Intrinsic bounds on complexity and definability at limit levels*

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Abstract

We show that for every computable limit ordinal α, there is a computable structure $\mathcal{A}$ that is Δ^0_α categorical, but not relatively Δ^0_α categorical, i.e., does not have a formally Σ^0_α Scott family. We also show that for every computable limit ordinal α, there is a computable structure $\mathcal{A}$ with an additional relation R that is intrinsically Σ^0_α on $\mathcal{A}$, but not relatively intrinsically Σ^0_α on $\mathcal{A}$, i.e., not definable by a computable Σ_α formula with

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finitely many parameters. Earlier results in [7], [10], [8] establish the same facts for computable successor ordinals α.

1 Introduction and Preliminaries

The languages we consider are all *computable*; that is, the set of symbols is computable, and we can effectively determine the type (relation or function) and the arity of a symbol. The structures we consider all have universes that are subsets of ω. When we measure complexity of a structure $\mathcal{A}$, we identify $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$. Thus, $\mathcal{A}$ is computable if $D(\mathcal{A})$ is computable, and computable relative to $\mathcal{A}$ means computable in $D(\mathcal{A})$.

1.1 Δ^0_α-categoricity and relative Δ^0_α-categoricity

Definition 1. *Let $\mathcal{A}$ be a computable structure, and let α be a computable ordinal.*

1. *The structure $\mathcal{A}$ is* Δ^0_α-categorical *if for all* computable *copies $\mathcal{B}$ of $\mathcal{A}$, there is a Δ^0_α isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.*
2. *The structure $\mathcal{A}$ is* relatively Δ^0_α-categorical *if for* all *copies $\mathcal{B}$ of $\mathcal{A}$ (not just the computable ones), there is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ that is* Δ^0_α relative to *$\mathcal{B}$.*

There are syntactical conditions that completely account for relative categoricity.

Definition 2. *A* formally Σ^0_α Scott family *for $\mathcal{A}$ is a Σ^0_α set Φ of computable Σ^0_α formulas $\varphi(\overline{c},\overline{x})$, with a fixed finite tuple $\overline{c}$ of parameters from $\mathcal{A}$, such that:*

1. *Every finite tuple $\overline{a}$ in $\mathcal{A}$ satisfies some formula $\varphi(\overline{c},\overline{x})$ in Φ;*
2. *For any tuples $\overline{a}$ and $\overline{b}$ in $\mathcal{A}$ and any formula $\varphi(\overline{c},\overline{x})$ in Φ, if $\overline{a}$ and $\overline{b}$ both satisfy $\varphi(\overline{c},\overline{x})$, then there is an automorphism of $\mathcal{A}$ mapping $\overline{a}$ to $\overline{b}$.*

The following result is in [4] and [6].

Theorem 1.1. *For a computable structure $\mathcal{A}$ and a computable ordinal α, the following are equivalent:*

1. *$\mathcal{A}$ is relatively Δ^0_α-categorical;*
2. *$\mathcal{A}$ has a formally Σ^0_α Scott family.*

For Δ^0_α-categoricity, the syntactical conditions in 2 of Theorem 1.1 are sufficient, but not necessary. Goncharov [7] showed the following.

Theorem 1.2. *There is a structure $\mathcal{A}$ that is computably categorical but not relatively computably categorical.*

Theorem 1.2 has been lifted to successor levels in [8].

Theorem 1.3. *For every computable successor ordinal α, there is a structure $\mathcal{A}$ that is Δ^0_α-categorical but not relatively Δ^0_α-categorical.*

In the present paper, we extend Theorem 1.3 to computable limit ordinals.

1.2 Intrinsically Σ^0_α relations

Definition 3. *Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on $\mathcal{A}$.*

1. *The relation R is* intrinsically Σ^0_α *on $\mathcal{A}$ if for every isomorphism F from $\mathcal{A}$ onto a computable structure, $F(R)$ is Σ^0_α.*
2. *The relation R is* relatively intrinsically Σ^0_α *on $\mathcal{A}$ if for every isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is Σ^0_α relative to $\mathcal{B}$.*

There are syntactical conditions that completely account for relative intrinsically Σ^0_α relations. The following result is in [4] and [6].

Theorem 1.4. *The relation R is relatively intrinsically Σ^0_α on $\mathcal{A}$ iff R is definable by a computable Σ_α formula $\varphi(\overline{c}, \overline{x})$ with a finite tuple $\overline{c}$ of parameters from $\mathcal{A}$.*

For a relation to be intrinsically Σ^0_α, the syntactical condition in Theorem 1.4 is sufficient but not necessary. Manasse [10] showed the following.

Theorem 1.5. *There is a computable structure $\mathcal{A}$ with an additional relation R that is intrinsically c.e. but not relatively intrinsically c.e.*

In [8], Manasse's result is lifted to successor levels.

Theorem 1.6. *For every computable successor ordinal α, there is a computable structure $\mathcal{A}$ with an additional relation R that is intrinsically Σ^0_α but not relatively intrinsically Σ^0_α on $\mathcal{A}$.*

In the present paper, we extend Theorem 1.6 to computable limit ordinals.

1.3 Enumerations

The results of Goncharov (Theorem 1.2) and Manasse (Theorem 1.5) were both based on an enumeration theorem, which was proved independently by Badaev and Selivanov. To state this theorem, we need some definitions.

Definition 4. *Let $\mathcal{S} \subseteq \omega^\omega$.*

1. *The family $\mathcal{S}$ is* discrete *if for every $g \in \mathcal{S}$, there exists $p \in \omega^{<\omega}$ such that g is the unique extension of p in S.*

2. *The family $\mathcal{S}$ is* effectively discrete *if there is a c.e. set $\Gamma \subseteq \omega^{<\omega}$ such that:*
   (a) *every $g \in \mathcal{S}$ extends some $p \in \Gamma$, and*
   (b) *for any $g, g' \in \mathcal{S}$ and $p \in \Gamma$, if g and g' both extend p, then $g = g'$.*

Definition 5. *Let $\mathcal{S} \subseteq \omega^\omega$.*

1. *A function $G : \omega^2 \to \omega$ is an* enumeration *of $\mathcal{S}$ if $\mathcal{S}$ is the family of functions of the form $g_a(t) = G(a, t)$ for $a \in \omega$—we refer to a as a G-*index *for g_a.*
2. *An enumeration G of $\mathcal{S}$ is* Friedberg *if every $g \in \mathcal{S}$ has a unique G-index.*
3. *Two Friedberg enumerations G and H are* computably equivalent *if there is a computable function k such that $G(a, t) = H(k(a), t)$; i.e., $k(a)$ is the H-index for the function with G-index a.*

Here is the result of Badaev [5] and Selivanov [12].

Theorem 1.7 (Badaev, Selivanov). *There is a family $\mathcal{S} \subseteq \omega^\omega$ such that:*

1. *$\mathcal{S}$ is discrete but not effectively discrete, and*
2. *$\mathcal{S}$ has a unique computable Friedberg enumeration, up to computable equivalence.*

The results in [8], lifting the results of Goncharov and Manasse to successor levels, involved relativizing and coding. Theorem 1.7, relativized but otherwise unchanged, was the basis of the proof. This method fails for limit ordinals. In the present paper, we vary the enumeration theorem. We consider functions $g(t)$, computed using a sequence of oracles the strength of which increases with t. In this section, we consider the limit ordinal $\alpha = \omega$.

1.4 Enumeration theorems for $\alpha = \omega$

Definition 6. *Let $\mathcal{S} \subseteq \omega^\omega$.*

1. *The family $\mathcal{S}$ is* ω-discrete *if there is a Σ^0_ω set $\Gamma \subseteq \omega^{<\omega}$ such that every $g \in \mathcal{S}$ extends some $p \in \Gamma$, and if $g, g' \in \mathcal{S}$ both extend the same $p \in \Gamma$, then $g = g'$.*
2. *An enumeration G is* anti-Friedberg *if every $g \in \mathcal{S}$ has infinitely many G-indices.*
3. *Enumerations G and H are* strongly Δ^0_ω-equivalent *if there is a Δ^0_ω permutation k of ω such that $G(a, t) = H(k(a), t)$.*

4. *Let $f : \omega \to \omega$ be a strictly increasing computable function with values ≥ 1. An ω-f-*enumeration *of $\mathcal{S}$ is an enumeration G such that for some fixed e, for all a and t, $G(a,t) = \varphi_e^{\Delta^0_{f(t)}}(a,t)$.*

It is helpful to fix some notation. Starting with a standard Σ^0_ω enumeration of all Σ^0_ω sets, we obtain a Σ^0_ω enumeration of the family of all Σ^0_ω subsets of $\omega^{<\omega}$. We identify the elements of $\omega^{<\omega}$ with their codes. Let Γ_e be the intersection of $\omega^{<\omega}$ with the Σ^0_ω set having index e in the standard enumeration. We call e an *index* of Γ_e. We write $\Gamma_{e,t}$ for the finite part of Γ_e enumerated in at most t steps, where the oracle questions are all answered by $\Delta^0_{f(t')}$ for $t' \leq t$.

Let $f \in \omega^\omega$ be a strictly increasing computable function with values ≥ 1. We want an enumeration of all partial functions computed in the same way as an ω-f-enumeration; i.e., using the same sequence of oracles. Let

$$E(e,a,t) = \begin{cases} \varphi_e^{\Delta^0_{f(t)}}(a,t) & \text{if this value is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

We write H_e for the partial ω-f-enumeration H such that $H(a,t) = E(e,a,t)$, and we refer to e as an *index* of H_e.

The result below (Theorem 1.8) is a natural analogue of the result of Badaev and Selivanov. In proving this result, we will actually prove a stronger result.

Theorem 1.8 (Enumeration Theorem I, for $\alpha = \omega$). *Let $f \in \omega^\omega$ be a strictly increasing computable function with values ≥ 1. There is a family $\mathcal{S} \subseteq \omega^\omega$ such that:*

1. *$\mathcal{S}$ is discrete but not ω-discrete,*
2. *$\mathcal{S}$ has a unique anti-Friedberg ω-f-enumeration, up to strong Δ^0_ω-equivalence.*

Proof. We construct an anti-Friedberg ω-f-enumeration G of a family $\mathcal{S}$. We proceed by induction on levels t, using oracle $\Delta^0_{f(t)}$ to define the value of $G(c,t)$ for all c. We have the following requirements.

R_{2e}: $\Gamma_e \subseteq \omega^{<\omega}$ does not witness the ω-discreteness of $\mathcal{S}$.

R_{2i+1}: If H_i is an anti-Friedberg enumeration of $\mathcal{S}$, then there is a Δ^0_ω permutation k_i of ω such that for all c, $G(c,t) = H_i(k_i(c),t)$.

We will call the requirements of the form R_{2e} even and those of the form R_{2i+1} odd.

For an even requirement R_{2e}, we set aside an infinite set

$$C_e = \{c_e^0, c_e^1, \ldots\}$$

of G-indices, an initial value a_e for the functions with indices in C_e, and an infinite set B_e of possible alternative values. The set C_e will provide indices for at most two functions. One function, g, will be constant, with $g(t) = a_e$ for all t. If there is a second function, g', then there is a single $t = t_0$ such that $g'(t_0) \in B_e$, and for all other t, $g'(t) = a_e$. We start by letting $G(c, 0) = a_e$ for all $c \in C_e$. We continue by letting $G(c, t) = a_e$ for all $c \in C_e$ until we come to t such that, using $\Delta^0_{f(t)}$, we see $p \in \Gamma_e$, of length $\leq t$, with constant value a_e. We will satisfy the requirement by making a split, at level t or higher.

For an odd requirement R_{2i+1}, we watch the enumeration H_i. At level t, assuming that H_i is an ω-f-enumeration of $\mathcal{S}$, we choose a finite partial permutation k_i^t, including all $c \leq t$ in the domain and all $d \leq t$ in the range, such that if $k_i^t(c) = d$, then for all $t' < t$, $G(c, t') = H_i(d, t')$ and if $c \in C_e$, then $H_i(d, t) = a_e$. We make sure that there is some t_i such that the functions k_i^t for $t > t_i$ form a chain. Then $k_i = \cup_{t>t_i} k_i^t$ will be a Δ^0_ω function satisfying the requirement.

There is a possible conflict between the even and odd requirements. Suppose that for all $t' \leq t$ and all $c \in C_e$, $G(c, t') = a_e$, and at level $t + 1$, we would like to make a split for requirement R_{2e}. Suppose also that H_i appears to be an ω-f-enumeration of $\mathcal{S}$, and we have $k_i^t(c) = d$. Suppose we make the split, letting $G(c, t + 1) = b$, where $b \in B_e$, and then we see that $H_i(d, t + 1) = a_e$. In this case, we cannot define k_i^{t+1} so that it extends k_i^t. We allow R_{2e} to injure R_{2i+1} in this way if $e \leq i$. However, if $i < e$, then we postpone splitting for R_{2e} until we have seen the value of $H_i(d, t + 1)$ for all $d \in ran(k_i^t)$. If $k_i^t(c) = d$, where $c \in C_e$, we will let $G(c, t + 1) = a_e$, and in case $H_i(d, t + 1) = b$, where $b \neq a_e$, our alternative value b_e will be different from b. Thus, either $H_i(d, t + 1) = a_e$ or else H_i is not an enumeration of $\mathcal{S}$,

At level t, we consider R_n for $n < t$. Thus, at level 0, we ignore all requirements. We let $k_i^0 = \emptyset$ for all i, and we let $G(c, 0) = a_e$ for all $c \in C_e$. At level $t + 1$, we consider R_n for $n \leq t$. We use some terminology. We say that R_{2e} is *forbidden* at level $t + 1$ if we have already made a split for R_{2e} at some level $t' \leq t$. We say that R_{2i+1} is *forbidden* at level $t + 1$ if k_i^t is not defined—this means that H_i is not an ω-f-enumeration of $\mathcal{S}$.

For $2i + 1 > t$, we define k_i^{t+1} to be $\emptyset$, ignoring R_{2i+1}. Suppose $2i + 1 \leq t$, and R_{2i+1} is not forbidden. Then we give H_i some tests.

Tests for H_i at level $t+1$. We require that there is a finite partial permutation k of ω such that:

1. $dom(k)$ includes all $c \leq t$ and $ran(k)$ includes all $d \leq t$,
2. if $k(c) = d$, then for all $t' \leq t$, $G(c, t') = H_i(d, t')$, and if $c \in C_e$, then $H_i(d, t + 1) \downarrow= a_e$,
3. if there is no $e \leq i$ such that R_{2e} splits at level t, then $k \supseteq k_i^t$.

If at some stage s_i, we see that H_i has passed all of the level $t + 1$ tests, witnessed by the function k, then we let $k_i^{t+1} = k$. If there is no such stage

s_i, then k_i^{t+1} will be undefined. In this case, H_i will not be an anti-Friedberg ω-f-enumeration of $\mathcal{S}$, and R_{2i+1} will be forbidden at all levels $> t+1$.

Suppose for some $e \leq t$, R_{2e} is not forbidden at level $t+1$, and we have found an appropriate p of length $\leq t$. Suppose we come to a stage at which for all $i < e$ such that R_{2i+1} is not forbidden, H_i has passed the level $t+1$ tests. Then we will split for R_{2e} at level $t+1$. If we do not come to such a stage, then we will not split at level $t+1$. We will try again at the next level, when there will be fewer R_{2i+1}, for $i < e$, to consider. Eventually, we will come to a level where for all $i < e$ such that R_{2i+1} is not forbidden, H_i passes the new tests. Then we can make the split to satisfy R_{2e}.

With these guidelines on the action for various requirements, we can see how to define $G(c, t+1)$ for all c. For $2e > t$, we let $G(c_e^j, t+1) = a_e$ for all j. Suppose that $2e \leq t$. If R_{2e} is forbidden, or if it is not forbidden, but we are not ready to split for R_{2e} at level $t+1$, then we let $G(c_e^j, t+1) = a_e$ for all j. Suppose we are ready to split for R_{2e}. We have $p \in \Gamma_{e,t}$ of length $\leq t$, with constant value a_e, and we come to a stage s such that for all $i < e$, either R_{2i+1} is forbidden, or it has passed the level $t+1$ tests, so k_i^{t+1} is defined. We choose $b_e \in B_e$ such that for all $i < e$ such that k_i^{t+1} is defined, for all $d \in ran(k_i^{t+1})$, $H_i(d, t+1) \neq b_e$. Let C_e^* consist of all c_e^j such that either $j \leq t+1$, or else there is some $i < e$ such that k_i^{t+1} is defined and $c_e^j \in dom(k_i^{t+1})$. We let $G(c_e^j, t+1) = a_e$ if j is even or $c_e^j \in C_e^*$. For other j (odd, and with $c_e^j \notin C_e^*$), we let $G(c_e^j, t+1) = b_e$.

This completes the construction. We can see that R_{2e} is satisfied. If we find an appropriate p, we will split when we come to a level at which all R_{2i+1} that are not already forbidden pass their new tests. We can also see that R_{2i+1} is satisfied. There are only finitely many $e \leq i$. Each R_{2e} splits at most once, so there is some level t_i after which there are no further splits for R_{2e} for $e \leq i$. If R_{2i+1} is never forbidden, then k_i, where $k_i = \cup_{t > t_i} k_i^t$, witnesses that G and H_i are strongly Δ_ω^0-equivalent. If R_{2i+1} becomes forbidden at level $t+1$ because H_i fails to pass the level $t+1$ tests, then H_i is not an anti-Friedberg ω-f-enumeration of the family $\mathcal{S}$. This completes the proof. □

The proof of Theorem 1.8 yields the following stronger result.

Theorem 1.9 (Enumeration Theorem II, for $\alpha = \omega$). *Let $f \in \omega^\omega$ be a strictly increasing computable function. There is a family $\mathcal{S} \subseteq \omega^\omega$ with an ω-f-enumeration G, which has the following properties:*

1. *every anti-Friedberg ω-f-enumeration of $\mathcal{S}$ is strongly Δ_ω^0-equivalent to G,*

2. *$\mathcal{S}$ is discrete,*

3. *there is a set $\mathcal{S}' \subseteq \mathcal{S}$ such that*

   (a) *the set of G-indices for elements of $\mathcal{S}'$ is Σ_ω^0, and*

   (b) *for every Σ_ω^0 set $\Gamma \subseteq 2^{<\omega}$, if each $g \in \mathcal{S}'$ extends some $p \in \Gamma$, then some $p \in \Gamma$ has extensions g_1, g_2 with $g_1 \in \mathcal{S}'$ and $g_2 \in \mathcal{S} - \mathcal{S}'$.*

Remark 1. Suppose that $\mathcal{S}$, G, and $\mathcal{S}'$ are as in Enumeration Theorem II. If H is any anti-Friedberg ω-f-enumeration of $\mathcal{S}$, then the set of H-indices of functions in $\mathcal{S}'$ is Σ^0_ω.

Proof. Let k be the Δ^0_ω permutation witnessing that G and H are strongly Δ^0_ω-equivalent. This function maps the G-indices of elements of $\mathcal{S}'$ onto the H-indices. $\square$

Remark 2. Enumeration Theorem II implies Enumeration Theorem I.

Proof. Let $\mathcal{S}$, G, and $\mathcal{S}'$ be as in Enumeration Theorem II. We must show that $\mathcal{S}$ is not ω-discrete. Suppose it is, and let $\Gamma \subseteq \omega^{<\omega}$ witness this. Each element of $\mathcal{S}$ must extend some element of Γ. In particular, each element of $\mathcal{S}'$ extends some element of Γ. Then we have $p \in \Gamma$ and $g_1, g_2 \supseteq p$ such that $g_1 \in \mathcal{S}'$ and $g_2 \in \mathcal{S} - \mathcal{S}'$. This is a contradiction. $\square$

1.5 Δ^0_ω-categoricity but not relative Δ^0_ω-categoricity

We now present our first main result, the analogue of Theorems 1.2 and 1.3, in the case when $\alpha = \omega$.

Theorem 1.10. *There is a computable structure $\mathcal{A}$ that is Δ^0_ω-categorical but not relatively Δ^0_ω-categorical.*

Proof. Let $f(t) = 2t + 1$. Let $\mathcal{S}$ be as in Theorem 1.8, and let G be an anti-Friedberg ω-f-enumeration of $\mathcal{S}$. We let

$$\mathcal{A} = (A, I, U, V, W, Q, P, <),$$

where the universe A is the disjoint union of sets I, U, V, W, and I, U, V, W are predicates for the respective sets. We identify the a^{th} element of I with $a \in \omega$, and think of a as an index for a function in S. The relation Q maps U onto I so as to partition U into infinite sets U_a, corresponding to $a \in I$, it maps V onto U so as to partition V into infinite sets $V_{a,t}$, and it maps W onto V so as to partition W into sets $W_{a,t,x}$. The relation P acts as a predecessor function on the sets U_a and $V_{a,t}$, making each of these sets into a copy of ω with the usual predecessor function p, where $p(0) = 0$, and $p(n+1) = n$. For the element playing the role of x in $V_{a,t}$, the relation $<$ is an ordering on $W_{a,t,x}$ of type

$$\begin{cases} \omega^t \cdot 2 & \text{if } G(a,t) = x, \\ \omega^t & \text{otherwise.} \end{cases}$$

The following is a well-known result. A precise statement may be found in [1]. In [2], there is a similar result, with $\mathbb{Z}$ replacing ω. These results are based on an idea due to Watnick [13].

Lemma 1.11. *Given a Δ^0_{2t+1} index for a linear ordering $\mathcal{L}$, we can effectively pass to a computable index for an ordering of type $\omega^t \cdot \mathcal{L}$.*

Using Lemma 1.11, we show the following.

Lemma 1.12. *The structure $\mathcal{A}$ has a computable copy.*

Proof. Recall that $G(a,t)$ is computed by a uniform procedure using Δ^0_{2t+1}. We have a uniform procedure for computing from a,t and x a Δ^0_{2t+1}-index for an ordering that has two elements if $G(a,t)=x$, and one element otherwise. Applying Lemma 1.11, we get a uniformly computable family of linear orderings $\mathcal{L}_{a,t,x}$ such that

$$\mathcal{L}_{a,t,x} \cong \begin{cases} \omega^t \cdot 2 & \text{if } G(a,t)=x, \\ \omega^t & \text{otherwise.} \end{cases}$$

Using this family of orderings, we get a computable copy of the structure $\mathcal{A}$ described above. $\square$

Lemma 1.13. *The structure $\mathcal{A}$ is Δ^0_ω-categorical.*

Proof. By Lemma 1.12, we may suppose that the structure $\mathcal{A}$ is computable. From any computable copy

$$\mathcal{B} = (B, I', U', V', W', Q', P', <'),$$

we obtain an anti-Friedberg ω-f-enumeration H of $\mathcal{S}$ as follows. We identify the b^{th} element of B with b, and think of it as an index. The fact that H and G are strongly Δ^0_ω-equivalent gives us a Δ^0_ω function k mapping $1-1$ the G-indices of every function in $\mathcal{S}$ onto the H-indices for the same function. Using Δ^0_2, we can match the elements of U_a with those of $U'_{k(a)}$, and we can match the elements of $V_{a,t}$ with those of $V'_{k(a),t}$. To match the elements of $W_{a,t,x}$ with those of $W'_{k(a),t,x}$, we use the oracle Δ^0_ω. We have c.e. sets of computable infinitary formulas $\{\psi_\beta(x) : \beta < \omega^t \cdot 2\}$ for all t, where $\psi_\beta(x)$ says that the set of predecessors of x has order type β. We match each element of $W_{a,t,x}$ with the corresponding element of $W'_{k(a),t,x}$ satisfying the same formula. $\square$

We need a further fact about the structure $\mathcal{A}$, stated in terms of the standard "back-and-forth relations $\leq_\alpha$".

Definition 7. *Let $\mathcal{A}$ and $\mathcal{B}$ be structures, with tuples $\overline{a}$ in $\mathcal{A}$ and $\overline{b}$ in $\mathcal{B}$.*

1. *We have $(\mathcal{A},\overline{a}) \leq_1 (\mathcal{B},\overline{b})$ if the existential formulas true of $\overline{b}$ in $\mathcal{B}$ are true of $\overline{a}$ in $\mathcal{A}$.*
2. *For countable $\alpha > 1$, $(\mathcal{A},\overline{a}) \leq_\alpha (\mathcal{B},\overline{b})$ if for all $1 \leq \beta < \alpha$ and all $\overline{d}$ in $\mathcal{B}$, there exists $\overline{c}$ in $\mathcal{A}$ such that $(\mathcal{B},\overline{b},\overline{d}) \leq_\beta (\mathcal{A},\overline{a},\overline{c})$.*

By a well-known result of Karp, for any countable ordinal α, $(\mathcal{A},\overline{a}) \leq_\alpha (\mathcal{B},\overline{b})$ iff the Σ_α formulas (of $L_{\omega_1\omega}$) true of $\overline{b}$ in $\mathcal{B}$ are also true of $\overline{a}$ in $\mathcal{A}$. The back-and-forth relations for well orderings are known (see [3]).

Proposition 1.14. $\omega^t \cdot 2 \leq_{2t+1} \omega^t$

Returning to the structure $\mathcal{A}$, suppose $a, a' \in I$, where a, a' are indices for functions g, g' that agree on an initial segment of length T. Using the proposition above, it is easy to see that a and a' satisfy the same Σ_{2T} formulas in $\mathcal{A}$.

Lemma 1.15. *The structure $\mathcal{A}$ has no formally Σ^0_ω Scott family.*

Proof. Suppose there is a formally Σ^0_ω Scott family Φ, with parameters $\overline{c}$. The orbit of $\overline{c}$ is defined by a computable Σ_n formula for some n. Therefore, we may suppose that Φ has no parameters. If $a, a' \in I$ are indices for functions that agree on initial segment of length T, where $2T \geq n$, then $(\mathcal{A}, a) \leq_n (\mathcal{A}, a')$. Using Δ^0_ω, we can associate with each $a \in I$ a formula $\varphi(x)$ of Φ satisfied by a. In fact, we can find a disjunct of $\varphi(x)$, say $\varphi_a(x)$, which is computable Σ_n for some n. Choose T such that $2T \geq n$, and let p_a be the finite sequence of length T that is an initial segment of the function with G-index a. The set of these chosen p_a is Σ^0_ω, and it witnesses that $\mathcal{S}$ is ω-discrete. This is a contradiction. □

The above lemmas complete the proof of Theorem 1.10. □

1.6 Intrinsically Σ^0_ω but not relatively intrinsically Σ^0_ω relations

We now present our second main result, the analogue of Theorems 1.5 and 1.6, in the case when $\alpha = \omega$.

Theorem 1.16. *There is a computable structure $\mathcal{A}$ with an additional relation R that is intrinsically Σ^0_ω but not relatively intrinsically Σ^0_ω on $\mathcal{A}$.*

Proof. Let $f(t) = 2t + 1$. Let $\mathcal{S}$, $\mathcal{S}'$, and G be as in Theorem 1.9. Then $\mathcal{S}$ satisfies the conclusion of Theorem 1.8, with anti-Friedberg ω-f-enumeration G. Let

$$\mathcal{A} = (A, I, U, V, W, Q, P, <)$$

be the structure obtained from $\mathcal{S}$ and G as in Theorem 1.10. We effectively identify the elements of I with natural numbers, and we think of them as G-indices for functions in $\mathcal{S}$. Let $R \subseteq I$ be the set of G-indices for functions in $\mathcal{S}'$. Then R is Σ^0_ω. If $\mathcal{B}$ is a computable copy of $\mathcal{A}$, we have a Δ^0_ω isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$. Then $F(R)$ is Σ^0_ω. We have shown that R is intrinsically Σ^0_ω.

We will now show that R is not relatively intrinsically Σ^0_ω. Suppose it is. Then R is defined in $\mathcal{A}$ be a computable Σ_ω formula. As in the previous proof, the orbit of any tuple $\overline{c}$ is defined by a formula that is computable Σ_n for some n, so we may suppose that R is defined by a computable Σ_ω formula with no parameters. Using a Δ^0_ω oracle, we choose for each $a \in R$, a disjunct of $\varphi(x)$, say $\psi_a(x)$, which is satisfied by a. If $\psi_a(x)$ is computable Σ_m, we choose T such that $2T \geq m$, and we let p_a be the restriction to length T of the function with G-index a. The set Γ of these chosen p_a is Σ^0_ω. Each function in $\mathcal{S}'$ extends some element of Γ. By the properties of $\mathcal{S}$, $\mathcal{S}'$, and G in Theorem 1.9, there exist $p_a \in \Gamma$, with extensions $g \in \mathcal{S}'$ and $g' \in \mathcal{S} - \mathcal{S}'$. Say $\psi_a(x)$ is computable

Σ_m, and p_a has length T, where $2T \geq m$. Let b be a G-index for g'. Then b satisfies the Σ_m formulas true of a, including $\psi_a(x)$, so b must be in R. This is a contradiction. $\square$

2 Intrinsic complexity at arbitrary computable limit ordinals

In this section, we extend the results of the previous section to arbitrary computable limit ordinals. We refer to Kleene's system of ordinal notation $\mathcal{O}$, which is described in Rogers' classic text [11]. If α is a successor ordinal, then from the notation for α, we can find the notation for its predecessor. If α is a limit ordinal, then from the notation for α, we can effectively find a computable sequence of notations for ordinals α_n such that $\alpha_n < \alpha_{n+1}$ and $\lim_n \alpha_n = \alpha$.

We identify computable ordinals with their unique notations on a fixed path through $\mathcal{O}$. We can do some ordinal arithmetic effectively, without leaving our fixed path. Given the notation a for α, we can effectively find the notation for $\alpha + 1$—it is 2^a. We can also effectively find the notation for 2α. We show this by computable transfinite recursion. If α is either 0 or a limit ordinal, then $2\alpha = \alpha$, so the notation for α is the same as the notation for 2α. If $\alpha = \beta + 1$, a successor ordinal, then $2\alpha = 2\beta + 2$, so if b is the notation for β, then 2^{2^b} is the notation for 2α.

Definition 8 (Δ^0_α oracles). *To every computable ordinal α (or its notation on the fixed path through $\mathcal{O}$), we associate a Turing complete Δ^0_α set. We call this set Δ^0_α. For finite $n \geq 1$, we may let $\Delta^0_n = \emptyset^{(n-1)}$. For infinite α, we use $H(a)$, where a is our chosen notation for α.*

Definition 9. *Let α be a computable limit ordinal, and let $(\alpha_n)_{n\in\omega}$ be the increasing sequence with limit α, obtained from our notation for α. Let $\mathcal{S} \subseteq \omega^\omega$.*

1. *The family $\mathcal{S}$ is* α-discrete *if there is a Σ^0_α set $\Gamma \subseteq \omega^{<\omega}$ such that every $g \in \mathcal{S}$ extends some $p \in \Gamma$, and if $g, g' \in \mathcal{S}$ both extend $p \in \Gamma$, then $g = g'$.*

2. *Enumerations G and H are* strongly Δ^0_α-equivalent *if there is a Δ^0_α permutation k of ω such that $G(a,t) = H(k(a),t)$.*

3. *Let f be a computable sequence of notations (on the fixed path through $\mathcal{O}$) for a strictly increasing sequence of ordinals with limit α. An α-f-*enumeration *of $\mathcal{S}$ is an enumeration G such that for some fixed e, for all a and t, $G(a,t) = \varphi_e^{\Delta^0_{f(t)}}(a,t)$.*

We will now state the general case of Enumeration Theorem I (Theorem 1.8). The proof is the same as for the special case (Theorem 1.8), so we omit it.

Theorem 2.1 (Enumeration Theorem I). *Let f be a computable function giving the notations (on the fixed path through $\mathcal{O}$) for a strictly increasing sequence of ordinals with limit α. There is a family $\mathcal{S} \subseteq \omega^\omega$ such that:*

1. *$\mathcal{S}$ is discrete but not α-discrete,*

2. *$\mathcal{S}$ has a unique anti-Friedberg α-f-enumeration, up to strong Δ^0_α equivalence.*

Here is the generalization of Enumeration Theorem II (Theorem 1.9). Again, we omit the proof.

Theorem 2.2 (Enumeration Theorem II). *Let f be a computable function giving the notations for a strictly increasing sequence of ordinals with limit α. There is a family $\mathcal{S} \subseteq \omega^\omega$ with an α-f-enumeration G, which has the following properties:*

1. *every anti-Friedberg α-f-enumeration of $\mathcal{S}$ is strongly Δ^0_α-equivalent to G,*

2. *$\mathcal{S}$ is discrete,*

3. *there is a set $\mathcal{S}' \subseteq \mathcal{S}$ such that*

    (a) *the set of G-indices for elements of $\mathcal{S}'$ is Σ^0_α, and*

    (b) *for every Σ^0_α set $\Gamma \subseteq 2^{<\omega}$, if each $g \in \mathcal{S}'$ extends some $p \in \Gamma$, then some $p \in \Gamma$ has extensions g_1, g_2 with $g_1 \in \mathcal{S}'$ and $g_2 \in \mathcal{S} - \mathcal{S}'$.*

2.1 Δ^0_α-categoricity but not relative Δ^0_α-categoricity

We now present the general case of our first main result, extending Theorem 1.10 to arbitrary computable limit ordinals.

Theorem 2.3. *Let α be a computable limit ordinal. There is a structure $\mathcal{A}$ that is Δ^0_α-categorical but not relatively Δ^0_α-categorical.*

Proof. From the notation for α_t on the fixed path through $\mathcal{O}$, we can effectively compute the notation for $2\alpha_t$, and for $2\alpha_t + 1$. We let $f(t) = 2\alpha_t + 1$. Let $\mathcal{S}$ be as in Theorem 2.1, and let G be an anti-Friedberg α-f-enumeration. We let

$$\mathcal{A} = (A, I, U, V, W, Q, P, <),$$

where, as in the proof of Theorem 1.10, I, U, V, W partition A into infinite sets, Q maps U onto I, V onto U, and W onto V, and P acts as a predecessor function on every U_a and every $V_{a,t}$, making these sets into copies of ω. For the element playing the role of x in $V_{a,t}$, the relation $<$ is an ordering on $W_{a,t,x}$ of type

$$\begin{cases} \omega^{\alpha_t} \cdot 2 & \text{if } G(a,t) = x, \\ \omega^{\alpha_t} & \text{otherwise.} \end{cases}$$

The following lemma is in [1]. A related result, with $\mathbb{Z}$ replacing ω, is in [2].

Lemma 2.4. *Given a $\Delta^0_{2\alpha_t+1}$ index for a linear ordering $\mathcal{L}$, we can effectively pass to a computable index for an ordering of type $\omega^{\alpha_t} \cdot \mathcal{L}$.*

Lemma 2.5. *The structure $\mathcal{A}$ has a computable copy.*

Proof. Recall that $G(a,t)$ is computed by a uniform procedure using $\Delta^0_{2\alpha_t+1}$. Using $\Delta^0_{2\alpha_t+1}$, we can find an index for a 2-element ordering if $G(a,t) = x$, and a 1-element ordering otherwise. Applying Lemma 2.4, we get a uniformly computable family of linear orderings $\mathcal{L}_{a,t,x}$, having type $\omega^{\alpha_t} \cdot 2$ if $G(a,t) = x$ and ω^{α_t} otherwise. Using this family of orderings, we get a computable copy of the structure $\mathcal{A}$ described above. ☐

Lemma 2.6. *The structure $\mathcal{A}$ is Δ^0_α-categorical.*

Proof. For any computable copy

$$\mathcal{B} = (B, I', U', V', W', Q', P', <')$$

of $\mathcal{A}$, we get an anti-Friedberg α-f-enumeration H of $\mathcal{S}$. We think of elements of I' as indices. For each $a \in I'$, we have a set $U'_a \subseteq U'$, which we identify with the natural numbers. For each $t \in U'_a$, we have a set $V'_{a,t} \subseteq V'$, which we identify with the natural numbers. For each $x \in V'_{a,t,x}$, we have a set $W'_{a,t,x} \subseteq W'$ ordered by $<'$. Given a and t, and using $\Delta^0_{2\alpha_t+1}$, we can find the unique $x \in V'_{a,t}$ such that $W'_{a,t,x}$ has type $\omega^{\alpha_t} \cdot 2$, as opposed to ω^{α_t}. We let $H(a,t)$ be this x. The fact that H and G are strongly Δ^0_α-equivalent gives us a Δ^0_α function matching the G-indices in I with the H-indices in I'. Then, using Δ^0_α, we can match the rest of the structures. ☐

Lemma 2.7. *The structure $\mathcal{A}$ has no formally Σ^0_α Scott family.*

Proof. Suppose there is a formally Σ^0_α Scott family Φ. In principle, there may be a finite tuple $\overline{c}$ of parameters. However, the orbit of $\overline{c}$ is defined by a formula that is computable Σ_β for some $\beta < \alpha$, so we may suppose that Φ has no parameters. It is well known that $\omega^{\alpha_t} \cdot 2 \leq_{2\alpha_t+1} \omega^{\alpha_t}$ (see [3]). From this, it follows that if $a, a' \in I$ are indices for functions in $\mathcal{S}$ that agree on all t such that $2\alpha_t < \beta$, then $(\mathcal{A}, a) \leq_\beta (\mathcal{A}, a')$. Using an oracle for Δ^0_α, we can associate with each $a \in I$ a formula of Φ satisfied by a. In fact, we can find a disjunct of this formula, say $\varphi_a(x)$, which is computable Σ_β for some $\beta < \alpha$. Choose T such that $2\alpha_t > \beta$, and let p_a be the initial segment of length T for the function with G-index a. The set of these p_a's is Σ^0_α, and it witnesses that $\mathcal{S}$ is α-discrete, which is a contradiction. ☐

This completes the proof of Theorem 2.3. ☐

2.2 Intrinsically Σ^0_α but not relatively intrinsically Σ^0_α relations

We now present the general case of our second main result, extending Theorem 1.16 to arbitrary computable limit ordinals.

Theorem 2.8. *Let α be a computable limit ordinal. There is a computable structure $\mathcal{A}$ with an additional relation R that is intrinsically Σ^0_α but not relatively intrinsically Σ^0_α.*

Proof. Let f be as in the proof of Theorem 2.3; i.e., $f(t) = 2\alpha_t + 1$. Let $\mathcal{S}$, $\mathcal{S}'$, and G be as in Theorem 2.2. Let $\mathcal{A}$ be the structure obtained from $\mathcal{S}$ and G as in Theorem 2.3. Let $R \subseteq I$ consist of the set of G-indices for functions in $\mathcal{S}'$. Then R is Σ^0_α. If $\mathcal{B}$ is a computable copy of $\mathcal{A}$, we have a Δ^0_α isomorphism F from $\mathcal{A}$ onto $\mathcal{B}$. Then $F(R)$ is Σ^0_α. Therefore, R is intrinsically Σ^0_α.

Suppose $\varphi(\overline{c}, x)$ is a computable Σ_α formula. As in the proof of Theorem 2.3, we can define the orbit of $\overline{c}$ by a formula that is computable Σ_β for some $\beta < \alpha$, so we may assume that there are no parameters in the formula $\varphi(x)$. Using an oracle for Δ^0_α, we choose for each $a \in R$, a disjunct of $\varphi(x)$, say $\psi_a(x)$, which is satisfied by a. If $\psi_a(x)$ is computable Σ_β, we take T such that $2\alpha_T > \beta$, and we let p_a be the restriction of the function g with G-index a to T. Let Γ be the set of these p_a's. Then Γ is Σ^0_α. Since every function in $\mathcal{S}'$ extends some element of Γ, there exist $p_a \in \Gamma$ and $g \in \mathcal{S} - \mathcal{S}'$ such that $g \supseteq p_a$. Say b is a G-index for g. Then b must satisfy $\psi_a(x)$, but it does not, which is a contradiction. $\square$

3 Conclusion

In this section, we mention some problems that remain open. We also recall some definitions needed to state these problems .

Definition 10. *Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on $\mathcal{A}$.*

1. *The relation R is* intrinsically Δ^1_1 *on $\mathcal{A}$ if in every computable copy of $\mathcal{A}$, the image of R is Δ^1_1 (that is, Δ^0_α for some computable ordinal α).*

2. *The relation R is* relatively intrinsically Δ^1_1 *if for every isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is Δ^1_1 relative to $\mathcal{B}$.*

The following result is due to Soskov [12], and is re-worked in [9].

Theorem 3.1. *For an additional relation R on a computable structure $\mathcal{A}$, the following are equivalent.*

1. *The relation R is* intrinsically Δ^1_1.

2. *The relation R is* relatively intrinsically Δ^1_1*; in fact, there is a computable ordinal α such that R is relatively intrinsically Δ^0_α.*

3. *The relation R is definable by some computable infinitary formula $\varphi(\overline{c}, \overline{x})$ with a finite tuple $\overline{c}$ of parameters from $\mathcal{A}$.*

Thus, for relations, we have a fairly complete picture. For every computable ordinal α, being intrinsically Σ^0_α does not imply being relatively intrinsically Σ^0_α. However, intrinsically hyperarithmetical and relatively intrinsically hyperarithmetical relations are the same. A similar problem for hyperarithmetical categoricity is open.

Problem 1. *For a computable structure $\mathcal{A}$, are the following equivalent?*

1. *The structure $\mathcal{A}$ is Δ^1_1-*categorical*; i.e., for all computable copies $\mathcal{B}$ of $\mathcal{A}$, there is a Δ^1_1 isomorphism.*
2. *The structure $\mathcal{A}$ is* relatively Δ^1_1-categorical*. (This implies that $\mathcal{A}$ is relatively Δ^0_α-categorical for some computable ordinal α.)*

We now recall the definition of effective stability of structures.

Definition 11. *Let $\mathcal{A}$ be a computable structure.*

1. *The structure $\mathcal{A}$ is Δ^0_α* stable *if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, all isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$ are Δ^0_α.*
2. *The structure $\mathcal{A}$ is* relatively Δ^0_α stable *if for every copy $\mathcal{B}$ of $\mathcal{A}$, all isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$ are Δ^0_α relative to $\mathcal{B}$.*

Goncharov's example [7] of a computable structure that is computably categorical but not relatively computably categorical is rigid. Therefore, the structure is computably stable but not relatively computably stable. In [8], it was asked whether this result lifts to computable successor ordinals. We may also ask the question for computable limit ordinals.

Problem 2. *For computable ordinals $\alpha > 1$, is there a computable structure $\mathcal{A}$ that is Δ^0_α stable but not relatively Δ^0_α stable?*

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