

# $\Sigma_1^0$ and $\Pi_1^0$ equivalence structures\*

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## Abstract

We study computability theoretic properties of  $\Sigma_1^0$  and  $\Pi_1^0$  equivalence structures and how they differ from computable equivalence structures or equivalence structures that belong to the Ershov difference hierarchy. Our investigation includes the complexity of isomorphisms between  $\Sigma_1^0$  equivalence structures and between  $\Pi_1^0$  equivalence structures.

**Keywords:** computability theory, equivalence structures, effective categoricity, computable model theory

## 1 Introduction

Computable model theory deals with the algorithmic properties of effective mathematical structures and the relationships between such structures. Perhaps the most basic kind of relationship between two structures is that of

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isomorphism. It is natural to study the isomorphism problem in the context of computable mathematics by investigating the following question.

Given two effective structures which are isomorphic, what is the least complex isomorphism between them?

In what follows, we restrict our attention to countable structures for computable languages. Hence, if a structure is infinite, we can assume that its universe is the set of natural numbers,  $\omega$ . We recall some basic definitions. If  $\mathcal{A}$  is a structure with universe  $A$  for a language  $\mathcal{L}$ , then  $\mathcal{L}^A$  is the language obtained by expanding  $\mathcal{L}$  by constants for all elements of  $A$ . The *atomic diagram* of  $\mathcal{A}$  is the set of all quantifier-free sentences of  $\mathcal{L}^A$  true in  $\mathcal{A}$ . The *elementary diagram* of  $\mathcal{A}$  is the set of all first-order sentences of  $\mathcal{L}^A$  true in  $\mathcal{A}$ . A structure  $\mathcal{A}$  is *computable* if its atomic diagram is computable, and a structure  $\mathcal{A}$  is *decidable* if its elementary diagram is computable. We call two structures *computably isomorphic* if there is a computable function that is an isomorphism between them. A computable structure  $\mathcal{A}$  is *relatively computably isomorphic* to a possibly noncomputable structure  $\mathcal{B}$  if there is an isomorphism between them that is computable in the atomic diagram of  $\mathcal{B}$ . A computable structure  $\mathcal{A}$  is *computably categorical* if every computable structure that is isomorphic to  $\mathcal{A}$  is computably isomorphic to  $\mathcal{A}$ . A computable structure  $\mathcal{A}$  is *relatively computably categorical* if every structure that is isomorphic to  $\mathcal{A}$  is relatively computably isomorphic to  $\mathcal{A}$ . Similar definitions arise for other naturally definable classes of structures and their isomorphisms. For example, for any  $n \in \omega$ , a structure is  $\Delta_n^0$  if its atomic diagram is  $\Delta_n^0$ ; two structures are  $\Delta_n^0$  *isomorphic* if there is a  $\Delta_n^0$  isomorphism between them; and a computable structure  $\mathcal{A}$  is  $\Delta_n^0$  *categorical* if every computable structure that is isomorphic to  $\mathcal{A}$  is  $\Delta_n^0$  isomorphic to  $\mathcal{A}$ .

Among the simplest nontrivial structures are equivalence structures, i.e., structures of the form  $\mathcal{A} = (\omega, E)$  where  $E$  is an equivalence relation. The study of the complexity of isomorphisms between computable equivalence structures was recently carried out by Calvert, Cenzer, Harizanov, and Morozov in [2]. Similarly, the study of structures and functions within the Ershov difference hierarchy has been recently carried out by Khousainov, Stephan, and Yang in [9], and by Cenzer, LaForte, and Remmel in [3] where they investigated equivalence structures in particular. In this paper, we study  $\Sigma_1^0$  and  $\Pi_1^0$  equivalence structures. Here, we say that an equivalence structure  $\mathcal{A} = (\omega, E)$  is  $\Sigma_1^0$  (or *c.e.*) if  $E$  is a c.e. set, and, similarly,  $\mathcal{A}$  is  $\Pi_1^0$  (or *co-c.e.*) if  $E$  is a  $\Pi_1^0$  set. It is also the case that  $\Sigma_1^0$  and  $\Pi_1^0$  structures

have been studied since the beginning of modern computable model theory. For example, in [11], Metakides and Nerode studied c.e. vector spaces, which consist of a structure  $V$  over the natural numbers such that the operations of vector addition and scalar multiplication are computable but where there is a c.e. equivalence relation  $\equiv$  the equivalence classes of which form a vector space under the vector addition and scalar multiplication. Similarly, in [13], Remmel studied co-c.e. structures where the underlying operations are computable.

Equivalence relations play an important role in mathematical logic and many other areas of mathematics. For example, isomorphism and elementary equivalence, as well as their effective versions such as computable isomorphism or  $\Sigma_n^0$ -equivalence, are equivalence relations. Similarly, a number of interesting applications of equivalence arise from the so-called classification problems where two structures are termed equivalent if they possess certain *invariant* properties.

We shall see that the complexity of isomorphisms between  $\Sigma_1^0$  equivalence structures and between  $\Pi_1^0$  equivalence structures is different from the complexity of isomorphisms between computable equivalence structures or between equivalence structures that lie in the Ershov difference hierarchy. Before we can state our results, we need some notation and definitions. For an equivalence structure  $\mathcal{A} = (A, E)$  where  $A = \omega$ , we let  $[a]^{\mathcal{A}}$  denote the equivalence class of  $a$ , i.e.,  $[a]^{\mathcal{A}} = \{b \in A : aEb\}$ . In computability theory, it is useful to split  $\mathcal{A}$  into two parts,  $Inf^{\mathcal{A}}$  and  $Fin^{\mathcal{A}}$ , where  $Inf^{\mathcal{A}}$  consists of elements with infinite equivalence classes, and  $Fin^{\mathcal{A}}$  consists of elements with finite equivalence classes. It is natural to consider different sizes of the equivalence classes of the elements in  $Fin^{\mathcal{A}}$  since these sizes code information into the equivalence relation. The *character* of an equivalence structure  $\mathcal{A}$  is the set

$$\chi(\mathcal{A}) = \{(k, n) : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } n \text{ equivalence classes of size } k\}.$$

This set provides a kind of skeleton for  $Fin^{\mathcal{A}}$ . Any set  $K \subseteq (\omega - \{0\}) \times (\omega - \{0\})$  such that for all  $n > 0$  and  $k$ ,  $(k, n + 1) \in K$  implies  $(k, n) \in K$ , is called a *character*. We say a character  $K$  is *bounded* if there is some finite  $k_0$  such that for all  $(k, n) \in K$ , we have  $k < k_0$ . Khisamiev [8] introduced the concepts of an  $s$ -function and an  $s_1$ -function as a means of computably approximating the characters of equivalence relations.

**Definition 1.1.** *Let  $f : \omega^2 \rightarrow \omega$ . The function  $f$  is an  $s$ -function if the following hold:*

1. for every  $i, s \in \omega$ ,  $f(i, s) \leq f(i, s + 1)$ , and
2. for every  $i \in \omega$ , the limit  $m_i = \lim_s f(i, s)$  exists.

We say that  $f$  is an  $s_1$ -function if, in addition:

3. for every  $i \in \omega$ ,  $m_i < m_{i+1}$ .

Calvert, Cenzer, Harizanov and Morozov [2] gave conditions under which a given character  $K$  can be the character of a computable equivalence structure. In particular, they observed that if  $K$  is a bounded character and  $\alpha \leq \omega$ , then there is a computable equivalence structure with character  $K$  and exactly  $\alpha$  infinite equivalence classes. To prove the existence of computable equivalence structures for unbounded characters  $K$ , they needed additional information given by  $s$ - or  $s_1$ -functions. They showed that if  $K$  is a  $\Sigma_2^0$  character,  $r < \omega$ , and either

- (a) there is a computable  $s$ -function  $f$  such that

$$(k, n) \in K \Leftrightarrow \text{card}(\{i : k = \lim_{s \rightarrow \infty} f(i, s)\}) \geq n, \quad \text{or}$$

- (b) there is a computable  $s_1$ -function  $f$  such that for every  $i \in \omega$ ,  $(\lim_s f(i, s), 1) \in K$ , then there is a computable equivalence structure with character  $K$  and exactly  $r$  infinite equivalence classes. In addition to these positive results, in [2], the authors also constructed an infinite  $\Delta_2^0$  set  $D$  such that for any computable equivalence structure  $\mathcal{A}$  with unbounded character and no infinite equivalence classes,  $\{k : (k, 1) \in K\}$  is not a subset of  $D$ .

$\Sigma_1^0$  equivalence structures were first considered by Ershov [5] where they are called *positive* equivalence relations. Bernardi and Sorbi [1] referred to  $\Sigma_1^0$  equivalence structures as *ceers* (computably enumerable equivalence relations) and they developed a notion of reducibility between ceers. Computably isomorphic structures are equivalent under this reducibility but the converse does not hold. This notion was developed further by Gao and Gerdes [6]. C.e. equivalence relations have also been studied by Lachlan [10] and Nies [12].

**Definition 1.2.** Let  $\alpha \leq \omega$ .

1. We say the structure  $\mathcal{A}$  is weakly  $\alpha$ -c.e. isomorphic to the structure  $\mathcal{B}$  if there are  $\alpha$ -c.e. functions  $f$  and  $g$  such that  $f$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{B}$  and  $g$  is an isomorphism from  $\mathcal{B}$  to  $\mathcal{A}$ .

2. We say the structure  $\mathcal{A}$  is  $\alpha$ -c.e. isomorphic to the structure  $\mathcal{B}$  if there is an  $\alpha$ -c.e. function  $f$  such that  $f^{-1}$  is  $\alpha$ -c.e. and  $f$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ .
3. We say the structure  $\mathcal{A}$  is graph- $\alpha$ -c.e. isomorphic to the structure  $\mathcal{B}$  if there is a graph- $\alpha$ -c.e. function  $f$  such that  $f$  is an isomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ , where a function  $f$  is graph- $\alpha$ -c.e. if the graph of  $f$  is an  $\alpha$ -c.e. set.

In [3], Cenzer, LaForte, and Remmel obtained the following results. First they proved the following basic properties of  $\alpha$ -c.e. and graph- $\alpha$ -c.e. functions.

- (a) Any nonempty  $\Sigma_2^0$  set is the range of a 2-c.e. function.
- (b) For every  $n \in \omega$ , there is an  $(n+1)$ -c.e. function that is not graph- $n$ -c.e.
- (c) There is a graph-2-c.e. function that is not  $\omega$ -c.e.
- (d) There is a 2-c.e. bijection  $f$  such that  $f^{-1}$  is not  $\omega$ -c.e.

Cenzer, LaForte, and Remmel established the following results about characters in the Ershov hierarchy.

- (i) For any  $n$ -c.e. character  $K$ , there is a computable equivalence structure with character  $K$  and without infinite equivalence classes.
- (ii) There is an  $\omega$ -c.e. character  $K$  such that any equivalence structure with character  $K$  must have infinite equivalence classes.
- (iii) For any  $\Delta_2^0$  character  $K$ , there exists a d.c.e. equivalence structure with no infinite equivalence classes and with character  $K$ .

Cenzer, LaForte, and Remmel proved the following results about isomorphisms between equivalence structures in the Ershov hierarchy.

- (I) For every  $n \in \omega$ , there exist two computable equivalence structures that are  $(n+1)$ -c.e. isomorphic, but not weakly  $n$ -c.e. isomorphic.
- (II) There are two computable equivalence structures that are graph-2-c.e. isomorphic, but not weakly  $\omega$ -c.e. isomorphic.

Cenzer, LaForte, and Remmel [3] also proved that a computable equivalence structure is computably categorical if and only if it is weakly  $\omega$ -c.e. categorical. Furthermore, they showed that any computable equivalence structure with bounded character is relatively graph-2-c.e. categorical, and that any computable equivalence structure with a finite number of infinite equivalence classes is relatively graph- $\omega$ -c.e. categorical. It then follows that a computable equivalence structure is  $\Delta_2^0$  categorical if and only if it is graph- $\omega$ -c.e. categorical.

We will prove a number of results about the complexity of isomorphisms of  $\Sigma_1^0$  and of  $\Pi_1^0$  equivalence structures. For example, in Section 2, we show that any  $\Sigma_1^0$  equivalence structure  $\mathcal{A}$  with infinitely many infinite equivalence classes is isomorphic to a computable structure. On the other hand, there are  $\Sigma_1^0$  equivalence structures with finitely many infinite equivalence classes, which are *not* isomorphic to any computable structure. We show that if  $\Sigma_1^0$  equivalence structures  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are isomorphic to a computable structure  $\mathcal{A}$  that is computably categorical or even relatively  $\Delta_2^0$  categorical, then  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are  $\Delta_2^0$  isomorphic. In Section 3, we first observe that if  $\mathcal{B}$  is a computably categorical computable equivalence structure any  $\mathcal{A}$  is a  $\Pi_1^0$  equivalence structure which is isomorphic to  $\mathcal{B}$ , then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\Delta_2^0$  isomorphic. If  $\mathcal{B}$  is a computable equivalence structure which is *not* computably categorical, then in several cases we construct a  $\Pi_1^0$  structure  $\mathcal{A}$  which is isomorphic to  $\mathcal{B}$  but is *not*  $\Delta_2^0$  isomorphic to  $\mathcal{B}$ . The simplest case is when  $\mathcal{B}$  consists of infinitely many equivalence classes of sizes 1 or 2, and no other classes; if  $\mathcal{B}$  is  $\Delta_2^0$  categorical, then we show that the  $\Pi_1^0$  structure  $\mathcal{A}$  is not even  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure. In Section 4, we consider the *spectrum question*, which is to determine the possible sets (or degrees of sets) that can be the sets of elements in equivalence classes of size  $k$ , for some fixed  $k$ , in a computable equivalence structure of a given isomorphism type. For example, we show that for any infinite c.e. set  $B$ , there is a computable equivalence structure with infinitely many equivalence classes of size 1, infinitely many classes of size 2, and no other equivalence classes, such that  $B = \{x : \text{card}([x]) = 2\}$ . In Section 5, we consider the complexity of the theory  $Th(\mathcal{A})$  of a computable equivalence structure  $\mathcal{A}$ , as well as the complexity of its elementary diagram  $FTh(\mathcal{A})$ . We explore the connection between the complexity of the character  $\chi(\mathcal{A})$  and the theory  $Th(\mathcal{A})$ . We show that if  $Th(\mathcal{A})$  is decidable, then the character  $\chi(\mathcal{A})$  is computable. We show that if an equivalence structure  $\mathcal{B}$  has a computable character, then there is a decidable structure  $\mathcal{A}$  isomorphic to  $\mathcal{B}$ .

A preliminary version [4] of this paper appeared in the Proceedings of the 5th Conference on Computability in Europe.

## 2 $\Sigma_1^0$ equivalence structures

In this section, we consider properties of  $\Sigma_1^0$  equivalence structures and their existence and categoricity. It is easy to show that the complexity of the character for  $\Sigma_1^0$  equivalence structures is at the same level of the arithmetical hierarchy as for computable equivalence structures.

**Lemma 2.1.** *For any  $\Sigma_1^0$  equivalence structure  $\mathcal{A}$ , we have:*

- (a)  $\{(k, a) : \text{card}([a]^{\mathcal{A}}) \geq k\}$  is a  $\Sigma_1^0$  set;
- (b)  $\text{Inf}^{\mathcal{A}}$  is a  $\Pi_2^0$  set;
- (c)  $\chi(\mathcal{A})$  is a  $\Sigma_2^0$  set.

Thus, if  $\mathcal{A}$  is a  $\Sigma_1^0$  equivalence structure with infinitely many infinite equivalence classes, then it follows from Lemma 2.1 above and Lemma 2.3 of [2] that  $\mathcal{A}$  is isomorphic to a computable equivalence structure. However, it was shown in [2] that there is a  $\Delta_2^0$  character  $K$  such that any computable equivalence structure with character  $K$  must have infinitely many infinite equivalence classes. It was shown in [3] that for *any*  $\Delta_2^0$  character  $K$ , there is a d.c.e. equivalence structure  $\mathcal{A}$  with character  $K$  and with no infinite equivalence classes. Hence there is a d.c.e.  $\mathcal{A}$  that is not isomorphic to any computable equivalence structure. Now, for  $\Sigma_1^0$  equivalence structures we have the following existence result.

**Theorem 2.2.** *For any  $\Sigma_2^0$  character  $K$  and any finite  $m \geq 1$ , there is a  $\Sigma_1^0$  equivalence structure  $\mathcal{A}$  with character  $K$  and with exactly  $m$  infinite equivalence classes.*

*Proof.* Let  $K$  be a  $\Sigma_2^0$  character. Let  $\mathcal{B}$  be the equivalence structure given by Lemma 2.3 in [2] such that  $\mathcal{B}$  has character  $K$  and infinitely many infinite equivalence classes, and, in addition,  $\text{Fin}^{\mathcal{B}}$  is a  $\Pi_1^0$  set. Simply define  $\mathcal{A} = (\omega, E^{\mathcal{A}})$  by  $E^{\mathcal{A}} = E^{\mathcal{B}} \cup (\text{Inf}^{\mathcal{B}} \times \text{Inf}^{\mathcal{B}})$ . Then the structure  $\mathcal{A}$  is  $\Sigma_1^0$  since  $\text{Inf}^{\mathcal{B}}$  is a  $\Sigma_1^0$  set. Furthermore,  $\mathcal{A}$  has the same character  $K$  as  $\mathcal{B}$ , and the infinitely many infinite equivalence classes of  $\mathcal{B}$  collapse into a single equivalence class  $\text{Inf}^{\mathcal{A}}$  in  $\mathcal{A}$ . For  $m > 1$ , we can then append  $(m - 1)$  computable infinite equivalence classes.  $\square$

**Corollary 2.3.** *There exists a  $\Sigma_1^0$  equivalence structure  $\mathcal{A}$  that is not isomorphic to any computable equivalence structure.*

*Proof.* Let  $K$  be a  $\Sigma_2^0$  character that does not have a computable  $s_1$ -function. Then, by Lemma 2.6 of [2], there is no computable structure with character  $K$  and with finitely many infinite equivalence classes.  $\square$

We will next consider the effective categoricity of  $\Sigma_1^0$  equivalence structures. It was shown in [2] that a computable equivalence structure  $\mathcal{A}$  is computably categorical if and only if  $\mathcal{A}$  is relatively computably categorical, and that a computable equivalence structure  $\mathcal{A}$  is computably categorical if and only if one of the following conditions is satisfied:

1.  $\mathcal{A}$  has only finitely many finite equivalence classes, or
2.  $\mathcal{A}$  has finitely many infinite equivalence classes and bounded character, and there is at most one finite  $k$  such that  $\mathcal{A}$  has infinitely many equivalence classes of size  $k$ .

It is also shown in [2] that a computable equivalence structure  $\mathcal{A}$  is relatively  $\Delta_2^0$  categorical if and only if  $\mathcal{A}$  has finitely many infinite equivalence classes or  $\mathcal{A}$  has a bounded character.

Clearly, a noncomputable  $\Sigma_1^0$  structure cannot be computably isomorphic to a computable structure, but we have the following best possible result.

**Theorem 2.4.** *Let  $\mathcal{A}$  be a  $\Sigma_1^0$  equivalence structure. Let  $\mathcal{B}$  be a computable equivalence structure isomorphic to  $\mathcal{A}$  such that  $\mathcal{B}$  is computably categorical or even relatively  $\Delta_2^0$  categorical. Then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\Delta_2^0$  isomorphic.*

*Proof.* As a warm-up, suppose first that  $\mathcal{B}$  is computably categorical. It follows from Theorem 3.16 of [2] that  $\mathcal{B}$  is relatively computably categorical. Hence there is an isomorphism  $f$  from  $\mathcal{B}$  to  $\mathcal{A}$ , which is computable in  $\mathcal{A}$ . Since  $\mathcal{A}$  is  $\Sigma_1^0$ , it follows that  $f$  is  $\Delta_2^0$ .

Now, suppose that  $\mathcal{B}$  is relatively  $\Delta_2^0$  categorical. Then:

- (i)  $\mathcal{B}$  has finitely many infinite equivalence classes, or
- (ii)  $\mathcal{B}$  has bounded character.

First, consider the computable structure  $\mathcal{B}$ . In Case (i), it is immediate that both  $Inf^{\mathcal{B}}$  and  $Fin^{\mathcal{B}}$  are computable. In Case (ii), it follows from Lemma 2.4 of [2] that there is a computable structure  $\mathcal{B}'$  isomorphic to  $\mathcal{B}$  such



that  $Inf^{\mathcal{B}'}$  is computable. Thus, we may assume, without loss of generality, that  $Inf^{\mathcal{B}}$  and  $Fin^{\mathcal{B}}$  are computable.

Next, consider the  $\Sigma_1^0$  structure  $\mathcal{A}$ . In Case (i),  $Inf^{\mathcal{A}}$  is  $\Sigma_1^0$  since there is a finite set of representatives  $\{c_1, \dots, c_m\}$  for the infinite classes, so  $a \in Inf^{\mathcal{A}} \iff aE^{\mathcal{A}}c_1 \vee \dots \vee aE^{\mathcal{A}}c_m$ . In Case (ii),  $Inf^{\mathcal{A}}$  is also  $\Sigma_1^0$ . That is, if  $n$  is an upper bound for the size of a finite equivalence class, then  $a \in Inf^{\mathcal{A}} \iff card([a]) > n$ .

Thus, both  $Fin^{\mathcal{A}}$  and  $Inf^{\mathcal{A}}$ , and  $Fin^{\mathcal{B}}$  and  $Inf^{\mathcal{B}}$  are computable in  $\emptyset'$ . Moreover, it is easy to see that if  $x \in Fin^{\mathcal{A}}$ , then we can find the equivalence class  $[x]^{\mathcal{A}}$  computably in  $\emptyset'$ . That is, we simply search until we find an  $n$  such that  $\{y : yE^{\mathcal{A}}x \ \& \ y > n\}$  is empty, which we can decide from an  $\emptyset'$ -oracle. Then we know that  $[x]^{\mathcal{A}} = \{z : zE^{\mathcal{A}}x \ \& \ z \leq n\}$ , which can also be computed from an  $\emptyset'$ -oracle. Similarly, we can find the equivalence class  $[y]^{\mathcal{B}}$  computably in  $\emptyset'$  for any  $y \in Fin^{\mathcal{B}}$ . Then we can use a simple back-and-forth argument to define an isomorphism  $f : Fin^{\mathcal{A}} \rightarrow Fin^{\mathcal{B}}$  that is computable in  $\emptyset'$ . That is, computably in  $\emptyset'$ , we can obtain enumerations  $a_0 < a_1 < \dots$  of  $Fin^{\mathcal{A}}$ , and  $b_0 < b_1 < \dots$  of  $Fin^{\mathcal{B}}$ . We then define  $f$  in stages, and let  $f_s$  denote the finite function defined at the end of stage  $s$ .

*Stage 0.* Search for the least  $b_i$  such that  $card([a_0]^{\mathcal{A}}) = card([b_i]^{\mathcal{B}})$  and define  $f_0$  so that it maps  $[a_0]^{\mathcal{A}}$  onto  $[b_i]^{\mathcal{B}}$  in an increasing fashion. If  $i > 0$ , then we search for the least  $a_j$  such that  $card([a_j]^{\mathcal{A}}) = card([b_0]^{\mathcal{B}})$ , and then define  $f_0$  so that it maps  $[a_j]^{\mathcal{A}}$  onto  $[b_0]^{\mathcal{B}}$  in an increasing fashion.

*Stage  $s+1$ .* Assume we have defined  $f_s$  so that its domain and range are finite unions of equivalence classes in  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, and that  $\bigcup_{i=0}^s [a_i]^{\mathcal{A}}$  is contained in the domain of  $f_s$ , and  $\bigcup_{i=0}^s [b_i]^{\mathcal{B}}$  is contained in the range of  $f_s$ . Then, to extend  $f_s$  to  $f_{s+1}$ , we search for the least  $i$  such that  $a_i$  is not in the domain of  $f_s$ . We then search for the least  $b_k$  not in the range of  $f_s$  such that  $card([a_i]^{\mathcal{A}}) = card([b_k]^{\mathcal{B}})$ , and define  $f_{s+1}$  so that it maps  $[a_i]^{\mathcal{A}}$  onto  $[b_k]^{\mathcal{B}}$  in an increasing fashion. Next, we search for the least  $n$  such that  $b_n$  is not in the range of  $f_s$  and  $n \neq k$ . We then search for the least  $m$  such that  $a_m$  is not in the domain of  $f_s$ , and  $m \neq i$ , and  $card([a_m]^{\mathcal{A}}) = card([b_n]^{\mathcal{B}})$ . Then define  $f_{s+1}$  so that it maps  $[a_m]^{\mathcal{A}}$  onto  $[b_n]^{\mathcal{B}}$  in an increasing fashion.

It is easy to see that the construction is computable in  $\emptyset'$ , and that at each stage we can find the appropriate elements since we are assuming that  $Fin^{\mathcal{A}}$  and  $Fin^{\mathcal{B}}$  are isomorphic. Thus,  $f$  will be a  $\Delta_2^0$  function, which is an isomor-

phism from  $Fin^A$  to  $Fin^B$ .

Similarly, it is easy to construct a function  $g$  that is computable in  $\emptyset'$  and which is an isomorphism from  $Inf^A$  to  $Inf^B$ . Hence  $\mathcal{A}$  is  $\Delta_2^0$  isomorphic to  $\mathcal{B}$ .  $\square$

**Corollary 2.5.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be isomorphic  $\Sigma_1^0$  equivalence structures that satisfy one of the following conditions:*

- (i)  $\mathcal{A}$  has bounded character, or
- (ii)  $\mathcal{A}$  has only finitely many infinite equivalence classes.

*Then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\Delta_2^0$  isomorphic.*

Next, we briefly discuss the connection with bi-reducibility of equivalence structures as studied by Bernardi and Sorbi [1], Lachlan [10], and Gao and Gerdes [6].

We say that one equivalence relation  $R$  is *strongly reducible* to another equivalence relation  $S$  (written  $R \leq S$ ) if and only if there exists a computable function  $f$  such that for all  $x, y \in \omega$ ,

$$xRy \Leftrightarrow f(x)Sf(y).$$

If  $R \leq S$  and  $S \leq R$ , then  $R$  and  $S$  are *bi-reducible*. If  $(\omega, R)$  is computably isomorphic to  $(\omega, S)$ , then they are certainly bi-reducible. The converse does not hold. That is, Gerdes and Gao [6] proved that every computable equivalence relation is bi-reducible to one of the following types:

1. for some finite  $n$ , the equivalence relation  $x \equiv y \pmod n$ , which defines a computable equivalence structure with  $n$  infinite equivalence classes and without finite classes;
2. the equality relation, which defines a computable equivalence structure with infinitely many classes of size one, and no other classes.

Thus, the partial ordering  $(C, <)$  of the computable equivalence structures modulo strong reducibility, is isomorphic to  $\omega + 1$ . In fact, it is easy to see that two computable equivalence structures are bi-reducible if and only if they have the same number of equivalence classes. Thus, in particular, bi-reducible structures need not be isomorphic. For example, if  $\mathcal{A}$  consists of

infinitely many classes of size 1, and  $\mathcal{B}$  consists of infinitely many classes of size 2, then  $\mathcal{A}$  and  $\mathcal{B}$  are bi-reducible but not isomorphic. Furthermore, we have already seen that, even if two computable equivalence structures are isomorphic, they need not be computably isomorphic. For computable equivalence structures, the given effective notion of bi-reducibility is identical to the noneffective version.

A  $\Sigma_1^0$  equivalence relation (or ceer)  $S$  is said to be *universal* if  $R \leq S$  for any  $\Sigma_1^0$  equivalence relation  $R$ . Bernardi and Sorbi [1] showed that universal ceers exist.

### 3 $\Pi_1^0$ equivalence structures

In this section, we show that even simple  $\Pi_1^0$  equivalence structures do not have to be  $\Delta_2^0$  isomorphic to computable structures. Note that if  $\mathcal{B}$  is a  $\Pi_1^0$  equivalence structure, and  $\mathcal{A}$  is an isomorphic computable structure that is computably categorical, then, since  $\mathcal{A}$  is also relatively computably categorical,  $\mathcal{A}$  and  $\mathcal{B}$  are  $\Delta_2^0$  isomorphic. Thus, we have the following result.

**Theorem 3.1.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be isomorphic  $\Pi_1^0$  equivalence structures such that  $\mathcal{A}$  satisfies one of the following conditions:*

- (i)  $\mathcal{A}$  has only finitely many finite equivalence classes, or
- (ii)  $\mathcal{A}$  has finitely many infinite equivalence classes and bounded character, and there is at most one finite  $k$  such that  $\mathcal{A}$  has infinitely many equivalence classes of size  $k$ .

*Then  $\mathcal{A}$  and  $\mathcal{B}$  are  $\Delta_2^0$  isomorphic.*

However, our next two results show that Theorem 3.1 does not extend to all equivalence structures that are isomorphic to computable, relatively  $\Delta_2^0$  categorical structures.

**Theorem 3.2.** *Suppose that  $\mathcal{B}$  is a computable equivalence structure with a bounded character for which there exist  $k_1 < k_2 \leq \omega$  such that  $\mathcal{B}$  has infinitely many equivalence classes of size  $k_1$  and infinitely many equivalence classes of size  $k_2$ . Then there exists an isomorphic  $\Pi_1^0$  structure  $\mathcal{A}$  that is not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$  and, moreover,  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.*

*Proof.* We first suppose that  $\mathcal{B}$  has no other equivalence classes. It suffices to build a  $\Pi_1^0$  equivalence structure  $\mathcal{A}$  such that  $\{a : \text{card}([a]^{\mathcal{A}}) = k_2\}$  is not a  $\Delta_2^0$  set. That is, it follows from Lemma 2.1 that for any  $\Sigma_1^0$  structure, the set of elements that belong to an equivalence class of (finite) size  $k$  is a  $\Delta_2^0$  set. So if  $\mathcal{A}$  were  $\Delta_2^0$  isomorphic to a  $\Sigma_1^0$  structure, then  $\mathcal{A}$  would also have this property.

For simplicity of the construction, we let  $\mathcal{A}$  have universe  $\omega \setminus \{0\}$ . Let  $\phi : \omega^3 \rightarrow \{0, 1\}$  be a computable function such that for every  $\Delta_2^0$  set  $D$ , there is some  $e$  for which for all  $n \in \omega$ , the limit  $\delta_e(n) =_{\text{def}} \lim_{t \rightarrow \infty} \phi(t, e, n)$  exists and  $\delta_e$  is the characteristic function of  $D$ . If  $\delta_e(n)$  is defined for all  $n$ , we let  $D_e = \{n : \delta_e(n) = 1\}$ . The function  $\phi$  exists by the Limit Lemma (see [14]). We will construct the equivalence relation  $E = E^{\mathcal{A}}$  so that for each  $e$ , if  $D_e$  exists, then  $\text{card}([2^e]^{\mathcal{A}}) = k_2$  if and only if  $2^e \notin D_e$ .

We construct  $E^{\mathcal{A}}$  in stages. That is, at each stage  $s$ , we shall define a computable equivalence relation  $E_s$  so that  $E_{s+1} \subseteq E_s$  for all  $s$ , and  $E^{\mathcal{A}} = \bigcap_s E_s$ . Let  $[a]_s$  denote the equivalence class of  $a$  in  $E_s$ . At each stage  $s$ , we shall also define an *intended* equivalence class  $I_s[2^e]$ , either of size  $k_1$  or of size  $k_2$ . We will ensure that for each  $e$ , there is some stage  $s_e$  such that for all  $s \geq s_e$ , we have  $[2^e] = I_s[2^e]$ . Furthermore, for all  $s$ ,  $[2^e]_{s+1} \subseteq [2^e]_s$ , and  $\bigcap_s [2^e]_s = [2^e]$ . We shall also define a number of *permanent* classes  $[a]$  of size  $k_1$  at each stage  $s$ .

### Construction

*Stage 0.* We start with the equivalence classes  $\{2^e(2k+1) : k \in \omega\}$  for  $e \geq 0$ . For each  $e \geq 0$ , we let  $I_0[2^e] = \{2^e, 3 \cdot 2^e, 5 \cdot 2^e, \dots, (2k_1 - 1) \cdot 2^e\}$ .

*Stage  $s+1$ .* At the end of stage  $s$ , assume that for each  $e$ , we have defined the intended equivalence class  $I_s[2^e]$  so that  $I_s[2^e]$  is an initial subset of  $[2^e]_s$  with cardinality either  $k_1$  or  $k_2$ . Moreover, assume that if  $\phi(s, e, 2^e) = 1$ , then  $I_s[2^e]$  has cardinality  $k_1$ , and if  $\phi(s, e, 2^e) = 0$ , then  $I_s[2^e]$  has cardinality  $k_2$ . For each  $e$ , we say that the element  $2^e$  *requires attention* at stage  $s+1$  if  $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$ . We can assume this occurs for exactly one  $e$ . Let  $[2^e]_s = \{2^e, a_1, a_2, \dots\}$ .

If  $2^e$  requires attention at stage  $s+1$ , we take the following action according

to whether  $I_s[2^e]$  has cardinality  $k_1$  or  $k_2$ .

*Case (i).* If  $\text{card}(I_s[2^e]) = k_2$ , then let  $I_{s+1}[2^e] = \{2^e, a_1, \dots, a_{k_1-1}\}$ , let  $[2^e]_{s+1} = \{2^e, a_1, \dots, a_{k_1-1}, a_{2k_1}, a_{2k_1+1}, \dots\}$ , and create a permanent equivalence class  $\{a_{k_1}, a_{k_1+1}, \dots, a_{2k_1-1}\}$  of size  $k_1$ .

*Case (ii).* If  $\text{card}(I_s[2^e]) = k_1$ , then do the following. First suppose that  $k_2$  is finite. Then we let  $I_{s+1}[2^e] = \{2^e, a_1, \dots, a_{k_2-1}\}$ , let  $[2^e]_{s+1} = \{2^e, a_1, \dots, a_{k_2-1}, a_{k_2+k_1}, a_{k_2+k_1+1}, \dots\}$ , and create a permanent equivalence class  $\{a_{k_2}, a_{k_2+1}, \dots, a_{k_2+k_1-1}\}$  of size  $k_1$ . If  $k_2 = \omega$ , then we simply let  $I_{s+1}[2^e] = [2^e]_{s+1} = [2^e]_s$ .

If  $2^e$  does not require attention, then, again, there are two cases. If  $k_2 = \omega$ , and  $I_s[2^e] = [2^e]_s$  is infinite, then we let  $I_{s+1}[2^e] = [2^e]_{s+1} = [2^e]_s$ . If  $\text{card}([I_s[2^e]]) = k_m$  is finite, then we let  $I_{s+1}[2^e] = \{2^e, a_1, \dots, a_{k_m-1}\}$ , let  $[2^e]_{s+1} = \{2^e, a_1, \dots, a_{k_m-1}, a_{k_m+k_1}, a_{k_m+k_1+1}, \dots\}$ , and create a permanent equivalence class  $\{a_{k_m}, a_{k_m+1}, \dots, a_{k_m+k_1-1}\}$  of size  $k_1$ .

Clearly, the equivalence relation  $E_s$  is uniformly computable, and we have  $E_{s+1} \subseteq E_s$  for every  $s$ . Thus,  $E = \bigcap_s E_s$  is a  $\Pi_1^0$  equivalence relation.

First, we show that every equivalence class in  $E$  has either  $k_1$  or  $k_2$  elements. The elements which are (ever) removed from  $[2^e]_s$  form permanent equivalence classes of size  $k_1$ . Thus, we only need to check the classes  $[2^e]_s$  for each  $e$ . By our construction,  $[2^e]_s$  is infinite for every  $s$ . There are two cases.

If  $\lim_{s \rightarrow \infty} \phi(s, e, 2^e)$  exists, then there is some stage  $s$  such that  $\phi(s, e, 2^e) = \phi(t, e, 2^e)$  for all  $t \geq s$ . Let  $[2^e]_s = I_s[2^e] \cup \{a_1 < a_2 < \dots\}$ . If  $I_s[2^e]$  has cardinality  $k_2$  and  $k_2 = \omega$ , then  $[2^e]_t = [2^e]_{t+1}$  for all  $t \geq s$  so that  $[2^e]$  is infinite. If  $I_s[2^e]$  has cardinality  $k_1$  or  $k_2$ , and  $k_2$  is finite, then  $[2^e]_{s+n} = I_s[2^e] \cup \{a_{k_n+1} < a_{k_n+2} < \dots\}$ , so  $[2^e] = I_s[2^e]$  which, by construction, has cardinality either  $k_1$  or  $k_2$ .

Next, suppose that there are infinitely many  $s$  such that  $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$ . Let  $s_0 < s_1 < \dots$  be the stages  $s+1$  such that  $\phi(s, e, 2^e) = 0$  and  $\phi(s+1, e, 2^e) = 1$  so that  $\text{card}(I_s[2^e]) = k_2$  and  $\text{card}(I_{s+1}[2^e]) = k_1$ . At each such stage  $s_n$ , we will remove the second  $k_1$  elements from  $[2^e]_{s_n}$  and make it a permanent equivalence class of size  $k_1$ . Thus, it follows that  $\bigcap_n [2^e]_{s_n} = \{2^e, 3 \cdot 2^e, \dots, (2k_1 - 1) \cdot 2^e\}$ , so that  $\text{card}([2^e]) = k_1$ .

Next, we check that  $A = \{n : \text{card}([2^n]) = k_2\}$  is not a  $\Delta_2^0$  set. If it were, then, for some  $e$ ,  $\chi_A(n) = \lim_{s \rightarrow \infty} \phi(s, e, n)$  exists for all  $n$ . Let  $t_0$  be large enough so that  $\chi_A(2^e) = \phi(t_0, e, 2^e)$  for all  $s \geq t_0$ . By the construction,  $\text{card}(I[2^e]_{t_0}) = k_2$  if and only if  $\phi(t_0, e, 2^e) = 0$ . By the definition of  $t_0$ ,  $\chi_A(2^e) = \phi(s, e, 2^e)$  and the element  $2^e$  never requires attention after stage  $t_0$ , so that  $\text{card}([2^e]) = \text{card}(I_{t_0}[2^e])$ . Thus,  $\text{card}([2^e]) = k_2$  if and only if  $\lim_{s \rightarrow \infty} \phi(s, e, 2^e) \neq \chi_A(2^e)$ .

Finally, suppose that  $\mathcal{B}$  is a structure that has bounded character and has infinitely many classes of size  $k_1$  and of size  $k_2$ , but also has other equivalence classes. Recall that the character  $\chi(\mathcal{B})$  is a  $\Sigma_2^0$  set. Now we may remove  $\{(k_i, n) : n \in \omega \ \& \ i \in \{1, 2\} \ \& \ k_i \text{ is finite}\}$  from  $\chi(\mathcal{B})$  and still have a  $\Sigma_2^0$  character  $K$ . We now have two cases.

First, suppose that  $k_1$  is finite and  $k_2 = \omega$ . Then  $K$  is a bounded character, and hence we can construct a computable equivalence structure  $\mathcal{C}$  with character  $K$ . Let  $\mathcal{A}$  be the  $\Pi_1^0$  structure that has infinitely many equivalence classes of size  $k_1$  and  $k_2$ , but no other equivalence classes, and which is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure. Then the disjoint union  $\mathcal{A} \oplus \mathcal{C}$  will be isomorphic to  $\mathcal{B}$ . We may assume that  $k_1$  is the largest size of an equivalence class that is finite and such that there are infinitely many equivalence classes of that size in  $\mathcal{B}$ . Thus, in any  $\Sigma_1^0$  structure  $\mathcal{B}^*$  that is isomorphic to  $\mathcal{B}$ , the set  $S$  of all elements that belong to finite equivalence classes of sizes bigger than  $k_1$  is finite. The set  $D$  of elements  $d$  in  $\mathcal{B}^*$  such that  $\text{card}([d]^{\mathcal{B}^*}) \geq k_1$  is clearly a c.e. set, so that  $D - S$  is a c.e. set consisting of all elements of  $\mathcal{B}^*$  the equivalence classes of which are of sizes  $k_1$  or  $k_2$ . Then, clearly,  $\mathcal{B}^* \upharpoonright (D - S)$  is computably isomorphic to a  $\Sigma_1^0$  structure. But then  $\mathcal{A} \oplus \mathcal{C}$  cannot be  $\Delta_2^0$  isomorphic to any such  $\Sigma_1^0$  structure  $\mathcal{B}^*$ , since any isomorphism would have to map  $\mathcal{A}$  onto  $\mathcal{B}^* \upharpoonright (D - S)$ .

Next, suppose that  $k_1$  and  $k_2$  are finite and  $\mathcal{B}$  has  $r < \omega$  infinite equivalence classes. It is easy to modify the construction to ensure that  $\mathcal{A}$  has  $r$  infinite equivalence classes, in addition to infinitely many equivalence classes of size  $k_1$  and infinitely many equivalence classes of size  $k_2$ , so that  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure. Since  $K$  is a bounded character, it is easy to construct a computable structure  $\mathcal{C}$  with character  $K$  and no infinite equivalence classes. Thus,  $\mathcal{A} \oplus \mathcal{C}$  will be a  $\Pi_1^0$  structure which is isomorphic to  $\mathcal{B}$ . We may assume that  $k_1$  and  $k_2$  are the two largest sizes of equivalence classes that are finite and such that there are infinitely many equivalence classes of those sizes in  $\mathcal{B}$ . Thus, in any  $\Sigma_1^0$  structure  $\mathcal{B}^*$  isomorphic to  $\mathcal{B}$ ,

there are only finitely many elements  $S$  that belong to finite equivalence classes the sizes of which are bigger than  $k_2$ . The set  $D$  of elements  $d$  in  $\mathcal{B}^*$  such that  $\text{card}([d]^{\mathcal{B}^*}) \geq k_1$  is clearly a c.e. set, so that  $D - S$  is a c.e. set consisting of all elements of  $\mathcal{B}^*$  the equivalence classes of which are of sizes  $k_1$  or  $k_2$ , together with  $r$  infinite equivalence classes in  $\mathcal{B}^*$ . Clearly,  $\mathcal{B}^* \upharpoonright D$  is computably isomorphic to a  $\Sigma_1^0$  structure. However, then  $\mathcal{A} \oplus \mathcal{C}$  cannot be  $\Delta_2^0$  isomorphic to any such  $\Sigma_1^0$  structure  $\mathcal{B}^*$ , since such an isomorphism would have to map  $\mathcal{A}$  onto  $\mathcal{B}^* \upharpoonright (D - S)$ .  $\square$

**Corollary 3.3.** *If  $\mathcal{B}$  is a computable equivalence structure with bounded character which is not computably categorical, then there exists an isomorphic  $\Pi_1^0$  structure  $\mathcal{A}$  that is not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$ , and, moreover,  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.*

*Proof.* Let the computable equivalence structure  $\mathcal{B}$  have bounded character such that  $\mathcal{B}$  is not computably categorical. Suppose first that  $\mathcal{B}$  has only finitely many infinite equivalence classes. It was proved in [2] that if  $\mathcal{B}$  has finitely many infinite equivalence classes, and at most one finite  $k$  such that there are infinitely many equivalence classes of size  $k$ , then  $\mathcal{B}$  is computably categorical (see Corollary 3.3 and Theorem 3.16 of [2]). Hence there exist finite  $k_1 < k_2$  such that  $\mathcal{B}$  has infinitely many equivalence classes of size  $k_1$  and infinitely many equivalence classes of size  $k_2$ . Next, suppose that  $\mathcal{B}$  has infinitely many infinite equivalence classes. If  $\mathcal{B}$  has a finite character, then  $\mathcal{B}$  is computably categorical. Thus,  $\chi(\mathcal{B})$  is both bounded and infinite, so that there must exist a finite  $k$  such that  $\mathcal{B}$  has infinitely many equivalence classes of size  $k$ , as well as infinitely many infinite equivalence classes. Thus, Theorem 3.2 applies in either case.  $\square$

Next, we shall consider structures with unbounded characters and with only finitely many infinite equivalence classes.

**Theorem 3.4.** *Suppose that  $\mathcal{B}$  is a computable equivalence structure that has an unbounded character and only finitely many infinite equivalence classes (and is therefore relatively  $\Delta_2^0$  categorical). Then there exists a  $\Pi_1^0$  structure  $\mathcal{A}$  that is isomorphic to  $\mathcal{B}$ , but not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$ , and, moreover,  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.*

*Proof.* Let  $\phi : \omega^3 \rightarrow \{0, 1\}$  be the computable function defined in Theorem 3.2, for which for every  $\Delta_2^0$  set  $D$ , there is some  $e$  such that for all  $n \in \omega$ ,

the limit  $\delta_e(n) =_{def} \lim_{t \rightarrow \infty} \phi(t, e, n)$  exists and  $\delta_e$  is the characteristic function of  $D$ .

By Lemma 2.6 of [2], there is a computable  $s_1$ -function  $f$  such that for each  $i$ , there exists finite  $\lim_s f(i, s) = m_i$  and  $\mathcal{B}$  has an equivalence class of size  $m_i$ . Note that  $M = \{m_i : i \in \omega\}$  is a  $\Delta_2^0$  set. Thus, by Lemma 2.8 of [2], there exists a computable equivalence structure which consists of exactly one equivalence class of size  $m_i$  for each  $i$ .

First, assume that  $\mathcal{B}$  has no other equivalence classes, i.e.,  $\mathcal{B}$  consists of exactly one equivalence class of size  $m_i$  for each  $i$ . It suffices to build an isomorphic  $\Pi_1^0$  equivalence structure  $\mathcal{A}$  such that  $\{a : \text{card}([a]^{\mathcal{A}}) = m_{2i} \text{ for some } i\}$  is not a  $\Delta_2^0$  set. That is, we observe that the functions  $f_E$  and  $f_O$ , defined by  $f_E(i, s) = f(2i, s)$  and  $f_O(i, s) = f(2i + 1, s)$  are also  $s_1$ -functions so it follows by Lemma 2.7 of [2] that the sets  $M_0 = \{m_{2i} : i \in \omega\}$  and  $M_1 = \{m_{2i+1} : i \in \omega\}$  are both  $\Delta_2^0$ , and hence there exist computable structures  $\mathcal{B}_0$  and  $\mathcal{B}_1$ , which consist of precisely one class of size  $m_{2i}$  for  $\mathcal{B}_0$  and of size  $m_{2i+1}$  for  $\mathcal{B}_1$ . Hence in the structure  $\mathcal{B}_0 \oplus \mathcal{B}_1$ , the set  $\{x : \text{card}([x]) \in M_0\}$  is computable. Since we have assumed that  $\mathcal{B}$  is relatively  $\Delta_2^0$  categorical, it follows from Theorem 2.4 that for any  $\Sigma_1^0$  equivalence structure with character  $\{(m, 1) : m \in M_0 \cup M_1\}$ , the set  $\{x : \text{card}([x]) \in M_0\}$  is  $\Delta_2^0$ .

The construction of  $E^{\mathcal{A}}$  is by stages, as usual. That is, at each stage  $s$  we shall define a computable equivalence relation  $E_s$  so that  $E_{s+1} \subseteq E_s$  for all  $s$ , and  $E^{\mathcal{A}} = \bigcap_s E_s$ . As in Theorem 3.2, we let  $[a]_s$  denote the equivalence class of  $a$  in  $E_s$ , and we let  $I_s[a]$  denote the *intended* equivalence class of  $a$  at stage  $s$ . At any given stage  $s$ , the intended classes have exactly the sizes  $f(i, s)$  for  $i \in \omega$ , and  $I_s[2^e]$  will be either of size  $f(2^e, s)$  or of size  $f(1 + 2^e(2j + 1), s)$  for some  $j$ . The construction will ensure that for each  $i$ , there exists  $t$  such that  $f(i, t) = m_i$  and the class of size  $m_i$  has become permanent. For each  $e$ , the element  $2^e$  belongs to the class of size  $f(2^e, s)$  at stage  $s$  if and only if  $\phi(s, e, 2^e) = 0$ . It follows that for each  $e$ , if  $D_e$  exists, then  $\text{card}([2^e]^{\mathcal{A}}) \in M_0$  if and only  $2^e \notin D_e$ .

### Construction

*Stage 0.* The relation  $E_0$  consists of the equivalence classes  $\{2^e(2k + 1) : k \in \omega\}$  for  $e \geq 0$ . For each  $e \geq 0$ , we let  $I_0[2^e] = \{2^e, 3 \cdot 2^e, 5 \cdot 2^e, \dots, (2f(2^e, 0) - 1) \cdot 2^e\}$ . We then partition the remaining elements of  $\{2^e(2k + 1) : k \in \omega\}$  consecutively into the intended classes  $I_0[2^e(2k + 1)]$  of sizes  $f(2^e(2k + 1), 0)$



for  $k > 0$ .

*Stage  $s + 1$ .* There are three tasks to accomplish at stage  $s + 1$ . We will perform them sequentially.

First, we suppose that  $f(i, s + 1) > f(i, s)$  for some  $i$ . In fact, we can construct  $f$  such that this occurs for exactly one  $i$  and that, in fact,  $f(i, s + 1) = f(i, s) + 1$ . Now the class  $I_s[a] = C_0$  intended to have size  $m_i$  lies in some infinite class  $[a]_s$ , where it is followed by intended classes  $C_1, C_2, \dots$ . Here we assume that for any pair  $i, j$  with  $0 \leq i < j$ , the elements of  $C_i$  are all smaller than the elements of  $C_j$ . The required action is to take for each  $i \geq 0$ , the first element of  $C_{i+1}$  and move it to  $C_i$ . This will make  $\text{card}(I_{s+1}[a]) = f(i, s + 1)$ , while leaving the other intended classes with the same cardinalities.

Second, we may have  $\phi(s + 1, e, 2^e) \neq \phi(s, e, 2^e)$ . Again, we assume this occurs for exactly one  $e$ . Here the class  $I_s[2^e]$  is an initial subset of the infinite class  $[2^e]_s$ , beginning with  $2^e$ , and is followed by intended classes  $C_1, C_2, \dots$  having cardinalities  $c_1, c_2, \dots$ , respectively. Let  $I_s[2^e] \cup C_1 \cup C_2 \cup \dots = \{a_1 < a_2 < \dots\}$ . Suppose that the previous requirements have changed the intended size from  $f(2^e, r)$  or from  $f(1 + 2^e(2i + 1), r)$  for each  $i < n$ . The required action now is to change the cardinality of  $I_s[2^e]$  either from  $f(2^e, s)$  to  $f(1 + 2^e(2n + 1), s)$  or *vice versa*. Suppose that intended cardinality  $I_{s+1}[2^e]$  is now going to be  $c_0 = f(j, s)$  where  $j \in \{2^e, 1 + 2^e(2n + 1)\}$ . Let  $d_i = c_0 + \dots + c_i$  for each  $i$ . Then we let  $I_{s+1}[2^e] = \{a_1, \dots, a_{c_0}\}$  and, for each  $i$ , we convert  $C_{i+1}$  into  $\{a_{d_i+1}, \dots, a_{d_i+c_{i+1}}\}$ , so that the classes  $C_1, C_2, \dots$  maintain their cardinality. Next, we declare that  $I_{s+1}[2^e]$  is intended for  $m_j$ . Observe once again that for any  $i < j$ , the elements of  $C_i$  are all smaller than the elements of  $C_j$ . Finally, suppose that  $I[2^e]$  was previously intended to have size  $m_k$  and now has size  $m_j$ . Then we have to work on the class  $C$  which was previously intended to have size  $m_j$  and change it over to size  $f(k, s + 1)$ . The intended class  $C$  lies in the middle of some infinite class and we proceed as we did for the class  $[2^e]$  above. Again, we observe that the elements of the class  $C$  are all larger than those of the class  $I_s[2^e]$ .

Third, we have to ensure that the actual classes  $[a]_s$  will converge to finite classes with the intended cardinality. We accomplish this as follows. For each class  $C = [a]_s$ , let  $C$  be the union of intended classes  $C_1, C_2, \dots$ . We partition  $C$  into new classes  $D_i = \cup_k C_{(2k+1) \cdot 2^i}$ . In this way we ensure that any two intended classes will eventually be separated.

This completes the construction.

We claim that for each  $a$ , the class  $I_s[a]$  eventually converges to the class  $[a]$  and is associated with some intended cardinality  $m_i$ .

First, consider the class  $[2^e]$ . Choose  $t$  such that  $f(2^e, t) = m_{2^e}$ . Suppose first that  $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$  infinitely often. Then infinitely often we have  $I_s[2^e] = C$  as the first  $m_{2^e}$  elements of  $[2^e]_s$ , and at the other type of stages we have  $C \subseteq I_s[2^e]$ , so that  $C \subseteq [2^e]$ . By the third type of action, all other elements are eventually not equivalent to  $2^e$ , so that  $[2^e] = C$  and  $[2^e]$  has cardinality  $m_{2^e}$ . Next, suppose that  $\phi(s+1, e, 2^e) \neq \phi(s, e, 2^e)$  only finitely many times. Then we may assume that after stage  $t$ ,  $I[2^e]$  is never changed by the second type of action.

There are two possibilities. If  $I_t[2^e]$  is intended to have size  $f(2^e, t) = m_{2^e}$ , then it cannot be affected by the first type of action (by the assumption above), and hence it cannot be affected by the third type of action, since it does not change any of the intended classes. If  $I_t[2^e]$  is intended to have size  $f(1+2^e(2j+1), t)$  for some  $j$ , then the intended size of this class will not be changed again by any action of the first type. Hence, once  $f(1+2^e(2j+1), s)$  stops changing, it will have a fixed size. Since  $I_s[2^e]$  is always an initial segment of  $[2^e]_s$  it is never affected by any other type of action. Thus, it will stabilize to a class of size  $m_k$ , where  $k = 2^e(2j+1)$ . Finally, actions of the third type will eventually remove all other elements from  $[2^e]$ .

Now, consider elements  $a$  which do not end up in  $[2^e]$  for any  $e$ . It follows from the construction (by the third type of action) that  $I_s[a]$  is eventually an initial segment of  $[a]_s$ . Take  $t$  large enough so that:

1.  $a \notin [2^e]_s$  for any  $e$  and any  $s > t$ ,
2.  $I_s[a]$  is an initial segment of  $[a]_s$  for any  $s > t$ ,
3.  $f(k, t) = m_k$ , where  $I_t[a]$  is intended to have size  $m_k$ .

Then for any  $s > t$ ,  $I[a]_s$  will be an initial segment of  $C = [a]_s$  of size  $m_k$ , and hence  $C \subseteq [a]$ . By the third type of action, no other elements will belong to  $[a]$ , and thus  $[a] = C$ .

Finally, suppose that  $D_e(x) = \lim_s \phi(s, e, x)$  is a  $\Delta_2^0$  set. Then by the construction,  $I[2^e]$  will stabilize once  $\phi(s, e, x)$  has stabilized and we will have  $\text{card}([2^e]) \in M_0 \iff 2^e \notin D_e$ .

Thus, in our  $\Pi_1^0$  structure  $\mathcal{A}$ ,  $\{x : \text{card}([x]) \in M_0\}$  is not a  $\Delta_2^0$  set and therefore  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.

Now, suppose that  $\mathcal{B}$  does not consist only of one equivalence class of size  $m_i$  for each  $i \geq \omega$ . Moreover, suppose that  $\mathcal{B}$  has  $r$  infinite equivalence classes for some  $r < \omega$ . In this case, we will take the corresponding  $s_1$ -function  $f$  and let  $N_0 = \{m_{3i} : i \in \omega\}$ ,  $N_1 = \{m_{3i+1} : i \in \omega\}$ , and  $N_2 = \{m_{3i+2} : i \in \omega\}$ . Then  $N_0$ ,  $N_1$ , and  $N_2$  are  $\Delta_2^0$  sets. The sets  $K_0 = \{(k, n) \in K : k \in N_0\}$ ,  $K_1 = \{(k, n) \in K : k \in N_1\}$ , and  $K_2 = \{(k, n) \in K : k \notin N_0 \cup N_1\}$ , are  $\Sigma_2^0$  characters. Each of these sets has an  $s_1$ -function since  $g_0(i) = f(3i)$  is an  $s_1$ -function for  $K_0$ ,  $g_1(i) = f(3i + 1)$  is an  $s_1$ -function for  $K_1$ , and  $g_2(i) = f(3i + 2)$  is an  $s_1$ -function for  $K_2$ . By Lemma 2.8 of [2], there exist computable equivalence structure  $\mathcal{R}$ ,  $\mathcal{S}$ , and  $\mathcal{T}$  such that:

1.  $\mathcal{R}$  has character  $K_0$  and no infinite equivalence classes,
2.  $\mathcal{S}$  has character  $K_1$  and no infinite equivalence classes, and
3.  $\mathcal{T}$  has character  $K_2$  and  $r$  infinite equivalence classes.

Thus,  $\mathcal{B}$  is isomorphic to  $\mathcal{B}^* = \mathcal{R} \oplus (\mathcal{S} \oplus \mathcal{T})$ . Clearly, in  $\mathcal{B}^*$ ,  $\{x : \text{card}([x]^{\mathcal{B}^*}) \in N_0\}$  is a computable set. Thus, by the relative  $\Delta_2^0$  categoricity of  $\mathcal{B}$ , it must be the case that in any  $\Sigma_1^0$  structure  $\mathcal{D}$  isomorphic to  $\mathcal{B}$ , the set  $\{x : \text{card}([x]^{\mathcal{D}}) \in N_0\}$  is  $\Delta_2^0$ .

However, we can clearly modify the construction so that we obtain a  $\Pi_1^0$  equivalence structure  $\mathcal{A}$  such that  $\mathcal{A}$  has exactly one equivalence class of size  $m_{3i}$  and one equivalence class of size  $m_{3i+1}$  for all  $i \in \omega$ , and  $\{x \in \mathcal{A} : \text{card}([x]^{\mathcal{A}}) \in N_0\}$  is not a  $\Delta_2^0$  set. Next, observe that, since  $\mathcal{B}$  has a  $\Sigma_2^0$  character  $K$ , the set  $K^* = \{(k, n) \in K : k \notin N_0 \cup N_1\} \cup \{(k, n) : k \in N_0 \cup N_1 \ \& \ (k, n + 1) \in K\}$  is also a  $\Sigma_2^0$  character, which has an  $s_1$ -function witnessed by  $g_2$ . Thus, by Lemma 2.8 of [2], there is a computable structure  $\mathcal{C}$  such that  $\mathcal{C}$  has character  $K^*$  and  $r$  infinite equivalence classes. Hence  $\mathcal{A} \oplus \mathcal{C}$  is a  $\Pi_1^0$  structure that is isomorphic to  $\mathcal{B}$ . Now, if  $\mathcal{A} \oplus \mathcal{C}$  were  $\Delta_2^0$  isomorphic to a  $\Sigma_1^0$  structure  $\mathcal{B}^*$ , which is isomorphic to  $\mathcal{B}$ , then, since in  $\mathcal{B}^*$ , the set  $\{x : \text{card}([x]^{\mathcal{B}^*}) \in N_0\}$  is a  $\Delta_2^0$  set, it would follow that  $\{x : \text{card}([x]^{\mathcal{A} \oplus \mathcal{C}}) \in N_0\}$  is a  $\Delta_2^0$  set. However, if that were the case, then  $\{2x : \text{card}([2x]^{\mathcal{A} \oplus \mathcal{C}}) \in N_0\}$  would also be a  $\Delta_2^0$  set, which it is not by the construction of  $\mathcal{A}$ . Thus,  $\mathcal{A} \oplus \mathcal{C}$  cannot be  $\Delta_2^0$  isomorphic to a  $\Sigma_1^0$  structure.  $\square$

By Corollary 4.8 of [2], a computable structure  $\mathcal{A}$  is relatively  $\Delta_2^0$  categorical if and only if  $\mathcal{A}$  has finitely many equivalence classes or  $\mathcal{A}$  has a bounded character. Thus we can combine the previous two theorems to conclude the following.

**Theorem 3.5.** *Suppose that  $\mathcal{B}$  is a computable equivalence structure that is relatively  $\Delta_2^0$  categorical, but not computably categorical. Then there exists an isomorphic  $\Pi_1^0$  structure  $\mathcal{A}$  that is not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$  and, moreover,  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.*

*Proof.* If  $\mathcal{B}$  has bounded character, this follows from Corollary 3.3. If  $\mathcal{B}$  has unbounded character, then this follows from Theorem 3.4.  $\square$

We note that Theorem 3.5 does not cover all  $\Delta_2^0$  categorical computable equivalence structures since Kach and Turetsky [7] showed that there exists a computable  $\Delta_2^0$  categorical equivalence structure  $\mathcal{B}$  which has infinitely many infinite equivalence classes and an unbounded character, but has no computable  $s_1$ -function, and has only finitely many equivalence classes of size  $k$  for any finite  $k$ . The next result will cover this case. In such a case, we shall show that there exists a  $\Pi_1^0$  structure  $\mathcal{A}$  which is isomorphic to  $\mathcal{B}$  such that  $\text{Inf}^{\mathcal{A}}$  is a  $\Pi_2^0$  complete set.

**Theorem 3.6.** *Let  $\mathcal{B}$  be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character such that for each finite  $k$ , there are only finitely many equivalence classes of size  $k$ . Then there is a  $\Pi_1^0$  structure  $\mathcal{A}$  which is isomorphic to  $\mathcal{B}$  such that  $\text{Inf}^{\mathcal{A}}$  is  $\Pi_2^0$  complete. Furthermore, if  $\mathcal{B}$  is  $\Delta_2^0$  categorical, then  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any computable structure.*

*Proof.* We fix a computable bijection  $t : \omega^3 \rightarrow \omega$ . For any subset  $S \subseteq \omega^3$ , the function  $t$  induces a total ordering of type  $\omega$  on  $S$  by defining for  $(a_1, b_1, c_1), (a_2, b_2, c_2) \in S$ ,  $(a_1, b_1, c_1) < (a_2, b_2, c_2)$  if and only if  $t(((a_1, b_1, c_1))) < t(((a_2, b_2, c_2)))$ .

Let  $\mathcal{B}$  have character  $K$ . Since  $K$  is  $\Sigma_2^0$ , there is a computable relation  $Q$  such that

$$(k, m) \in K \iff (\exists w)(\forall s)Q(s, w, k, m).$$

We may assume, without loss of generality, that for each  $(k, m)$ , there is at most one  $w$  such that  $(\forall s)Q(s, w, k, m)$ . Let  $C = \{(w, k, m) : (\forall s)Q(s, w, k, m)\}$ . Then there is a one-to-one correspondence between  $C$  and  $K$  given by mapping  $(w, k, m)$  to  $(k, m)$ .

For each  $s$ , we have the uniformly computable set

$$C_s = \{(w, k, m) : (\forall t < s) C(t, w, k, m)\},$$

which can be enumerated, relative to the order induced by  $t$  described above, as  $\{(w, k, m)_{i,s} : i \in \omega\}$ . For each  $s$ ,  $C_{s+1} \subseteq C_s$  and  $\bigcap_s C_s = C$ . For each  $k$  and  $m$ , we know that  $(k, m+1) \in K \implies (k, m) \in K$ , so that we may assume that if  $(w, k, m) \notin C_s$ , then for all  $v$  and all  $n > m$ ,  $(v, k, n) \notin C_s$ .

We will construct the  $\Pi_1^0$  equivalence relation  $E$  as the intersection  $\bigcap_s E_s$  of uniformly computable equivalence relations defined at stage  $s$ . At stage  $s$ , we will have for each  $(w, k, m) \in C_s$ , some equivalence class that is intended to have cardinality  $k$ . For example, if  $I[2e]$  is the intended equivalence class associated with  $(w, k, m)$  as stage  $s$ , then just as we did in the proof of Theorems 3.2 and 3.4, we will attempt to ensure that  $[2e]$  has the correct cardinality by gradually removing all but the first  $k$  elements in  $[2e]_s$ . That is, if at stage  $t > s$ ,  $[2e]_t = \{a_0 < a_1 < \dots\}$ , then we remove all elements of the form  $a_{k+2i}$  for  $i \geq 0$  from  $[2e]_t$  and have these elements form a new permanent infinite equivalence class.

Now, let  $P$  be a complete  $\Pi_2^0$  set such that for some computable relation  $R$  we have for all  $n$ ,  $(n \in P \iff \{x : R(n, x)\}$  is infinite). We may assume, without loss of generality, that for each  $x$ , there is exactly one  $n$  such that  $R(n, x)$  and that  $n \leq x$ .

At each stage  $s$  of the construction, we will define an ordering  $n_{0,s}, n_{1,s}, \dots$  of  $\omega$  of type  $\omega$  so that the intended equivalence class  $I_s[2n_{i,s}]$  is associated with  $(w, k, m)_{i,s}$ , which means that at stage  $s$ , we intend  $[2n_{i,s}]$  to have cardinality  $k$ .

The construction will ensure that for each  $i \notin P$ , the class  $[2i]$  eventually becomes associated with a fixed  $(w, k, m) \in C$  and thus has finite cardinality  $k$  in  $\mathcal{A}$ . For each  $i \in P$ , the construction will ensure that  $[2i]$  is associated with an increasing sequence of triples  $(w, k, m)_s$  of larger and larger size so that in the limit,  $[2i]$  is infinite. Thus, it will follow that  $i \in P \iff 2i \in \text{Inf}^{\mathcal{A}}$ , and, hence,  $\text{Inf}^{\mathcal{A}}$  will be a  $\Pi_2^0$  complete set.

### Construction

*Stage 0.* At stage 0, we have  $C_0$  enumerated as  $\{(w, k, m)_{i,0} = (w_i, k_i, m_i) : i \in \omega\}$ . For each  $i$ , we let  $n_{i,0} = 2i$  for all  $i$  so that the intended class  $I[2i]$  is to have cardinality  $k_i$  at stage 0. The odd numbers are partitioned among the

classes  $[2i]_0$  in some computable fashion, say  $[2i]_0 = \{2i\} \cup \{1 + 2^{i+1}(2n+1) : n \in \omega\}$ . Thus,  $(x, y) \in E_0$  if and only if  $x$  and  $y$  belong to the same class  $[2i]_0$  for some  $i$ .

*Stage  $s+1$ .* After stage  $s$ , we have an equivalence relation  $E_s$  and an ordering  $2n_{1,s}, 2n_{2,s}, \dots$  of the even numbers, so that  $2n_{i,s}$  is associated with the triple  $(w, k, m)_{i,s}$ . At stage  $s+1$ , let  $i \leq s+1$  be the unique number such that  $R(s+1, n_{i,s})$ . Let  $j > s+1$  be large enough so that no number  $n_{p,s}$  with  $p \geq j$  has been used during the construction. Then we simply move  $n_{i,s}$  to location  $j$  and let all of the  $n_{r,s}$  in between move down one position. That is, we let  $n_{j,s+1} = n_{i,s}$ ,  $n_{r,s+1} = n_{r+1,s}$  for all  $r$  with  $i \leq r < j$  and  $n_{r,s+1} = n_{r,s}$  for all  $r$  such that either  $r < i$  or  $r > j$ . Finally, for all  $i$ , we define  $E_{s+1}$  as follows. For each class  $[2n]$ , let  $n = n_{i,s+1}$ , and let  $(w, k, m) = (w, k, m)_{i,s+1}$ . Suppose that  $[2n]_s = \{2n < a_1 < a_2 < \dots < a_{k-1} < a_k < \dots\}$ . Then we let  $[2n]_{s+1} = \{2n, a_1, \dots, a_{k-1}, a_{k+1}, a_{k+3}, \dots\}$  and we create a new, permanent, infinite class  $\{a_k, a_{k+2}, \dots\}$ . Previously created permanent, infinite classes are left untouched.

**Claim 1.** If  $n \notin P$ , then there exist  $i$  and  $s$  such that for all  $t \geq s$ ,  $n = n_{i,t}$ .

*Proof of Claim 1.* Since  $\{x : R(n, x)\}$  is finite, we may choose  $t$  to be large enough so that for all  $x > t$ ,  $\neg R(n, x)$ . Let  $n = n_{i,t}$ . It follows from the construction that for all  $s > t$ , if  $n = n_{j,s}$ , then  $j \leq i$  and hence  $j$  can only decrease a finite number of times before becoming fixed at some stage  $s$ .

For  $n \notin P$ , let  $i(n)$  be the limit of  $\{i : n = n_{i,s}\}$ , as shown to exist in Claim 1. Let  $I = \{i(n) : n \notin P\}$  and denote  $n$  by  $N_i$  if  $i = i(n)$ .

**Claim 2.**  $I = \omega$

*Proof of Claim 2.* Observe that  $I$  is infinite since  $\omega - P$  is infinite. Now suppose that  $I \neq \omega$ . Then there must be some  $i$  such that  $i+1 \in I$  but  $i \notin I$ . Let  $t > i$  be a stage such that  $N_{i+1} = n_{i+1,s}$  for all  $s \geq t$ . Then for any  $s > t$ , it can never happen that  $R(s+1, n_{i,s})$ . Otherwise, the construction would make  $n_{i,s+1} = N_{i+1}$ , contrary to the choice of  $t$ .

**Claim 3.** For each  $(k, m) \in K$ , there exist at least  $m$  classes in  $\mathcal{A}$  of size exactly  $k$ .

*Proof of Claim 3.* For each  $(k, m) \in K$ , we have some  $(w, k, m) \in C$ . After some stage  $s$ , we will have a fixed  $i$  such that  $(w, k, m) = (w, k, m)_{i,t}$  for all

$t > s$  and a fixed  $n \notin P$  (by Claim 2) such that  $n = N_i = n_{i,t}$  for all  $t > s$ . Suppose that  $[2n]_s = \{2n, a_1, a_2, \dots\}$ . Then,  $I_t[2n] = \{2n, a_1, \dots, a_{k-1}\}$  for all  $t > s$ . It is easy to see that in such a situation  $a_{k+r} \notin [2n]_{s+r+1}$  for all  $r$ . Hence  $[2n] = \{2n, a_1, \dots, a_{k-1}\}$  and has size  $k$ , as desired. Similarly for  $1 \leq p < m$ , we will have a class of size  $k$  and these classes will all be distinct.

**Claim 4.** If  $n \in P$ , then for any  $r$ , there exists  $s$  such that for all  $t > s$ ,  $n = n_{i,t}$  with  $i > r$ .

*Proof of Claim 4.* Given  $r$ , just let  $s$  be large enough so that for all  $i \leq r$  and all  $t > s$ , we have  $n_{i,t} = N_i$ . Since each  $N_i \notin P$ , it follows that  $n \neq n_{i,t}$  for any  $i \leq r$ .

**Claim 5.** If  $n \in P$ , then  $[2n]$  is infinite.

*Proof of Claim 5.* Suppose that  $n \in P$ . We will define an infinite sequence  $\{a_1, a_2, \dots\}$  such that each  $a_i$  is in  $[2n]$ . Since there are only finitely many classes in  $\mathcal{B}$  of any fixed finite size, there is only a finite number of elements in  $C$  of the form  $(w, 1, r)$ , say  $(w^1, 1, 1), (w^2, 1, 2), \dots, (w^m, 1, m)$ . Let  $r$  be large enough so that each of these elements is among the first  $r$  elements of  $C$ . Let  $s$  be large enough, by Claim 4, so that for all  $t \geq s$ ,  $n = n_{i,t}$  with  $i > r$ , and let the intended class be  $I_s[2n] = \{2n, b_1, b_2, \dots\}$ . It follows from the construction that  $b_1 \in [2n]_t$  for all  $t \geq s$ , so we can define  $a_1 = b_1$ .

To determine  $b_{j+1}$ , we similarly find  $s_j$  large enough so that the intended class of  $[2n]$  has at least  $j + 2$  elements for all  $t \geq s_j$ , and let  $b_{j+1}$  be the  $(j + 2)$ -nd element of  $[2n]_{s_j}$ .

It follows that  $\mathcal{A}$  has infinitely many infinite classes.

**Claim 6.**  $\chi(\mathcal{A}) = K$

*Proof of Claim 6.* By Claim 3, we have  $K \subseteq \chi(\mathcal{A})$ . For the other direction, we have, by Claim 5, that  $[2n]$  is infinite for  $n \in P$ . By Claim 1, we see that for  $n \notin P$ , there is some  $i$  such that  $n = N_i$ , and hence some  $(w, k, m)$  such that  $I_s[2n]$  has size  $k$  for all sufficiently large  $s$ . So, by the construction,  $[2n]$  is the unique class of size  $k$  corresponding to  $(k, m)$ .

Finally, suppose that  $\mathcal{B}$  is  $\Delta_2^0$  categorical. By Lemma 2.3 of [2], there is a computable equivalence structure  $\mathcal{D}$  which has character  $K$  and infinitely many infinite equivalence classes such that  $Fin^{\mathcal{D}}$  is a  $\Pi_1^0$  set. Thus,  $\mathcal{D}$  is isomorphic to  $\mathcal{B}$  and  $Inf^{\mathcal{D}}$  is  $\Delta_2^0$ . Hence in any computable equivalence structure  $\mathcal{C}$  that is isomorphic to  $\mathcal{B}$ ,  $Inf^{\mathcal{C}}$  must be  $\Delta_2^0$  and, hence,  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to  $\mathcal{C}$ .  $\square$

There is one final result in order to cover all possible computable equivalence structures.

**Theorem 3.7.** *Suppose that  $\mathcal{B}$  is a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character and that there is some finite  $k$  such that  $\mathcal{B}$  has infinitely many equivalence classes of size  $k$ . Then there is an isomorphic  $\Pi_1^0$  structure  $\mathcal{A}$  which is not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$  and, moreover,  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.*

*Proof.* By the proof of Theorem 3.2, there is a  $\Pi_1^0$  structure  $\mathcal{A}_0$  which consists of infinitely many infinite equivalence classes and infinitely many equivalence classes of size  $k$  such that  $Fin(\mathcal{A}_0) = \{c : card([c]) = k\}$  is not  $\Delta_2^0$ . Define the  $\Sigma_2^0$  character  $K$  to be  $\chi(\mathcal{B}) - (k \times \omega)$ , and let  $\mathcal{C}$  be a computable structure with character  $K$ . Now let  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{C}$ . Hence the set  $\{a \in \mathcal{A} : card([a]) = k\} = \{a \in \mathcal{A}_0 : card([a]) = k\} \times \{0\}$  is not a  $\Delta_2^0$  set. However, in any  $\Sigma_1^0$  structure  $\mathcal{D}$ , the set  $\{d \in \mathcal{D} : card([d]) = k\}$  is a  $\Delta_2^0$  set. Thus  $\mathcal{A}$  is a  $\Pi_1^0$  equivalence structure which is isomorphic to  $\mathcal{B}$ , but not  $\Delta_2^0$  isomorphic to any  $\Sigma_1^0$  structure.  $\square$

Thus we have the following.

**Theorem 3.8.** *Suppose that  $\mathcal{B}$  is a computable equivalence structure which is not computably categorical. Then there is a  $\Pi_1^0$  structure  $\mathcal{A}$  which is isomorphic to  $\mathcal{B}$  but is not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$ .*

*Proof.* There are three possible cases for  $\mathcal{B}$ .

*Case I.* Suppose that  $\mathcal{B}$  has bounded character. Then since  $\mathcal{B}$  is not computably categorical,  $\mathcal{B}$  must have infinitely many finite classes. Thus, there must exist a finite  $k_1$  such that  $\mathcal{B}$  has infinitely many classes of size  $k_1$ . Moreover, it must be the case that either  $\mathcal{B}$  has infinitely many infinite equivalence classes or there exists a finite  $k_2 \neq k_1$  such that  $\mathcal{B}$  also has infinitely many equivalence classes of size  $k_2$ . Then the result follows from Theorem 3.2.

*Case II.* Suppose that  $\mathcal{B}$  has unbounded character and has finitely many infinite equivalence classes. Then  $\mathcal{B}$  is relatively  $\Delta_2^0$  categorical and the result follows from Theorem 3.4.

*Case III.* Suppose that  $\mathcal{B}$  has unbounded character and infinitely many infinite equivalence classes. There are two possibilities. First, there may exist a



finite  $k$  such that  $\mathcal{B}$  has infinitely many equivalence classes of size  $k$ . Then the result follows from Theorem 3.7. Second, it may be that for each finite  $k$ , there are only finitely many classes of size  $k$ . Then the result follows from Theorem 3.6 if  $\mathcal{B}$  is  $\Delta_2^0$  categorical, and it follows easily if  $\mathcal{B}$  is not  $\Delta_2^0$  categorical, since then there is a computable structure  $\mathcal{A}$  which is not  $\Delta_2^0$  isomorphic to  $\mathcal{B}$ .  $\square$

For  $\Delta_2^0$  categorical structures, we have the following immediate corollary.

**Corollary 3.9.** *Suppose that  $\mathcal{B}$  is a  $\Delta_2^0$  categorical, but not computably categorical. Then there is a  $\Pi_1^0$  structure  $\mathcal{A}$  that is isomorphic to  $\mathcal{B}$  such that  $\mathcal{A}$  is not  $\Delta_2^0$  isomorphic to any computable equivalence structure.*

## 4 Spectra of relations on equivalence structures

In this section, we begin to examine the spectrum question for relations on equivalence structures. For a computable ( $\Sigma_1^0$ ,  $\Pi_1^0$ , respectively) equivalence structure  $\mathcal{A}$  and any cardinal  $k \leq \omega$ , we consider the possible Turing degrees of  $\{a : \text{card}([a]) = k\}$  and  $\{a : \text{card}([a]) \geq k\}$ . For example, we know that for any c.e. equivalence structure  $\mathcal{A}$ ,  $\text{Inf}^{\mathcal{A}}$  is  $\Pi_2^0$  and  $\text{Fin}^{\mathcal{A}}$  is  $\Sigma_2^0$ . Thus, a natural question is to ask whether there exists for any  $\Sigma_2^0$  Turing degree  $\mathbf{c}$ , a computable equivalence structure  $\mathcal{A}$  with  $\text{Fin}^{\mathcal{A}}$  of degree  $\mathbf{c}$ . We will not pursue such question in this paper. Instead, we shall prove two results about spectra of relations on computable equivalence structures.

We now give an initial result for computable equivalence structures with infinitely many equivalence classes of size 1, infinitely many equivalence classes of size 2, and with no other equivalence classes. Clearly, for such computable equivalence structure, the elements in classes of size 2 form a c.e. set, and the elements in classes of size 1 form a co-c.e. set. In this case, we obtain not only every c.e. degree, but also every c.e. set.

**Theorem 4.1.** *For any infinite c.e. set  $B$ , there is a computable equivalence structure  $\mathcal{A}$  with character  $\{1, 2\} \times (\omega - \{0\})$  and no infinite equivalence classes such that  $\{a : \text{card}([a]^{\mathcal{A}}) = 2\} = B$ .*

*Proof.* Let  $\{b_0, b_1, \dots\}$  be a computable 1 – 1 enumeration of  $B$ . We will first give an enumeration  $\{c_0, c_1, \dots\}$  of  $B$  such that for every  $n$  and each

$i < 2n + 1$ ,  $c_i < c_{2n+1}$ . Let  $c_0 = b_0$  and let  $c_1$  be equal to  $b_i$ , where  $i$  is the least  $j$  such that  $b_j > b_0$ . Then for  $n \geq 1$ , we inductively define:

- (1)  $c_{2n} = b_i$ , where  $i$  is the least such that  $b_i \notin \{c_0, c_1, \dots, c_{2n-1}\}$ , and
- (2)  $c_{2n+1} = b_k$ , where  $k$  is the least such that  $c_i < b_k$  for all  $i \leq 2n$ .

Now, consider the equivalence structure  $\mathcal{A} = (\omega, E)$ , where  $E = \{(n, n) : n \in \omega\} \cup \{(c_{2n}, c_{2n+1}) : n \in \omega\}$ . Then for each  $i$ ,  $\text{card}([c_i]) = 2$  and for  $a \notin B$ ,  $[a] = \{a\}$ . Thus, in  $\mathcal{A}$ , we have  $B = \{a : \text{card}([a]) = 2\}$ , as desired. It remains to show that  $E$  is a computable relation. Observe that  $c_1 < c_3 < \dots$ , so for every  $n$ ,  $c_{2n+1} \geq n$ . Now, given  $a < b$ , let  $n = \max\{a, b\}$ . Then it is easy to see that

$$aEb \iff (\exists m \leq n)[a = c_{2m} \wedge b = c_{2m+1}],$$

so  $E$  is computable. □

We note that it is easy to modify the proof of Theorem 4.1 to obtain an analogous result for computable equivalence structures which consist of infinitely many equivalence classes of size 1, infinitely many equivalence classes of size  $k > 1$ , and no other equivalence classes.

The analogue of Theorem 4.1 fails for structures with infinitely many classes of size  $k_1$  and  $k_2$  where  $k_2 > k_1 > 1$ . For example, we can prove the following.

**Theorem 4.2.** *There is a c.e. set  $B$  such that for any c.e. equivalence structure  $\mathcal{A}$  with character  $\{2, 3\} \times (\omega - \{0\})$ ,  $\{a : \text{card}([a]^{\mathcal{A}}) = 3\} \neq B$ .*

*Proof.* Let  $\mathcal{A}_e$  be the  $e$ -th equivalence structure. That is,  $\mathcal{A}_e$  has universe  $\omega$  and equivalence relation  $E_e$  which is the reflexive and transitive closure of the  $e$ -th c.e. set  $W_e$ . Thus,  $aE_e b$  holds if and only if

$$a = b \vee (\exists x_0, x_1, \dots, x_k)[x_0 = a \ \& \ x_k = b \ \& \ (\forall i < k)(\langle x_i, x_{i+1} \rangle \in W_e)].$$

Here, we let  $\langle x, y \rangle = 2^{x+1} \cdot 3^{y+1}$ . Let  $[a]_e$  be the equivalence class of  $a$  in  $\mathcal{A}_e$ . Let  $C_e = \{a : \text{card}([a]_e) = 3\}$ . Let  $E_{e,s}$  be the transitive closure of  $W_{e,s}$  for all  $e, s \geq 0$  and  $\mathcal{A}_{e,s}$  be the equivalence structure  $(\omega, E_{e,s})$ .

We will construct a desired c.e. set  $B$  by a finite injury priority argument. Our construction will meet the following requirements  $R_e$  for all  $e \geq 0$ .

$R_e$ : If  $\chi(\mathcal{A}_e) = \{2, 3\} \times (\omega - \{0\})$ , then  $C_e \neq B$ .

To satisfy a particular requirement  $R_e$ , we find a pair  $a_e, b_e$  such that  $a_e E_e b_e$ , but  $a_e \in B \iff b_e \notin B$ .

At every stage  $s$ , we will define  $a_{e,s}$  for each  $e \geq 0$ . We say that the equivalence structure  $\mathcal{A}_e$  is *active* at stage  $s$  as long as  $\mathcal{A}_{e,s}$  has no equivalence classes of size  $> 3$ . At stage  $s$ , if  $\mathcal{A}_e$  is not active, then we will say that the requirement  $R_e$  is *inactive*, and that  $R_e$  is permanently satisfied for all stages  $t \geq s$ . For certain requirements  $e < s$ , we will also define  $b_{e,s}$  such that  $a_{e,s} E_{e,s} b_{e,s}$  and one of  $a_{e,s}, b_{e,s}$  is in  $B_s$ , while the other is restrained by  $R_e$  from entering  $B$ . In this case, if  $\mathcal{A}_e$  is active, then we say that the requirement  $R_e$  is *inactive*, and otherwise that  $R_e$  is *active*.

We say that requirement  $R_e$  with  $e \leq s$  *requires attention* at stage  $s + 1$  if  $\mathcal{A}_e$  and  $R_e$  are both active at stage  $s$ ,  $b_{e,s}$  is undefined, and there exists  $b \leq s + 1$  such that  $b \neq a_{e,s}$  and  $a_{e,s} E_{e,s} b$ .

### Construction

*Stage 0.* For each  $e$ , set  $a_{e,0} = 2e$ . Let  $b_{e,0}$  be undefined for all  $e \geq 0$ .

*Stage  $s + 1$ .* Let  $e$  be the least  $f \leq s + 1$  such that  $R_f$  requires attention at stage  $s + 1$ . If there is no such  $e$ , then do nothing. Otherwise, let  $b$  be the least  $z \leq s + 1$  such that  $z \neq a_{e,s}$  and  $a_{e,s} E_{e,s} z$ . Then we take the following action.

**Case I.** Let  $b = a_{i,s}$  for some  $i < e$ . Then reset  $a_{e,s+1}$  to be the least  $x \neq b$  such that: (i)  $x \notin B_s$ , and (ii)  $x \neq a_{j,s}$  for any  $j$ , and  $x \neq b_{j,s}$  for any  $j$  for which  $b_{j,s}$  is defined. Note that such  $x$  always exists since initially each  $a_{k,s}$  is even, and only a finite number of odd elements will be used up to any stage  $s$  of the construction. We then let  $a_{i,s+1} = a_{i,s}$  for all  $i \neq e$ , and let  $b_{j,s} = b_{j,s+1}$  for all  $j$  such that  $b_{j,s}$  is defined. Then go to stage  $s + 2$ .

If we are not in Case I, then we know that  $b \neq a_{i,s}$  for every  $i \leq e$ .

**Case II.** Let  $b \notin B_s$ . Then put  $a_{e,s} \in B_{s+1}$ , set  $b_{e,s+1} = b$ , and let  $R_e$  restrain  $b$  from entering  $B$ . We then let  $a_{i,s+1} = a_{i,s}$  for all  $i$ , and let  $b_{j,s} = b_{j,s+1}$  for all  $j$  such that  $b_{j,s}$  is defined.

**Case III.** Let  $b \in B_s$ . In this case, we have  $R_e$  restrain  $a_{e,s}$  from entering  $B$  and set  $b_{e,s+1} = b$ . We then let  $a_{i,s+1} = a_{i,s}$  for all  $i$ , and let  $b_{j,s} = b_{j,s+1}$  for

all  $j$  such that  $b_{j,s}$  is defined.

In either Case I or Case II, if  $b = a_{i,s+1}$  for some  $i > e$ , then reset  $a_{i,s+1}$  to be the least  $x \neq b$  such that: (i)  $x \notin B_s$ , and (ii)  $x \neq a_{j,s+1}$  for any  $j \neq i$ , and  $x \neq b_{j,s+1}$  for any  $j$  such that  $b_{j,s+1}$  is defined. Note that such  $x$  always exists since initially each  $a_{k,s}$  is even, and only a finite number of odd elements will be used up to any stage  $s$  of the construction. Let  $b_{i,s+1}$  become undefined. This might injure the requirement  $R_i$ .

This completes the construction.

Since each requirement  $R_e$  can only be injured by the (higher priority) requirements  $R_i$  with  $i < e$ , it is clear that  $R_e$  will require attention only a finite number of times. Thus, the limit  $a_e = \lim_s a_{e,s}$  exists for each  $e$ . Similarly, for each  $e$ , there exists a stage  $s_e$  such that for all  $t \geq s_e$ , either: (a)  $b_{e,t}$  is undefined at stage  $t$ , or (b)  $b_{e,t} = b_{e,s_e}$ , and  $a_{e,t} = a_{e,s_e}$ , and  $a_{e,t}E_{e,t}b_{e,t}$ , and  $a_{e,t} \in B_{e,t} \iff b_{e,t} \notin B_{e,t}$ . If  $\mathcal{A}_e$  has characteristic  $\{2, 3\} \times (\omega - \{0\})$ , then  $\text{card}([a_e]) \geq 2$  and, hence, there exists  $b$  with  $a_e E_e b$ . Consider any stage  $s$  after which no action for any requirements  $R_i$  with  $i \leq e$  will take place, and there is  $b \leq s+1$  such that  $a_e E_{e,s} b$ . Then either  $\mathcal{A}_e$  is inactive at stage  $s+1$ , in which case requirement  $R_e$  is permanently satisfied, or  $\mathcal{A}_e$  is active at stage  $s+1$ . In the second case, requirement  $R_e$  must be inactive at stage  $s+1$  so that  $b_{e,s+1}$  is defined,  $a_{e,s+1}E_{e,s+1}b_{e,s+1}$ , and  $a_{e,s+1} \in B_{e,s+1} \iff b_{e,s+1} \notin B_{e,s+1}$ . Since no requirement  $R_i$  with  $i \leq e$  requires attention after stage  $s+1$ , we will never add either  $a_{e,s+1}$  or  $b_{e,s+1}$  to  $B$  after stage  $s$ , so that  $a_{e,s+1}$  and  $b_{e,s+1}$  will witness that  $B \neq C_e$ .

Note that the c.e. set  $B$  constructed in this proof has the property that for any c.e. structure  $\mathcal{A}$  with all equivalence classes of size  $\geq 2$  and for all  $k \leq \omega$ ,  $B \neq \{a : \text{card}([a]^{\mathcal{A}}) = k\}$ .  $\square$

For equivalence structures with equivalence classes of three or more different cardinalities  $k_1 < k_2 < \dots < k_n$ , the elements of an intermediate size equivalence class form a d.c.e. set. Thus it is natural to ask whether any d.c.e. set can be represented in this way. Similar questions can be asked for  $\Sigma_1^0$  and  $\Pi_1^0$  equivalence structures.

## 5 Decidability of structures and theories

Recall that for any structure  $\mathcal{A}$ ,  $Th(\mathcal{A})$  denotes the first-order theory of  $\mathcal{A}$ , and  $FTh(\mathcal{A})$  denotes the elementary diagram of  $\mathcal{A}$ . In this section, we consider the decidability of equivalence structures and their theories. The intuitive idea is that the character of an equivalence structure, together with the number of infinite classes, determines its theory. Similarly, the character, together with the function mapping any element to the size of its equivalence class, determines its elementary diagram.

**Proposition 5.1.** *If  $Th(\mathcal{A})$  is decidable, then the character  $\chi(\mathcal{A})$  is computable.*

*Proof.* It follows from the definition of  $\chi(\mathcal{A})$  that the character is uniformly definable by first-order formulas. That is, it is easy to write down first-order formulas  $\psi_{n,k}$  so that

$$(k, n) \in \chi(\mathcal{A}) \iff \mathcal{A} \models \psi_{n,k}.$$

□

It follows from the argument above that, in fact,  $\chi(\mathcal{A})$  is many-one reducible to  $Th(\mathcal{A})$ . Define the set  $K(\mathcal{A}) \subseteq \omega \times (\omega - \{0\})$  by

$$(a, k) \in K(\mathcal{A}) \iff \text{card}([a]) \geq k.$$

**Theorem 5.2.** *For any equivalence structure  $\mathcal{A}$ , the elementary diagram of  $\mathcal{A}$  is Turing reducible to the join of the set  $K(\mathcal{A})$  with the atomic diagram of  $\mathcal{A}$ .*

*Proof.* First, assume that  $\mathcal{A}$  has only finitely many equivalence classes. Then, clearly,  $FTh(\mathcal{A})$  is axiomatizable and hence computable. That is, for simplicity, let  $a_1, \dots, a_n$  be representatives of the  $n$  classes having cardinalities  $k_1, \dots, k_n$ , respectively. Then in the expanded language with names for  $a_1, \dots, a_n$ , we have the following axioms.

- (i) Every element is equivalent to one of  $a_1, \dots, a_n$ :

$$(\forall x)[xEa_1 \vee \dots \vee xEa_n].$$

- (ii) For every finite class  $[a]$  with the representative  $a$ , there is an axiom giving the size  $k$  of the class:

$$(\exists x_1, \dots, x_k) \left[ \bigwedge_{i=1}^k x_i E a \wedge \bigwedge_{\substack{i \neq j \\ 1 \leq i, j \leq k}} x_i \neq x_j \wedge (\forall z) (a E z \implies (z = x_1 \vee \dots \vee z = x_k)) \right].$$

- (iii) For every infinite class  $[a]$ , we have an axiom for every  $n$ :

$$(\forall x_1, \dots, x_n) (\exists y) [y E a \wedge y \neq x_1 \wedge \dots \wedge y \neq x_n].$$

Now, we assume that  $\mathcal{A}$  has infinitely many equivalence classes. We proceed by quantifier elimination. We first expand the language by adding the relation symbols  $\gamma_k$  such that  $\mathcal{A} \models \gamma_k(a)$  if and only if  $(a, k) \in K(\mathcal{A})$ . Let  $\psi(x, t_1, \dots, t_n)$  be any conjunction of literals in this expanded language, where  $t_1, \dots, t_n$  are either variables or elements of  $\mathcal{A}$ , and let  $\theta$  be  $(\exists x)\psi$ . Without loss of generality, we may assume that  $\psi$  includes either  $t_i = t_j$  or  $\neg(t_i = t_j)$  for all  $i, j$  where we set  $x = t_0$ . Similarly, we may assume that  $\psi$  includes either  $t_i E t_j$  or  $\neg(t_i E t_j)$  for all  $i, j$ . As usual, it suffices to eliminate the quantifier from  $\psi$ . There are three cases.

*Case 1.* If  $\psi$  has a conjunct  $x = t_i$  with  $i > 0$ , then  $\theta$  is logically equivalent to the quantifier-free formula  $\psi^-$  obtained from  $\psi$  by replacing all occurrences of  $x$  with  $t_i$ .

In the remaining cases,  $\psi$  has the conjuncts  $\neg(x = t_i)$  for all  $i > 0$ .

*Case 2.* Suppose that  $\psi$  has the conjunct  $x E t_m$  for some  $m$ . Let  $k$  be the number of distinct terms (modulo  $\psi \models t_i = t_j$ ) out of  $x, t_1, \dots, t_n$  such that  $t_i E t_m$ . Then  $\theta$  is logically equivalent to the quantifier-free formula  $\psi^- \wedge \gamma_{k+1}(t_m)$ . That is, the desired  $x$  will exist if and only if  $\text{card}([t_m]) \geq k+1$ , so that  $\mathcal{A}$  contains an additional element of  $[t_m]$ .

*Case 3.* Suppose that  $\psi$  has the conjuncts  $\neg x E t_i$  for all  $i$ . Then, again,  $\theta$  is equivalent to the formula  $\psi^-$ . This is true since  $\mathcal{A}$  has infinitely many distinct equivalence classes.

At the end of quantifier elimination, we can determine whether the reduced formula  $\psi$  holds in  $\mathcal{A}$  by consulting the diagram of  $\mathcal{A}$  as well as  $K(\mathcal{A})$ .  $\square$

**Theorem 5.3.** *For any equivalence structure  $\mathcal{B}$ , there is a structure  $\mathcal{A}$  isomorphic to  $\mathcal{B}$ , such that  $\mathcal{A}$  and  $K(\mathcal{A})$  are computable from  $\chi(\mathcal{A})$ .*

*Proof.* We may assume, without loss of generality, that  $\mathcal{B}$  has no infinite equivalence classes, since, if needed, we can simply adjoin either infinitely many or some fixed finite number of infinite equivalence classes. We may also assume that  $\mathcal{B}$  has infinitely many classes with at least two elements, since otherwise  $\mathcal{B}$  certainly has a decidable copy. The structure  $\mathcal{A}$  will contain a distinct equivalence class  $[\langle k, n \rangle]$  for each  $(k, n) \in \chi(\mathcal{B})$ , where we let  $\langle k, n \rangle = 2^{k+1} \cdot 3^{n+1}$ . Let  $\chi(\mathcal{B})$  be enumerated numerically as  $\langle k_0, n_0 \rangle, \langle k_1, n_1 \rangle, \dots$  and let  $b_0, b_1, \dots$  enumerate  $\omega - \chi(\mathcal{B})$ . Then  $E = E^{\mathcal{A}}$  is defined by using the elements  $b_0, b_1, \dots$  to fill out the equivalence classes  $[\langle k_0, n_0 \rangle], [\langle k_1, n_1 \rangle], \dots$  in order, as needed. It is easy to see that  $\mathcal{A}$  and  $K(\mathcal{A})$  are computable from  $\chi(\mathcal{A})$ .  $\square$

Putting these results together, we have the next two theorems along with some immediate corollaries.

**Theorem 5.4.** *For any equivalence structure  $\mathcal{A}$ ,  $Th(\mathcal{A})$  and  $\chi(\mathcal{A})$  have the same Turing degree.*

*Proof.* It follows from the argument in Proposition 5.1 that  $\chi(\mathcal{A})$  is Turing reducible to  $Th(\mathcal{A})$ . Conversely, let  $\mathcal{B}$  be an equivalence structure and let  $\mathcal{A}$ , isomorphic to  $\mathcal{B}$ , be given by Theorem 5.3, so that  $\mathcal{A}$  and  $K(\mathcal{A})$  are both computable from  $\chi(\mathcal{A})$  (which, of course, equals  $\chi(\mathcal{B})$ ). It follows from Theorem 5.2 that  $FTh(\mathcal{A})$  is computable from  $\chi(\mathcal{B})$ . Now  $Th(\mathcal{B}) = Th(\mathcal{A})$  is computable from  $FTh(\mathcal{A})$ , and, hence, is computable from  $\chi(\mathcal{B})$  as desired.  $\square$

**Corollary 5.5.** *For any equivalence structure  $\mathcal{A}$ ,  $Th(\mathcal{A})$  is decidable if and only if  $\chi(\mathcal{A})$  is computable.*

**Theorem 5.6.** *For any equivalence structure  $\mathcal{B}$  with computable character  $\chi(\mathcal{B})$ , there is a decidable structure  $\mathcal{A}$  isomorphic to  $\mathcal{B}$ . (Hence  $Th(\mathcal{B})$  is decidable.)*

*Proof.* Again, it suffices to assume that  $\mathcal{B}$  has no infinite equivalence classes. By Theorem 5.3, there is a structure  $\mathcal{A}$  isomorphic to  $\mathcal{B}$ , which is computable from  $\chi(\mathcal{A})$ , and hence  $\mathcal{A}$  and  $K(\mathcal{A})$  are also computable. It now follows from Theorem 5.2 that  $FTh(\mathcal{A})$  is decidable, and hence  $Th(\mathcal{A})$ , which equals  $Th(\mathcal{B})$ , is decidable.  $\square$

Clearly, any bounded character is computable.

**Corollary 5.7.** *If the equivalence structure  $\mathcal{A}$  has bounded character, then  $Th(\mathcal{A})$  is decidable.*

For computably categorical structures, we can say more.

**Corollary 5.8.** *If  $\mathcal{A}$  is a computably categorical equivalence structure, then  $\mathcal{A}$  is decidable.*

*Proof.* Let  $\mathcal{A}$  be computably categorical. Then  $\mathcal{A}$  has bounded character, so  $\chi(\mathcal{A})$  is computable. Hence by Theorem 5.6, there is a structure  $\mathcal{B}$  isomorphic to  $\mathcal{A}$ , which is decidable. Since  $\mathcal{A}$  is computably categorical,  $\mathcal{A}$  is computably isomorphic to  $\mathcal{B}$  and, therefore,  $\mathcal{A}$  is also decidable.  $\square$

Note that there are equivalence structures that are not computably categorical, which have decidable theories. For example, fix  $k_1 < k_2 \leq \omega$  and let  $\mathcal{A}$  have infinitely many equivalence classes of size  $k_1$  and infinitely many classes of size  $k_2$  and no other classes. Then  $\chi(\mathcal{A})$  is computable and, thus,  $Th(\mathcal{A})$  is decidable. We note that in all considered cases of decidable theories, one could, in fact, give a complete set of axioms for the theory.

## References

- [1] C. Bernardi and A. Sorbi, Classifying positive equivalence relations, *Journal of Symbolic Logic* 48 (1983), pp. 529–538.
- [2] W. Calvert, D. Cenzer, V. Harizanov, and A. Morozov, Effective categoricity of equivalence structures, *Annals of Pure and Applied Logic* 141 (2006), pp. 61–78.
- [3] D. Cenzer, G. LaForte, and J. Remmel, Equivalence structures and isomorphisms in the difference hierarchy, *Journal of Symbolic Logic* 74 (2009), pp. 535–556.
- [4] D. Cenzer, V. Harizanov, and J.B. Remmel,  $\Sigma_1^0$  and  $\Pi_1^0$  equivalence structures, in *Mathematical Theory and Computational Practice, CIE 2009*, K. Ambos-Spies, B. Löwe and W. Merkle, editors, Lecture Notes in Computer Science, vol. 5635, (Springer, 2009), pp. 99–108.



- [5] Yu.L. Ershov, Positive equivalence relations, *Algebra and Logic* 10 (1973), pp. 378–394 (English translation).
- [6] S. Gao and P. Gerdes, Computably enumerable equivalence relations, *Studia Logica* 67 (2001), pp. 27–59.
- [7] A.M. Kach and D. Turetsky,  $\Delta_2^0$ -categoricity of equivalence structures, to appear in the *New Zealand Journal of Mathematics*.
- [8] N.G. Khisamiev, Constructive Abelian groups, in: Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors, *Handbook of Recursive Mathematics*, vol. 2 (North-Holland, Amsterdam, 1998), pp. 1177–1231.
- [9] B. Khossainov, F. Stephan, and Y. Yang, Computable categoricity and the Ershov hierarchy, *Annals of Pure and Applied Logic* 156 (2008), pp. 86–95.
- [10] A. Lachlan, A note on positive equivalence relations, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* 33 (1987), pp. 43–46.
- [11] G. Metakides and A. Nerode, Recursively enumerable vector spaces, *Annals of Mathematical Logic* 11 (1977), pp. 141–171.
- [12] A. Nies, Computably enumerable equivalence relations modulo finite differences, *Math. Logic Quarterly* 40 (1994), pp. 490–518.
- [13] J.B. Remmel, Combinatorial functors on co-r.e. structures, *Annals of Mathematical Logic* 10 (1976), pp. 261–287.
- [14] R.I. Soare, *Recursively Enumerable Sets and Degrees* (Springer, Berlin, 1987).