \ldots, b_{n-1})$ of the putative range structure.

Of course, there is no guarantee that the strings σ and τ can be extended to elementary diagrams of models of T. However, with our T-oracle, we can check whether

$$(\exists x_0)\cdots(\exists x_{n-1})\alpha_\sigma \in T \qquad \text{and} \qquad (\exists u_0)\cdots(\exists u_{n-1})\gamma_\tau \in T.$$

If either is not in T, then we ignore this pair of strings σ and τ. However, if both belong to T, then we enumerate $\alpha_\sigma(x_0, \ldots, x_{n-1})$ into our list of complete formulas. (It would be harmless to enumerate γ_τ as well, but this will also occur at some other stage of the enumeration using a different σ.)

Finally, still using the T-oracle, we enumerate on our list every formula $\beta(x_0, \ldots, x_{n-1})$ such that $T \vdash \forall\overrightarrow{x}(\alpha_\sigma \Leftrightarrow \beta)$. Of course, once we have proven (below) that α_σ is complete over T, all these formulas β will also be complete. This completes our program for enumerating formulas, leaving us to prove that we have enumerated precisely those formulas in $C(\mathcal{M})$.

If we enumerated α_σ, then $(\exists x_0)\cdots(\exists x_{n-1})\alpha_\sigma$ is consistent with T. Therefore, there exists a model $\mathcal{A}$ of T in which the domain elements $0, \ldots, n-1$ realize α_σ. Lemma 11, applied to $\Gamma^{E(\mathcal{A})\oplus E(\mathcal{A})}$, makes it clear that α_σ is in fact a complete formula for T, and therefore so are the formulas T-equivalent to α_σ that were also enumerated. So we have enumerated a subset of $C(\mathcal{M})$.

On the other hand, fix any n-tuple of elements from $\mathcal{M}$. By switching to an isomorphic structure $\mathcal{A}$, we may assume that the n-tuple is just $(0, 1, \ldots, n-1)$.

By uniform decidable categoricity, $\Gamma^{E(\mathcal{A})\oplus E(\mathcal{A})}(i)$ must converge for each $i < n$, so let σ be a finite initial segment of $\mathcal{A}$ long enough that $\Gamma^{\sigma\oplus\sigma}(i)\downarrow$ for all these i. Then the formula $(\exists x_0)\cdots(\exists x_{n-1})\alpha_\sigma$ holds in $\mathcal{A}$, hence is consistent with T, hence lies in T. The formula γ_σ will not necessarily be the same as α_σ (since $\Gamma^{\sigma\oplus\sigma}(i)$ may not equal i), but once again $(\exists u_0)\cdots(\exists u_{n-1})\gamma_\sigma$ will hold in $\mathcal{A}$, hence lies in T. So our procedure will eventually have found $\sigma\oplus\sigma$ and will have enumerated α_σ. This shows that for every type realized in the (prime) model $\mathcal{M}$ of T, some generator of that type was enumerated. So every formula in $C(\mathcal{M})$ eventually appeared on our list, since the type it generates is principal and thus must be realized in $\mathcal{M}$. As remarked above, this shows that $C(\mathcal{M}) \leq_T T$, completing the proof. $\square$

Theorem 7 can be viewed as an analogue of a known result about uniform computable categoricity (see [47]), which we state here. The result should be considered as folklore, but a formal statement appears in [23], and relevant discussion appears in [43].

Proposition 3 (as in [23])**.** *A countable structure $\mathcal{A}$ is uniformly computably categorical if and only if it has a Scott family $\mathcal{S}$ of (finitary) Σ^0_1 formulas such that, when we regard $\mathcal{S}$ as a subset of ω using Gödel codes, $\mathcal{S}$ is enumeration reducible to the (finitary) Σ^0_1-theory of $\mathcal{A}$.*

The situation for decidable categoricity is simpler, in that relative decidable categoricity is equivalent to being uniformly decidably categorical after an expansion by constants, and also in that decidable categoricity for decidable structures implies relative decidable categoricity. From these equivalences, we get an immediate corollary, generalizing Theorem 1 of Nurtazin to undecidable theories and structures.

Corollary 1. *A countable structure $\mathcal{M}$ is relatively decidably categorical if and only if some finite tuple of new constant symbols $\overrightarrow{c}$ can be added to name elements of $\mathcal{M}$ in such a way that $C(\mathcal{M},\overrightarrow{c}) \leq_T Th(\mathcal{M},\overrightarrow{c})$ and $(\mathcal{M},\overrightarrow{c})$ is a prime model of $Th(\mathcal{M},\overrightarrow{c})$.*

Notice that $C(\mathcal{M},\overrightarrow{c}) \leq_T \mathrm{Th}(\mathcal{M},\overrightarrow{c})$ is equivalent to enumeration reducibility $C(\mathcal{M},\overrightarrow{c}) \leq_e \mathrm{Th}(\mathcal{M},\overrightarrow{c})$, since $C(\mathcal{M},\overrightarrow{c})$ is always Π^0_1 in $\mathrm{Th}(\mathcal{M},\overrightarrow{c})$,

We can also relativize uniform decidable categoricity.

Definition 5. Let D be a set, of Turing degree $\boldsymbol{d}$. A countable structure $\mathcal{M}$ is *$\boldsymbol{d}$-uniformly decidably categorical* if there exists a Turing functional Γ such that, for all structures $\mathcal{A}$ and $\mathcal{B}$ with domain ω such that $\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}$, the function $\Gamma^{D\oplus E(\mathcal{A})\oplus E(\mathcal{B})}$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Lemma 11 relativizes to an arbitrary oracle set D as follows. The proof is essentially identical, recognizing that the fixed set D within the oracle for Γ does not change anything.

Lemma 12. *Suppose that* Γ *is the functional for a uniformly* $\boldsymbol{d}$*-decidably categorical structure* $\mathcal{M}$*. Whenever* $\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}$ *and*

$$(\forall i < n)\ [\Gamma^{D \oplus E(\mathcal{A}) \oplus E(\mathcal{B})}(i)\downarrow = b_i],$$

with use $\subseteq \rho \oplus \sigma \oplus \tau$*, the formula* $\alpha_\sigma(\overrightarrow{x})$ *generates (over the theory* $T = Th(\mathcal{M})$*) the type realized by the tuple* $(0, 1, \ldots, n-1)$ *in* $\mathcal{A}$*.*

Likewise, the formula defined analogously from τ *generates the type realized by* $\overrightarrow{b}$ *in* $\mathcal{B}$*.*

With this lemma, it is easy to show that the proof of the relativization of Theorem 8 goes through.

Theorem 8. *A countable structure* $\mathcal{M}$ *is* $\boldsymbol{d}$*-uniformly decidably categorical if and only if* $C(\mathcal{M}) \leq_T D \oplus Th(\mathcal{M})$ *and* $\mathcal{M}$ *is a prime model of* $Th(\mathcal{M})$*.*

Corollary 2. *Let* D *be a set of Turing degree* $\boldsymbol{d}$*. A countable structure* $\mathcal{M}$ *is relatively* $\boldsymbol{d}$*-decidably categorical if and only if some finite tuple of new constant symbols* $\overrightarrow{c}$ *can be added to name elements of* $\mathcal{M}$ *in such a way that* $C(\mathcal{M}, \overrightarrow{c}) \leq_T D \oplus Th(\mathcal{M}, \overrightarrow{c})$ *and* $(\mathcal{M}, \overrightarrow{c})$ *is a prime model of* $Th(\mathcal{M}, \overrightarrow{c})$*.*

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