The computable embedding problem

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July 15, 2011

Abstract

Calvert [3], [4], [5] calculated the complexity of the computable isomorphism problem for a number of familiar classes of structures. Rosendal suggested that it might be interesting to do the same for the computable embedding problem. By "computable isomorphism problem" and "computable embedding problem" we mean the difficulty of determining whether there exists an isomorphism (embedding) between two members of a class of computable structures. For some classes, such as the class of $\mathbb{Q}$-vector spaces and the class of linear orderings, it turns out that the two problems have the same complexity. Moreover, the calculations are essentially the same. For other classes, there are differences. We give examples in which the embedding problem is trivial (within the class) and the computable isomorphism problem is more complicated. We also give an example in which the embedding problem is more complicated than the isomorphism problem.

1 Introduction

An important approach to classification of structures from the point of view of computability theory is the computable isomorphism problem. This problem deals with the question of how hard, within the arithmetical or analytical hierarchy, is it to tell whether two computable structures are isomorphic. For a discussion of this and another two seemingly distinct but equivalent approaches, see [11]. Recently, Rosendal suggested that we investigate the related computable embedding problem—instead of asking whether two computable structures are isomorphic, we ask whether the first is isomorphic to a substructure of the second.

*The authors acknowledge partial support by the NSF binational grant DMS-0554841, Carson, Knight, Maher, Quinn, and Wallbaum by the NSF grants DMS-0139626 and DMS-0353748, and Harizanov by the NSF grant DMS-0704265. Fokina acknowledges support by a grant of the Russian Federation as a 2006-2007 research visitor to the University of Notre Dame.

In this section, we describe the general setting for computable embedding and isomorphism problems. In Section 2, we discuss classes for which the embedding and isomorphism problems have the same complexity. Section 3 gives some classes for which the embedding problem is simpler than the isomorphism problem. Section 4 gives a class in which the embedding problem is more complicated. Our structures are all countable, with universe a subset of ω. The language of each structure is computable. We work in computable infinitary logic (for background on infinitary formulas, see [2]). When we measure the complexity of a structure $\mathcal{A}$, we identify the structure with its atomic diagram $D(\mathcal{A})$, where the atomic sentences are identified with their Gödel numbers. If $(\varphi_e)_{e\in\omega}$ is an enumeration of the unary partial computable functions, then $\mathcal{A}$ is computable if there is some e such that the characteristic function of $D(\mathcal{A})$ is φ_e. We call e a *computable index* for $\mathcal{A}$. If e is a computable index for a structure, we may denote the structure by $\mathcal{A}_e$. Note that not all numbers e are indices for structures.

Definition 1. Let K be a class of structures, closed under isomorphism.

1. The *index set* for K is the set $I(K)$ of computable indices of structures in K.
2. The *computable isomorphism problem* for K is the set $E(K)$ of pairs $(a, b) \in I(K) \times I(K)$ such that $\mathcal{A}_a \cong \mathcal{A}_b$.
3. The *computable embedding problem* for K is the set $Em(K)$ of pairs $(a, b) \in I(K) \times I(K)$ such that $\mathcal{A}_a \hookrightarrow \mathcal{A}_b$, i.e., there is an embedding of $\mathcal{A}_a$ into $\mathcal{A}_b$ (that is, an isomorphism from $\mathcal{A}_a$ onto a substructure of $\mathcal{A}_b$).

Our calculations of complexity involve showing that a set is m-complete Γ, where Γ is a complexity class such as Π^0_3, or Σ^1_1. (For more on these complexity classes, see the classic text by Rogers [20].) Recall that A is Γ-*hard* if for all $S \in \Gamma$, we have $S \leq_m A$, and A is m-*complete* Γ if $A \in \Gamma$, and A is Γ-hard. For a class K, closed under isomorphism, such that $I(K)$ is hyperarithmetical, $E(K)$ and $Em(K)$ are both Σ^1_1.

In some of Calvert's calculations [3], [4], [5], the complexity of $I(K)$ is at least as great as that of $E(K)$. The same is true for $Em(K)$. We would like to focus on the difficulty of determining whether two structures are isomorphic, or whether one embeds in the other, given that we are looking at structures in the class. In cases where the complexity of $I(K)$ might interfere with this, we proceed as Calvert did, using notions of complexity "within".

Definition 2. Let Γ be a complexity class, and suppose $A \subseteq B$.

1. We say that A *is* Γ *within* B if there exists $R \in \Gamma$ such that $A = R \cap B$.
2. We say that A *is* Γ-*hard within* B if for any $S \in \Gamma$, there is a computable function $f : \omega \to B$ such that for all $n \in \omega$, we have $n \in S$ iff $f(n) \in A$.
3. We say that A *is* m-*complete* Γ *within* B if

(a) A is Γ within B, and

(b) A is Γ-hard within B.

One final introductory note: we sometimes use Δ^0_n to mean the oracle $\emptyset^{(n-1)}$ as well as the class.

2 Embedding and isomorphism may be alike

In this section, we describe some familiar classes for which the computable embedding and isomorphism problems have the same complexity. We begin with vector spaces and other examples where the complexity is arithmetical. We then give examples where the complexity is properly Σ^1_1.

2.1 Simple examples

Let VS be the class of nontrivial $\mathbb{Q}$-vector spaces. It is easy to see that $I(VS)$ is Π^0_2. In [3], it is shown that the computable isomorphism problem $E(VS)$ is m-complete Π^0_3.

Proposition 2.1. *The computable embedding problem $Em(VS)$ is m-complete Π^0_3.*

Proof. First, we show that $Em(VS)$ is Π^0_3. We write $dim(\mathcal{A})$ for the dimension of $\mathcal{A}$. For each k, we have a computable Σ_2 sentence saying (of a vector space) that the dimension is at least k, and we call that sentence φ_k. We have $(a,b) \in Em(VS)$ iff

$$a, b \in I(VS) \ \& \ \bigwedge\!\!\!\bigwedge_k (\mathcal{A}_a \models \varphi_k \rightarrow \mathcal{A}_b \models \varphi_k) \ .$$

This is clearly Π^0_3. Next, we show that $Em(VS)$ is Π^0_3-hard. Let S be a Π^0_3 set. Calvert [3] showed that there is a uniformly computable sequence $(\mathcal{B}_n)_{n\in\omega}$ of $\mathbb{Q}$-vector spaces such that

$$dim(\mathcal{B}_n) = \begin{cases} \infty & \text{if } n \in S, \\ \text{finite} & \text{otherwise.} \end{cases}$$

Fix a computable vector space $\mathcal{A}$ with $dim(\mathcal{A}) = \infty$. Then $\mathcal{A} \hookrightarrow \mathcal{B}_n$ iff $n \in S$. $\square$

We may generalize Proposition 2.1, using the following reasoning. Let K and K' be classes of structures, and let Φ be an effective transformation from K to K'. Suppose that for $\mathcal{A}, \mathcal{B} \in K$, $\mathcal{A} \hookrightarrow \mathcal{B}$ iff $\Phi(\mathcal{A}) \hookrightarrow \Phi(\mathcal{B})$. Then $Em(K) \leq_m Em(K')$. If $Em(K')$ is in some complexity class Γ, such as Π^0_3, then so is $Em(K)$. Also, if $Em(K)$ is Γ-hard, then so is $Em(K')$. If we have embeddability-preserving effective transformations from K to K' and from K' to K, then $Em(K)$ is m-complete Γ just in case $Em(K')$ is.

Corollary 2.2. *Let T be a strongly minimal theory which is not $\aleph_0$-categorical and has effective quantifier elimination. Then $Em(Mod(T))$ is m-complete Π^0_3.*

Proof. The theory of infinite $\mathbb{Q}$-vector spaces is an example of a theory satisfying our hypotheses. In [15], there is a result saying that if T and T' are two theories satisfying the hypotheses, then there is an effective transformation Φ from $Mod(T)$ to $Mod(T')$ such that for $\mathcal{A}, \mathcal{B} \in K$,

$$dim(\mathcal{A}) \leq dim(\mathcal{B}) \text{ iff } dim(\Phi(\mathcal{A})) \leq dim(\Phi(\mathcal{B})) \ .$$

Thus, we have embeddability-preserving effective transformations from VS to $Mod(T)$ and from $Mod(T)$ to VS. Then the conclusion follows from Proposition 2.1 and the comments above. □

Let AOF be the class of Archimedean ordered fields. It is not difficult to see that $I(AOF)$ is Π^0_2, and in [3], it is shown that $E(AOF)$ is m-complete Π^0_3.

Proposition 2.3. *$Em(AOF)$ is m-complete Π^0_3.*

Proof. First we show that $Em(AOF)$ is Π^0_3. A structure in AOF is determined by the rational cuts that are filled. If $\mathcal{A}, \mathcal{B} \in AOF$, then $\mathcal{A} \hookrightarrow \mathcal{B}$ iff all cuts that are filled in $\mathcal{A}$ are also filled in $\mathcal{B}$. We have $(a, b) \in Em(AOF)$ iff

$$a, b \in I(AOF) \ \& \ (\forall x \in \mathcal{A}_a)(\exists y \in \mathcal{B}_b) \bigwedge\!\!\!\bigwedge_{q \in \mathbb{Q}} (\mathcal{A}_a \models q < x \leftrightarrow \mathcal{A}_b \models q < y) \ .$$

This condition is Π^0_3.

To show that $Em(AOF)$ is Π^0_3-hard, let S be an arbitrary Π^0_3 set. We want a computable sequence of pairs of structures $(\mathcal{A}_n, \mathcal{B}_n)$ such that

$$\mathcal{A}_n \hookrightarrow \mathcal{B}_n \text{ iff } n \in S \ .$$

Calvert [4] showed that there is a computable Archimedean real closed field $\mathcal{A}$ of infinite transcendence degree such that for a Π^0_3 set such as our S, there is a uniformly computable sequence $(\mathcal{B}_n)_{n \in \omega}$ of Archimedean ordered real closed fields such that if $n \in S$, then $\mathcal{B}_n \cong \mathcal{A}$, and otherwise $\mathcal{B}_n$ has finite transcendence degree. Clearly, a field of infinite transcendence degree does not embed in one of finite transcendence degree, so $\mathcal{A} \hookrightarrow \mathcal{B}_n$ iff $n \in S$. □

Let TF be the class of torsion-free Abelian groups. These are the subgroups of $\mathbb{Q}$-vector spaces. The *rank* of such a group $\mathcal{G}$ is the least dimension of a vector space in which $\mathcal{G}$ can be embedded. We consider the class FTF of torsion-free Abelian groups of finite rank.

Proposition 2.4. *$I(FTF)$ is m-complete Σ^0_3.*

Proof. Note that there is a computable Π_2 sentence characterizing the torsion-free Abelian groups, and there is a computable Σ_3 sentence saying that the group has finite rank. From this, it follows that $I(FTF)$ is Σ^0_3. For hardness, it is enough to show that there is a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of torsion-free Abelian groups such that $\mathcal{A}_n$ has finite rank iff $n \in Cof$. We do not need a new construction. We let $(\mathcal{A}_n)_{n\in\omega}$ be a sequence of $\mathbb{Q}$-vector spaces, as in Proposition 2.1, such that $dim(\mathcal{A}_n)$ is finite iff $n \in Cof$. □

Calvert [5] showed that $E(FTF)$ is m-complete Σ^0_3 within $I(FTF)\times I(FTF)$.

Proposition 2.5. *$Em(FTF)$ is m-complete Σ^0_3 within $I(FTF) \times I(FTF)$.*

Proof. Let $\mathcal{A}$ be the group of rank 1 generated by the rationals of the form $\frac{1}{p}$, for p a prime. We may suppose that $\mathcal{A}$ is computable; in fact $\mathcal{A}$ is the additive group of the rationals. We produce a uniformly computable sequence of groups $(\mathcal{B}_n)_{n\in\omega}$, all of rank 1. Let $(p_n)_{n\in\omega}$ be the list of the primes in increasing order. We let $\mathcal{B}_n$ be generated by the elements $\frac{1}{p_k}$, where $k \in W_n$. If $n \in Cof$, then all but finitely many k are in W_n. Multiplying 1 in $\mathcal{B}_n$ by the product of the corresponding finite set of primes p_k we have an image for 1 in $\mathcal{A}$, so that $\mathcal{A} \hookrightarrow \mathcal{B}_n$. If $n \notin Cof$, then $\overline{W_n}$ is infinite, and there is no appropriate image for 1 in $\mathcal{B}_n$, that is, no element of $\mathcal{B}_n$ is divisible by all primes. Therefore, $\mathcal{A} \not\hookrightarrow \mathcal{B}_n$. □

2.2 Maximal complexity

Downey and Montalbán [9] showed that for the class TF of torsion-free Abelian groups, $E(TF)$ is m-complete Σ^1_1. In the process, they also showed that $Em(TF)$ is m-complete Σ^1_1. Downey and Montalbán also showed that the isomorphism relation on arbitrary (not necessarily computable) torsion-free Abelian groups with universe ω is analytic complete. We consider further classes K for which $E(K)$ and $Em(K)$ turn out to be maximally complicated. The following results are well-known. Proofs of these facts can be found in [11], [12].

Theorem 2.6. *For each of the following classes K, $I(K)$ is arithmetical (Π^0_2 for 1–5 and Π^0_3 for 6), while $E(K)$ is m-complete Σ^1_1.*

1. *linear orderings,*
2. *Boolean algebras,*
3. *Abelian p-groups,*
4. *undirected graphs,*
5. *fields of characteristic 0, or p,*
6. *2-step nilpotent groups.*

For all of the classes K above, of course $Em(K)$ is Σ^1_1. Therefore, to show that $Em(K)$ is m-complete Σ^1_1, it is enough to show that it is Σ^1_1-hard.

Proposition 2.7. *Let LO be the class of linear orderings. Then $Em(LO)$ is m-complete Σ^1_1.*

Proof. We show hardness. Let S be a Σ_1^1 set. Recall that the Harrison ordering is a computable ordering $\mathcal{H}$ of type $\omega_1^{CK}(1+\eta)$. There is a uniformly computable sequence of linear orderings $\mathcal{L}_n$ such that if $n \in S$, then $\mathcal{L}_n \cong \mathcal{H}$, while if $n \notin S$, then $\mathcal{L}_n$ is a well ordering. See [11] for a proof. Then $\mathcal{H} \hookrightarrow \mathcal{L}_n$ iff $n \in S$. □

Proposition 2.8. *Let BA be the class of Boolean algebras. Then $Em(BA)$ is m-complete Σ_1^1.*

Proof. We show hardness. Let S be a Σ_1^1 set. Let $(\mathcal{L}_n)_{n\in\omega}$ be as in the proof of Proposition 2.7, a uniformly computable sequence of linear orderings such that $\mathcal{L}_n$ is a Harrison ordering if $n \in S$, and a well ordering otherwise. Let $\mathcal{A}$ be the Boolean algebra obtained as the interval algebra of the Harrison ordering. Let $\mathcal{B}_n$ be the interval algebra of the ordering $\mathcal{L}_n$. Note that $\mathcal{B}_n$ is superatomic iff $n \notin S$. Any subalgebra of a superatomic Boolean algebra is superatomic. Then $\mathcal{A} \hookrightarrow \mathcal{B}_n$ iff $n \in S$. □

Proposition 2.9. *Let ApG be the class of Abelian p-groups. Then $Em(ApG)$ is m-complete Σ_1^1.*

Proof. From a tree T, we obtain an Abelian p-group $G(T)$, generated by the elements of T under the relations $\emptyset = 0$, and $pb = a$, where b is a successor of a. Note that $G(T)$ is reduced iff T has no (infinite) path. We have a uniform effective procedure for passing from T to a copy of $G(T)$ computable in T. Let T be the tree consisting of finite decreasing sequences in the Harrison ordering. Let S be a Σ_1^1 set, and let $\mathcal{L}_n$ be as in the proofs of Propositions 2.7 and 2.8. Let T_n be the tree of finite decreasing sequences in $\mathcal{L}_n$, and let $\mathcal{B}_n$ be a computable copy of $G(T_n)$, obtained by our uniform effective procedure. If $n \in S$, then $\mathcal{B}_n \cong G(T)$. If $n \notin S$, then T_n has no path, so $\mathcal{B}_n$ is reduced. A subgroup of a reduced group is reduced, so we have $G(T) \hookrightarrow \mathcal{B}_n$ iff $n \in S$. □

Proposition 2.10. *Let UG be the class of undirected graphs. Then $Em(UG)$ is m-complete Σ_1^1.*

Let R be a binary relation symbol. Marker's model theory book [18] describes an effective transformation Φ from $\{R\}$-structures into undirected graphs. (There are similar transformations described in other sources.) As in [7], we start with a large computable graph $\mathcal{G}$ with special points g_n for all $n \in \omega$, each attached to a triangle. In addition, for each pair (m, n), there is a special point $r_{m,n}$, connected to m directly and connected to n by a 2-chain. The special point $r_{m,n}$ is one vertex of a square and of a pentagon. For the structure $\mathcal{A} = (A, R)$, the corresponding graph $\Phi(\mathcal{A})$ consists of the special elements g_n, for $n \in A$, the triangles, the special points $r_{m,n}$, for $m, n \in A$, and the square if $(m, n) \in R$, or the pentagon if $(m, n) \notin R$.

From this description, it is clear that there are existential formulas $\varphi(x)$ and $\psi_1(x, y)$, $\psi_2(x, y)$, such that

1. for any structure $\mathcal{A} = (A, R)$ for the language, the formulas $\varphi(x)$, $\psi_1(x, y)$, and $\psi_2(x, y)$ define a copy of $(A, R, \neg R)$, in $\Phi(\mathcal{A})$, and

2. if $\mathcal{A} \subseteq \mathcal{A}'$, then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{A}')$.

To prove Proposition 2.10, we show the following.

Lemma 2.11. *The transformation Φ preserves embeddability, in the sense that for input structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \hookrightarrow \mathcal{B}$ iff $\Phi(\mathcal{A}) \hookrightarrow \Phi(\mathcal{B})$.*

Proof. We show the nontrivial direction; that is, if $\Phi(\mathcal{A}) \hookrightarrow \Phi(\mathcal{B})$, then $\mathcal{A} \hookrightarrow \mathcal{B}$. Let $\mathcal{B}'$ be the copy of $\mathcal{B}$ defined in $\Phi(\mathcal{B})$ by the given formulas. Let G be a subgraph of $\Phi(\mathcal{B})$ isomorphic to $\Phi(\mathcal{A})$. Let $\mathcal{A}'$ be the structure defined in G by the given formulas. Since $G \cong \Phi(\mathcal{A})$, $\mathcal{A}'$ is isomorphic to the copy of $\mathcal{A}$ defined in $\Phi(\mathcal{A})$ by the same formulas, so $\mathcal{A}' \cong \mathcal{A}$. Because φ is existential, $|\mathcal{A}'| \subseteq |\mathcal{B}'|$. Because ψ_1 and ψ_2 are both existential, for $x, y \in |\mathcal{A}'|$, we have $(x, y) \in R^{\mathcal{A}'}$ iff $(x, y) \in R^{\mathcal{B}'}$. (The formula ψ_2 assures that if $\mathcal{A}' \models \neg R(x, y)$, then $\mathcal{B}' \models \neg R(x, y)$.) $\square$

For later use, we give one more technical property of the transformation Φ.

Lemma 2.12. *Let $\mathcal{A}, \mathcal{B}$ be binary relation structures, with $G' = \Phi(\mathcal{A})$ and $G = \Phi(\mathcal{B})$. Suppose f maps G' one-to-one into G such that $G' \models R(x, y)$ implies $G \models R(f(x), f(y))$. Then f is an isomorphism.*

Proposition 2.13. *Let FLD be the class of fields of some fixed characteristic. Then $Em(FLD)$ is m-complete Σ^1_1.*

Friedman and Stanley [10] gave an effective transformation Ψ from graphs to fields which preserves isomorphism. We may describe it as follows. Let $\mathcal{F}$ be a computable algebraically closed field (of the desired characteristic) with an infinite computable sequence $(b_i)_{i \in \omega}$ of independent transcendentals. For a graph $\mathcal{G}$, let $\Psi(\mathcal{G})$ be the subfield of $\mathcal{F}$ generated by the elements of $acl(b_i)$, for $i \in \mathcal{G}$, and further elements $\sqrt{b_i + b_j}$, where i, j are elements of $\mathcal{G}$ joined by an edge. (For characteristic 2, we use cube roots instead of square roots to indicate edges.) We note that if $G' \hookrightarrow G$, then $\Psi(G') \hookrightarrow \Psi(G)$. However, the converse fails. For example, if G is totally disconnected and G' is complete, where both are infinite, then $G \not\hookrightarrow G'$, but $\Psi(G) \hookrightarrow \Psi(G')$. We have the following.

Lemma 2.14. *For graphs G', G with edge relation E, $\Psi(G') \hookrightarrow \Psi(G)$ iff there is a one-to-one function f from G' into G such that $G' \models xEy$ implies $G \models f(x)Ef(y)$.*

Proof of Proposition 2.13. For a Σ^1_1 set S, we take a uniformly computable sequence of linear orderings $(\mathcal{A}_n)_{n \in \omega}$ such that if $n \in S$, then $\mathcal{A}_n$ is a Harrison ordering, and otherwise $\mathcal{A}_n$ is a well ordering. Let $G_n = \Phi(\mathcal{A}_n)$, and let $F_n = \Psi(G_n)$. Let H be a Harrison ordering, let $G = \Phi(H)$, and let $F = \Psi(G)$. If $n \in S$, then $F \cong F_n$, so $F \hookrightarrow F_n$. If $n \notin S$, then $H \not\hookrightarrow \mathcal{A}_n$, $G \not\hookrightarrow G_n$, and $F \not\hookrightarrow F_n$. $\square$

We turn to groups.

Proposition 2.15. *Let NG be the class of 2-step nilpotent groups. Then $Em(NG)$ is m-complete Σ_1^1.*

For a field F, the *Heisenberg group* $H(F)$ consists of the matrices of the form

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with entries $a, b, c \in F$. We write (a, b, c) for the matrix with entries as above. The center $Z(H(F))$ consists of the elements of the form $(0, 0, c)$. We use the following result of Mal'cev [17], with a modification from [12].

Theorem 2.16. *Let F be a field. Then:*

1. *there is a non-commuting pair of elements $g, g' \in H(F)$, and*
2. *for any non-commuting pair of elements g, g', the given field F is isomorphic to*

   $$(Z(H(F)), *, \times_{g,g'}) ,$$

   where $Z(H(F))$ is the center of $H(F)$, the additive operation $*$ is the restriction of the group operation to the center, and the multiplicative operation $\times_{g,g'}$ is defined in $H(F)$ by an existential formula $\psi(g, g', x, y, z)$, with parameters g, g', saying that $[x', g'] = 1$, $[y', g'] = 1$, $[g, y'] = y$, $[x', g'] = x$, and $[x', y'] = z$.

Mal'cev's result is also true for arbitrary rings with unity. Mal'cev used the result to transform the ring of integers into a group the theory of which is hereditarily undecidable. The result has also been used in [14].

To prove Proposition 2.15, we use Theorem 2.16, together with the following lemma.

Lemma 2.17. *Suppose F is a field, and let g, g' be an arbitrary non-commuting pair in $H(F)$. Then for all $z \in H(F)$, $z \in Z(H(F))$ iff z commutes with both g and g'.*

Proof. Let $z = (a_1, b_1, c_1)$. Suppose $g = (a_2, b_2, c_2)$ and $g' = (a_3, b_3, c_3)$. If $z * g = g * z$ and $z * g' = g' * z$, then

1. $a_2b_3 \neq a_3b_2$ but
2. $a_1b_2 = a_2b_1$ and
3. $a_1b_3 = a_3b_1$.

Cross-multiplying the latter two equations (and noting that the field multiplication is commutative), we get $a_1b_1a_3b_2 = a_1b_1a_2b_3$. Since $a_2b_3 \neq a_3b_2$, we have $a_1b_1 = 0$. In fact, we can show that $a_1 = b_1 = 0$. If $b_1 \neq 0$, then $a_1 = 0$ and $a_2 = a_3 = 0$, while if $a_1 \neq 0$, then $b_1 = 0$ and $b_2 = b_3 = 0$. Either of these contradicts the first equation. □

To prove Proposition 2.15, it is enough to show the following.

Lemma 2.18. *For fields $\mathcal{A}, \mathcal{B}$, $\mathcal{A} \hookrightarrow \mathcal{B}$ iff $H(\mathcal{A}) \hookrightarrow H(\mathcal{B})$.*

Proof. First, suppose $\mathcal{A} \hookrightarrow \mathcal{B}$. Without loss of generality, we may suppose that $\mathcal{A} \subseteq \mathcal{B}$. It is clear that $H(\mathcal{A}) \subseteq H(\mathcal{B})$. Now, suppose $H(\mathcal{A}) \hookrightarrow H(\mathcal{B})$. Let G be a subgroup of $H(\mathcal{B})$ isomorphic to $H(\mathcal{A})$. There is a non-commuting pair g, g' in G. Let $\mathcal{B}'$ be the field $(Z(\mathcal{B}), *, \times_{g,g'})$. By Malcev's result, this is isomorphic to $\mathcal{B}$. Since G is isomorphic to $H(\mathcal{A})$, $\mathcal{A}' = (Z(G), *, \times_{g,g'})$ is a field isomorphic to $\mathcal{A}$. To show that $\mathcal{A}'$ is a subfield of $\mathcal{B}'$, we note that if $x \in Z(G)$, then x commutes with both g and g'. Then, by Lemma 2.17, x is in $Z(H(\mathcal{B}))$. Therefore, the universe of $\mathcal{A}'$ is contained in the universe of $\mathcal{B}'$. Addition in both $\mathcal{A}'$ and $\mathcal{B}'$ is the restriction of the group operation in $H(\mathcal{B})$. Multiplication in both $\mathcal{A}'$ and $\mathcal{B}'$ is $\times_{g,g'}$, defined in G and $H(\mathcal{B})$ by the existential formula with parameters $g, g' \in G$. □

3 Isomorphism may be more complicated

In this section, we describe some classes for which the isomorphism problem is more complicated than the embedding problem. The first examples are natural classes of groups.

3.1 Groups

We first consider the class of free groups on a finite number of generators, greater than one. The following is a well-known result of Higman, Neumann, and Neumann [16], [13] .

Lemma 3.1. *Let G_n be a free group on n generators. Then $G_n \hookrightarrow G_2$.*

Idea of proof: We give a sketch of the proof for $n = 3$. Suppose G_3 has generators c_1, c_2, c_3, and G_2 has generators a, b. Let $w_1 = aa$, $w_2 = ab$, $w_3 = ba$. There is a homomorphism h from G_3 to G_2, where $h(c_i) = w_i$. It is not difficult to see that the kernel of h is trivial. □

The above lemma together with the observation that the free group on two generators can easily be embedded into any free group on two or more generators gives the following result.

Proposition 3.2. *The embedding problem for free groups on finitely many generators, greater than* 1*, is computable within the class.*

However, the isomorphism problem is more complicated. The following lemma, found in [21] will be useful.

Lemma 3.3. *Any quantifier-free formula δ true in G_{n+1} of the generators $a_1, \ldots, a_n, a_{n+1}$ is also true in G_n of the generators $a_1, \ldots, a_n$ and some word w on $a_1, \ldots, a_n$.*

Idea of proof: Again, we give a sketch in the case where $n = 3$. We may suppose that δ is a conjunction saying of finitely many words on $a_1, \ldots, a_{n+1}$ that they are not equal to the identity. We let $w = a_1^k a_2^k$, where k is much larger than any of the words appearing in δ. □

Proposition 3.4. *The isomorphism problem for free groups on finitely many generators, greater than* 1*, is* m*-complete* Δ^0_3 *within the class.*

Proof. A free group on finitely many generators is determined by the number of generators. For each n, we have a computable Σ_3 sentence φ_n saying that there exist $a_1, \ldots, a_n$, generating the whole group, and with no non-trivial relations. Using Δ^0_3, we can determine which of these sentences is true in a given computable group. Thus, we can determine whether two free groups are isomorphic.

For hardness, let S be a Δ^0_3 set. We produce a uniformly computable sequence of pairs $\mathcal{A}_n, \mathcal{B}_n$ of free groups on finitely many generators such that $n \in S$ iff $\mathcal{A}_n \cong \mathcal{B}_n$. There is a Δ^0_2 function $g : \omega \times \omega \to 2$ such that for all $n \in \omega$, $\chi_S(n) = \lim_s g(n, s)$. Fix an n, and define a Δ^0_2 function $h : \omega \to \omega \times \omega$. The value of $h(s)$ will have the form (k, k) if $g(n, s) = 1$, and $(k, k + 1)$, if $g(n, s) = 0$. We let $h(0)$ be $(2, 2)$ if $g(n, 0) = 1$, or $(2, 3)$ if $g(n, 0) = 0$. At stage $s + 1$, if $g(n, s + 1) = g(n, s)$, we let $h(s + 1) = h(s)$. If the value of g has changed at stage $s+1$, then there are two cases. First, suppose $g(n, s) = 1$, but $g(n, s+1) = 0$. Then $h(s) = (k, k)$ for some k, and we let $h(s+1) = (k, k+1)$. Next, suppose $g(n, s) = 0$ but $g(n, s + 1) = 1$. Then $h(s) = (k, k + 1)$ for some k, and we let $h(s + 1) = (k + 1, k + 1)$.

Note that h has a limiting value. We shall arrange that the first component of this limiting value gives the number of generators for $\mathcal{A}_n$ and the second gives the number of generators for $\mathcal{B}_n$. We have a computable sequence of guesses at the function h. At each stage r, we have a guess at a finite initial segment, of length ℓ, and at stage $r + 1$, we will either extend our guess to length $\ell + 1$, or else back up and correct what we see as the first mistake. We enumerate the diagrams of $\mathcal{A}_n$ and $\mathcal{B}_n$ based on the guesses. At every stage, we have proposed generators for each group. When we believe that the number should be increased, we add new generators. When we believe that the number should be decreased, using Lemma 3.3 we collapse the generators, preserving the generators from the last stage we believe is correct. Note that if the limiting number given by h, for $\mathcal{A}_n$ or $\mathcal{B}_n$, is k, there will be some stage after which our guess is always $\geq k$. We may infinitely often believe that this number is greater than k. We will preserve a fixed set of k generators, adding more and then collapsing infinitely often. □

We now mention two other quick examples: reduced Abelian p-groups of length ω, and the class of equivalence structures in which there are no infinite classes and there are arbitrarily large finite classes. Calvert [3] showed that the isomorphism problem for the class of reduced Abelian p-groups of length ω is m-complete Π^0_3. Again the embedding problem is trivial within the class.

Lemma 3.5. *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *are countable reduced Abelian* p*-groups of length* ω*. Then* $\mathcal{A} \hookrightarrow \mathcal{B}$*.*

Proof. We can express $\mathcal{A}$ and $\mathcal{B}$ as direct sums of cyclic groups, each of order p^n, for some n, where the exponents n are unbounded. □

The class of equivalence structures in which there are no infinite classes and there are arbitrarily large finite classes resembles in many ways the class of Abelian p-groups of length ω. We mention that the embedding problem is again trivial, and the isomorphism problem is m-complete Π^0_3 (see [6]).

3.2 A subclass of linear orderings

The next example, a special class of linear orderings, is somewhat artificial. In what follows, we use η to represent the order type of the rationals, and $(\omega^* + \omega)$ to represent the order type of the integers. Let K consist of all orderings of types $n \cdot \eta$ (for $n \in \omega$), $\omega \cdot \eta$, and $(\omega^* + \omega) \cdot \eta$. It is easy to see that the embedding problem for K is trivial within the class. We show that the index set of K is m-complete Π^0_4. We will also show in Prop 3.10 that the isomorphism problem has an interesting form. A set is 3-Σ^0_3 if it has the form $S_1 - (S_2 - S_3)$, where S_1, S_2, S_3 are all Σ^0_3. The complement of such a set is co-3-Σ^0_3. The computable isomorphism problem is m-complete co-3-Σ^0_3 within the class.

Proposition 3.6. *$Em(K)$ is computable within $I(K) \times I(K)$.*

We will first determine the complexity of the index set for K. We use a result on pairs of structures proved in [1], and also discussed in [2]. We need some definitions.

Definition 3. For structures $\mathcal{A}$ and $\mathcal{B}$ with tuples $\overline{a}$ and $\overline{b}$ of the same length, we say $(\mathcal{A}, \overline{a}) \leq_1 (\mathcal{B}, \overline{b})$ if all finitary Σ^0_1 formulas true of $\overline{b}$ in $\mathcal{B}$ are also true of $\overline{a}$ in $\mathcal{A}$. Inductively, we define for $\alpha > 1$, $(\mathcal{A}, \overline{a}) \leq_\alpha (\mathcal{B}, \overline{b})$ if for each $\overline{d}$ in $\mathcal{B}$ and each β with $1 \leq \beta < \alpha$ there exists $\overline{c}$ in $\mathcal{A}$ such that $(\mathcal{B}, \overline{b}, \overline{d}) \leq_\beta (\mathcal{A}, \overline{a}, \overline{c})$.

Definition 4. A family of structures $\{\mathcal{A}_0, \mathcal{A}_1, ...\}$ is *α-friendly* if the structures are uniformly computable and for $\beta < \alpha$, the relations $(\mathcal{A}_i, \overline{a}_i) \leq_\beta (\mathcal{A}_j, \overline{a}_j)$ are c.e. uniformly in β.

Here is the result on pairs of structures.

Theorem 3.7. *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be structures such that $\mathcal{A}_1 \leq_\alpha \mathcal{A}_0$ and $\{\mathcal{A}_0, \mathcal{A}_1\}$ is α-friendly. Then for any Π^0_α set S, there is a uniformly computable sequence of structures $\mathcal{C}_n$ such that $\mathcal{C}_n \cong \mathcal{A}_0$ if $n \in S$ and $\mathcal{C}_n \cong \mathcal{A}_1$ if $n \notin S$.*

Proposition 3.8. *$I(K)$ is m-complete Π^0_4.*

Proof of Proposition 3.8. The index set for linear orderings is Π^0_2. Recall that in a linear order, any single element x lies in the trivial successor chain $\{x\}$. Then to say that a linear ordering $\mathcal{A}$ is in K, we say the following:

1. all maximal successor chains have the same size,
2. if one maximal successor chain has a first element then all do,

3. if a maximal successor chain has a last element, then it also has a first element,

4. between any two maximal successor chains, there is another,

5. there is no first or last successor chain.

We have computable Σ_2 formulas $\lambda_n(x)$, $\rho_n(x)$, $\varphi(x)$ saying that there is a successor chain of size n to the left of, right of, or including, x, respectively. For 1, we have a computable Π_3 sentence saying that for all n and for all x, y, $\varphi_n(x) \to \varphi_n(y)$. We have computable Σ_3 formulas $\psi(x)$ and $\psi'(x)$, saying that the maximal successor chain to the left of x has a first element, or that the one to the right of x has a last element. For 2, we have a computable Π_4 sentence saying that if some element has no predecessor, then for all x, $\psi(x)$ holds. For 3, we have a computable Π_4 sentence saying for all x, $(\psi'(x) \to \psi(x))$. We have a computable Σ_2 formula $x \sim y$ saying that there are only finitely many elements between x and y. For 4, we have a computable Π_4 sentence saying for all x, y, if $x < y$ and $x \not\sim y$, then there exists z such that $x < z < y$ and $x \not\sim z$ and $z \not\sim y$. Similarly, for 5, there are computable Π_4 sentences. Thus, we have a computable Π_4 description of the elements of K, and $I(K)$ is Π^0_4.

We turn to the proof that $I(K)$ is Π^0_4-hard. Let S be a Π^0_4 set. We show that there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{C}_n \cong (\omega^* + \omega) \cdot \eta$, and if $n \notin S$, then $\mathcal{C}_n \cong (\omega^* + \omega) \cdot (1 + \eta)$. We use Theorem 3.7. We must verify two claims.

Claim 1. $(\omega^* + \omega) \cdot (1 + \eta) \leq_4 (\omega^* + \omega) \cdot \eta$

Proof. Starting with a tuple $\overline{a}$ in $Z \cdot \eta$, we choose a corresponding tuple $\overline{b}$ in $(\omega^* + \omega) \cdot (1 + \eta)$, ordered the same, and such that, for $a_i, a_j \in \overline{a}$, if a_i and a_j lie on the same $(\omega^* + \omega)$-piece, with n elements in between, then b_i and $b_j \in \overline{b}$ also lie on the same $(\omega^* + \omega)$-piece, with n elements in between. The tuple $\overline{b}$ has no elements in the first copy of $(\omega^* + \omega)$. We must show that $((\omega^* + \omega) \cdot \eta, \overline{a}) \leq_3 ((\omega^* + \omega) \cdot (1 + \eta), \overline{b})$. Given a tuple $\overline{d}$ in $(\omega^* + \omega) \cdot (1 + \eta)$, we choose a tuple $\overline{c}$ from $(\omega^* + \omega) \cdot \eta$ so that the ordering is preserved, and intervals between corresponding elements have the same size. We will be able to do this, but we notice that now some elements of $\overline{d}$ may be in the first copy of $(\omega^* + \omega)$.

We must show that $((\omega^* + \omega) \cdot (1 + \eta), \overline{b}, \overline{d}) \leq_2 ((\omega^* + \omega) \cdot \eta, \overline{a}, \overline{c})$. Given a tuple $\overline{e}$ in $(\omega^* + \omega) \cdot \eta$, we must choose a tuple $\overline{f}$ in $(\omega^* + \omega) \cdot (1 + \eta)$ such that all existential statements true of $((\omega^* + \omega) \cdot (1 + \eta), \overline{b}, \overline{d}, \overline{f})$ are also true of $((\omega^* + \omega) \cdot \eta, \overline{a}, \overline{c}, \overline{e})$. Each tuple determines a partition of the ordering into subintervals. We need to assure that the intervals determined by the tuple $\overline{a}, \overline{c}, \overline{e}$ are no larger than those determined by the tuple $\overline{b}, \overline{d}, \overline{f}$. This is no problem. Notice that $\overline{e}$ may be to the left of $\overline{c}$, which was matching $\overline{d}$, possibly in the first copy of $(\omega^* + \omega)$. We match the sizes of the intervals exactly except where we are forced to choose f_i from the first $(\omega^* + \omega)$-piece, so that the interval between f_i and some d_j is finite, while the interval between the corresponding e_i and c_j is infinite. □

Claim 2. The family $\{(\omega^* + \omega) \cdot (1 + \eta), (\omega^* + \omega) \cdot \eta\}$ is 4-friendly.

Proof. Clearly, we have computable orderings of types $(\omega^* + \omega) \cdot \eta$ and $(\omega^* + \omega) \cdot (1 + \eta)$. Moreover, we may choose the orderings so that the elements correspond naturally to pairs (k, q), where $k \in (\omega^* + \omega)$ and q is a positive rational, or a non-negative rational. In these orderings, we can effectively determine whether two elements lie in the same $(\omega^* + \omega)$-piece, and if so, we can determine the size of the interval between them. If there is a first $(\omega^* + \omega)$-piece, then we can recognize its elements. From this, it follows that the back-and-forth relations $\leq_1$, $\leq_2$, and $\leq_3$ are c.e., so the pair is 4-friendly. □

Now, we are in a position to apply Theorem 3.7. For an arbitrary Π^0_4 set S, we get a uniformly computable sequence $(\mathcal{C}_n)_{n \in \omega}$ such that $\mathcal{C}_n$ has type $(\omega^* + \omega) \cdot \eta$, if $n \in S$, and $(\omega^* + \omega) \cdot (1 + \eta)$, otherwise. This shows that $I(K)$ is Π^0_4-hard, completing the proof of Proposition 3.8. □

We turn to the isomorphism problem. The following is easy to check.

Lemma 3.9.

1. *The* 3-Σ^0_3 *sets are those of the form* $S_1 \cup (S_2 - S_3)$, *where all* S_i *are* Σ^0_3.
2. *The co*-3-Σ^0_3 *sets are those of the form* $T \cup (S_1 - S_2)$, *where* T *is* Π^0_3 *and* S_1, S_2 *are both* Σ^0_3.

Proposition 3.10. $E(K)$ *is m-complete co*-3-Σ^0_3 *within* $I(K) \times I(K)$.

Proof. First, we show that $E(K)$ is co-3-Σ^0_3 within $I(K) \times I(K)$. The isomorphism type of a structure $\mathcal{A} \in K$ is determined by the size of the maximal chains if these are finite. If the maximal chains are infinite, then the isomorphism type of $\mathcal{A}$ is determined by the presence or absence of left limits, by which we mean a limit from the left, i.e. an element with predecessors but no immediate predecessor. Suppose $\mathcal{A}$ and $\mathcal{B}$ are computable members of K. Then $\mathcal{A} \ncong \mathcal{B}$ iff one of the following two things happens.

1. For some n, the maximal successor chains in one structure have size n, while in the other structure, there is a successor chain of size at least $n+1$.
2. Both structures have arbitrarily large finite successor chains, and one has a left limit point, while the other does not.

The statement in 1 is Σ^0_3. For each n, we have a computable Σ_2 sentence λ_n saying that there is a successor chain of size at least n. Using Δ^0_3, we can search for n such that λ_n holds in one of the structures but not in the other. The statement in 2 is $\Sigma^0_3 \cap \Pi^0_3$. We will now check that all of the following hold:

(a) there are arbitrarily large finite successor chains in both $\mathcal{A}$ and $\mathcal{B}$,

(b) $\mathcal{A}$ has a left limit point or $\mathcal{B}$ has a left limit point,

(c) it is not the case that both $\mathcal{A}$ and $\mathcal{B}$ have left limit points.

We can collapse (a) and (c) to a single Π_3^0 statement, while (b) is Σ_3^0. Therefore, the complement of $E(K)$ is 3-Σ_3^0 within $I(K)\times I(K)$, and $E(K)$ is co-3-Σ_3^0 within $I(K)\times I(K)$.

To prove hardness, we use the following lemma.

Lemma 3.11. *There is a uniform procedure which we can apply to any Δ_2^0 ordering $\mathcal{A}$ of type $(\omega^* + \omega)$, ω, or n to get a computable ordering of type $\mathcal{A}\cdot\eta$.*

Proof. Let $(d_k)_{k\in\omega}$ be a uniformly Δ_2^0 sequence of finite orderings such that $d_k \subseteq d_{k+1}$ and $\cup_k d_k = \mathcal{A}$. Let $\sigma_k = (d_0, \ldots, d_k)$. We have a computable sequence $(\rho_s)_{s\in\omega}$ of approximations to the Δ_2^0 sequence, where each ρ_s represents our stage s guess at some d_k. At stage $s+1$, we either add one term to the current sequence ρ_s, or else we back up and correct what appears to be the first mistake. Let $d_{k(s)}^s$ be the last term of ρ_s. This is our stage s approximation to $\mathcal{A}$. Note that for each k, there is some s such that $\rho_s = \sigma_k$ and for all $s' > s$, $\rho_s \supseteq \sigma_k$.

Our goal is to build an ordering of type $\mathcal{A}\cdot\eta$. At stage s, we have put into our computable ordering finitely many blocks, each of type $d_{k(s)}^s$. At stage $s+1$, if $k(s+1) = k(s)+1$, then we add elements to the old blocks, if necessary, so that they have type $d_{k(s+1)}^{s+1}$. In addition, we add blocks between existing ones, and before the first and after the last. If $k(s+1) \leq k(s)$, then we correct the existing blocks, putting any extra elements into new blocks. □

For hardness, we show that for any co-3-Σ_3^0 set S, there is a uniformly computable sequence of pairs of structures $(\mathcal{A}_n, \mathcal{B}_n)_{n\in\omega}$ in K such that $\mathcal{A}_n \cong \mathcal{B}_n$ iff $n \in S$. We may suppose that $S = T \cup (S_1 - S_2)$, where T is Π_3^0 and S_1, S_2 are both Σ_3^0.

Lemma 3.12. *There is a uniformly Δ_2^0 sequence of pairs of linear orderings $(\mathcal{A}_n, \mathcal{B}_n)_{n\in\omega}$ such that:*

1. *if $n \in T$, then $\mathcal{A}_n$ and $\mathcal{B}_n$ have type $(\omega^* + \omega)$,*
2. *if $n \notin T$ and $n \notin S_1$, then $\mathcal{A}_n$ has type ω and $\mathcal{B}_n$ has type $(\omega^* + \omega)$,*
3. *if $n \notin T$ and $n \in S_1 - S_2$, then $\mathcal{A}_n$ and $\mathcal{B}_n$ have type ω,*
4. *if $n \notin T$ and $n \in S_1 \cap S_2$, then $\mathcal{A}_n$ has type ω and $\mathcal{B}_n$ is finite.*

Proof. We consider Δ_2^0 approximations for T, S_1, and S_2. At stage s, if $n \in T_s$, then we add one element on each end of $\mathcal{A}_n$ and $\mathcal{B}_n$. If $n \notin T_s$ and $n \notin S_{1,s}$, then we add one element on the right of $\mathcal{A}_n$ and one element on each end of $\mathcal{B}_n$. If $n \notin T_s$ and $n \in S_{1,s} - S_{2,s}$, then for both $\mathcal{A}_n$ and $\mathcal{B}_n$, we add one element on the right. If $n \notin T_s$ and $n \in S_{1,s} \cap S_{2,s}$, then we add one element on the right of $\mathcal{A}_n$, leaving $\mathcal{B}_n$ unchanged.

Suppose $n \in T$. For infinitely many s, we have $n \in T_s$, and we add an element to each end of $\mathcal{A}_n$, and $\mathcal{B}_n$, thus we get $(\omega^* + \omega)$ for both $\mathcal{A}_n$ and $\mathcal{B}_n$.

Suppose $n \notin T$ and $n \notin S_1$. For all sufficiently large s, we have $n \notin T_s$, and we stop adding new elements on the left of $\mathcal{A}_n$. At infinitely many stages, we add a new element on the right of $\mathcal{A}_n$ and on both ends of $\mathcal{B}_n$. We get type ω for $\mathcal{A}_n$, while $\mathcal{B}_n$ has type $(\omega^* + \omega)$. Suppose $n \notin T$ and $n \in S_1 - S_2$. For all sufficiently large stages, we will not add elements on the left of either $\mathcal{A}_n$ or $\mathcal{B}_n$. Infinitely often, we will add new elements on the right of $\mathcal{A}_n$ and $\mathcal{B}_n$. Then both have type ω. Suppose $n \notin T$ and $n \in S_1 \cap S_2$. For all sufficiently large stages, we do not add new elements on the left of $\mathcal{A}_n$ and we do not add any elements to $\mathcal{B}_n$. Infinitely often, we add new elements on the right of $\mathcal{A}_n$. Then $\mathcal{A}_n$ has type ω, while $\mathcal{B}_n$ is finite.

□

To complete the proof of hardness, we take a uniformly Δ^0_2 sequence of pairs $(\mathcal{A}'_n, \mathcal{B}'_n)_{n\in\omega}$ as in Lemma 3.12. We apply Lemma 3.11 to get a uniformly computable sequence of pairs $(\mathcal{A}_n, \mathcal{B}_n)_{n\in\omega}$ such that $\mathcal{A}_n \cong \mathcal{A}'_n \cdot \eta$ and $\mathcal{B}_n \cong \mathcal{B}'_n \cdot \eta$. The structures $\mathcal{A}_n, \mathcal{B}_n$ are in K, and $\mathcal{A}_n \cong \mathcal{B}_n$ iff $\mathcal{A}'_n \cong \mathcal{B}'_n$ iff $n \in T \cup (S_1 - S_2)$. □

4 Embedding may be more complicated

In this section, we describe a class of graphs for which the isomorphism problem is less complicated than the embedding problem. We begin with a computable copy $\mathcal{R}$ of the random graph. The next result is a special case of a theorem of Nurtazin [19].

Lemma 4.1. *There is a uniformly computable sequence of graphs $(\mathcal{G}_e)_{e\in\omega}$ with representatives of all isomorphism types of computable graphs, including the empty graph.*

Sketch of proof. For each e, W_e is the universe of a subgraph of $\mathcal{R}$. We can pass effectively from e to a computable graph isomorphic to W_e. □

We define a graph $\mathcal{H}_e$ for each $e \in \omega$. The graph has a special point c, which forms one vertex of an $(e+3)$-gon. In addition, the graph includes a copy G of $\mathcal{G}_e$, which is disjoint from the $(e+3)$-gon, and the special point c is connected to all elements of G. There are no other points or edges in $\mathcal{H}_e$. Let H be the class of copies of the graphs $\mathcal{H}_e$, for $e \in \omega$.

Proposition 4.2. *$I(H)$ is m-complete Σ^1_1.*

Proof. To show that $I(H)$ is Σ^1_1, we consider a larger class H', consisting of graphs having a special point c, forming one vertex of a $(3+e)$-gon, for some e, such that if G is the set of points outside the $(3+e)$-gon, then c is connected to all elements of G, and the other elements of the $(3+e)$-gon are not connected to any elements of G. Clearly, $I(H')$ is arithmetical—it is d-Σ^0_2. For a structure $\mathcal{A} \in H'$, with special point c, forming one vertex of an $(e+3)$-gon, where G is the set of points outside the $(e+3)$-gon, we have $\mathcal{A} \in H$ if and only if $G \cong \mathcal{G}_e$. This is Σ^1_1.

For hardness, we take a computable graph G such that $I(G)$ is m-complete Σ^1_1, and we choose e such that $\mathcal{G}_e \cong G$. Let S be a Σ^1_1 set. We have a uniformly computable sequence of graphs $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n \cong G$ iff $n \in S$. For each n, we pass effectively to a structure $\mathcal{B}_n \in H'$, with a special point c forming one vertex of a $(3+e)$-gon (for our chosen e), and the points outside the $(3+e)$-gon forming a copy G of $\mathcal{A}_n$, where c is connected to all elements of G, and there are no other points or edges. Then $\mathcal{B}_n \in H$ if and only if $n \in S$. □

Proposition 4.3. *$E(H)$ is m-complete Δ^0_3 within $I(H) \times I(H)$.*

Proof. For each e, we have a natural computable Σ_2 sentence φ_e saying that there is an $(e+3)$-gon, one vertex of which is connected to everything outside the $(e+3)$-gon. Given a computable $\mathcal{A} \in H$, using Δ^0_3, we search for the unique e such that $\mathcal{A} \models \varphi_e$, and then we know that $\mathcal{A} \cong \mathcal{H}_e$. Therefore, $E(H)$ is Δ^0_3 within $I(H) \times I(H)$.

Toward hardness, take a Δ^0_3 set S. As in the proof of Proposition 3.4, we obtain a Δ^0_2 function $h(n,s)$ such that the limiting value of h has the form (k,k) if $n \in S$ and $(k,k+1)$ otherwise. Let $h_1(n,s)$, $h_2(n,s)$ be the first and second components of $h(n,s)$.

We build a uniformly computable sequence of pairs $(\mathcal{A}_n,\mathcal{B}_n)_{n\in\omega}$ in H such that $\mathcal{A}_n \cong \mathcal{B}_n$ if and only if $n \in S$. Note that for a given e, we can find $e' > e$ such that $\mathcal{G}_{e'} \cong \mathcal{H}_e$, and a copy of $\mathcal{H}_e$ extends to a copy of $\mathcal{H}_{e'}$. Let $\nu(e) = e'$. To construct $\mathcal{A}_n$, we start off copying $\mathcal{H}_0$. Suppose at stage s, we are copying $\mathcal{H}_e$. If $h_1(n,s+1) = h_1(n,s)$, then we continue copying $\mathcal{H}_e$, and if $h_1(n,s+1) \neq h_1(n,s)$, then we switch to copying $\mathcal{H}_{\nu(e)}$. Similarly, to construct $\mathcal{B}_n$, we start off copying $\mathcal{H}_0$. Suppose at stage s, we are copying $\mathcal{H}_e$. If $h_2(n,s+1) = h_2(n,s)$, then we continue copying $\mathcal{H}_e$, and if $h_2(n,s+1) \neq h_2(n,s)$, then we switch to copying $\mathcal{H}_{\nu(e)}$. After finitely many steps, the function $h(n,s)$ will reach its limiting value (k,k) or $(k,k+1)$. From then on, we will be building $\mathcal{A}_n \cong \mathcal{H}_p$ and $\mathcal{B}_n = \mathcal{H}_q$, where $p = \nu^k(0)$, and

$$q = \begin{cases} p & \text{if } n \in S, \\ \nu(p) = \nu^{k+1}(0) & \text{otherwise.} \end{cases}$$

□

Proposition 4.4. *$Em(H)$ is m-complete Σ^1_1 within $I(H) \times I(H)$.*

Proof. We show hardness. Let S be a Σ^1_1 set. We describe a fixed computable structure $\mathcal{A}$ and a uniformly computable sequence of structures $(\mathcal{B}_n)_{n\in\omega}$, all in H such that $\mathcal{A} \hookrightarrow \mathcal{B}_n$ iff $n \in S$. Let $\mathcal{H}$ be the Harrison ordering, and let T be the tree of finite decreasing sequences in $\mathcal{H}$, ordered by the relation of being an initial segment. This is a computable tree, but we consider it as a graph. There is a copy of T which is a c.e. subgraph of $\mathcal{R}$. We fix a c.e. index e. We form $\mathcal{G}_e$ and then $\mathcal{H}_e$, which is in H. This is the structure $\mathcal{A}$.

As in Section 2, corresponding to our Σ^1_1 set S, we have a uniformly computable sequence of linear orderings $(\mathcal{L}_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{L}_n \cong \mathcal{H}$, and if $n \notin S$, then $\mathcal{L}_n$ is a well-ordering. We pass effectively from $\mathcal{L}_n$ to the tree T_n of finite decreasing sequences in $\mathcal{L}_n$. We pass effectively from T_n to a graph $\mathcal{M}_n$, which is the result of building a copy of T_n, adding a special point c connected to every element of T, and then adding an $(e+3)$-gon for each e, where these are disjoint from T, and disjoint from each other except that they share the vertex c. We can pass effectively from $\mathcal{M}_n$ to an isomorphic copy that is a c.e. subgraph of $\mathcal{R}$, and, moreover, we know an index e_n. Then we get a computable graph $\mathcal{G}_{e_n}$, and we also get $\mathcal{H}_{e_n}$, which is in H. This is our $\mathcal{B}_n$. It is clear that $\mathcal{A} \hookrightarrow \mathcal{B}_n$ if and only if $n \in S$. (If $n \notin S$, then T_n has no path, and $\mathcal{B}_n$ will have no subgraph representing a path through a tree.) □

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