On Conrad's property for group orderings

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Crossings

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- $u \prec w \prec v$,
- $g^n u \prec v$ and $f^n v \succ u$ for every $n \in \mathbb{N}$,
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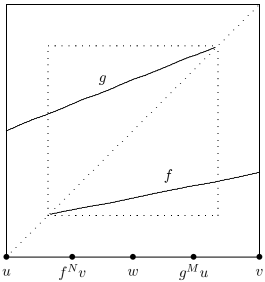

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– Hence, the space of Conradian orderings is closed [N].

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This implies that $g^{-1}fg^2$ is positive. $\square$

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Theorem [N]. A group Γ admits a Conradian order if and only if the following condition is satisfied: for every finite family of elements $g_1, \ldots, g_k$ which are different from the identity, there exists a family of exponents $\varepsilon_i \in \{-1, 1\}$ such that *id* does not belong to the smallest semigroup $\langle\langle g_1^{\varepsilon_1}, \ldots, g_k^{\varepsilon_k} \rangle\rangle$ which simultaneously satisfies the following two properties:

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Remark. The condition above is easy to test on locally indicable groups. Hence, these groups are C-orderable [Broskii, Rhemtulla-Rolfsen].

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– Think on a dynamical robust property and take its complement.

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In particular,

$$g \prec_{f^{-1}} id \quad \text{iff} \quad g(f) \prec f.$$

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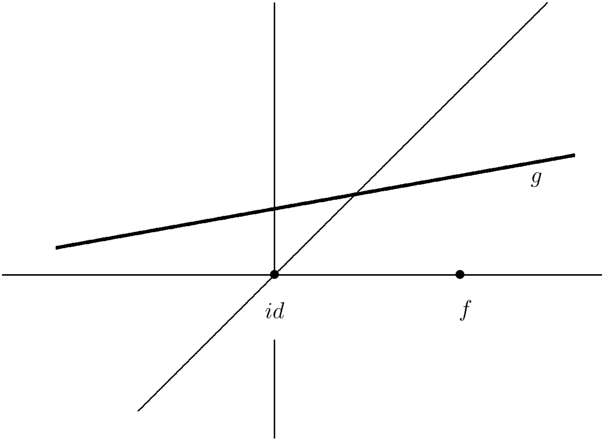

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given finitely many $g_i \succ id$, there exists $f \in \Gamma$ such that $g_i \succ_{f^{-1}} id$, but $g \prec_{f^{-1}} id$ for some $g \succ id$.

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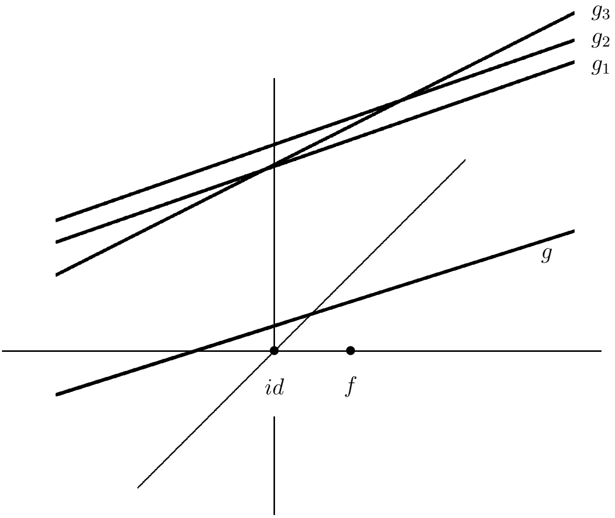

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Theorem [N, N-Rivas, Clay]. If the Conradian soul of an ordering is trivial, then this ordering is an accumulation point of its conjugates.

Approximation of Conradian orderings: algebra

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Consequence. If a group Γ has an isolated ordering with a convex Conradian subgroup, then this subgroup has finitely many orderings.

Proof. Use the convex extension procedure.

The intermediate case is subtle: two relevant examples

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This leads to an alternative proof of the fact that no space of group orderings can be countably infinite [Linnell].

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Corollary [N]. Dehornoy's ordering is an accumulation point of its conjugates.

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Problem. Give a concrete sequence of distinct orderings having $\preceq_C$ as a limit point.