On Automorphisms of Structures in Logic

and

Orderability of Groups in Topology

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Contents

List of Figures

Dedication

I dedicate this work to Ali Sabetian who opened my eyes to the joys of math.

And has not such a Story from of Old
Down Man's successive generations roll'd
    Of such a clod of saturated Earth
Cast by the Maker into Human mould?

For "Is" and "Is-not" though with Rule and Line
And "Up-and-down" by Logic I define,
    Of all that one should care to fathom, I
Was never deep in anything but–Wine.

Up from Earth's Centre through the Seventh Gate
I rose, and on the Throne of Saturn sate,
    And many a Knot unravel'd by the Road;
But not the Master-knot of Human Fate.

Omar Khayyam, Persian mathematician (1048–1122)

(Quoted from Edward FitzGerald's translation of Omar Khayyam's poetry)

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Abstract

We investigate properties of non-standard models of set theory, in particular, the *countable recursively saturated* models while having the automorphisms of such models in mind. The set of automorphisms of a model forms a group that in certain circumstances can give information about the model and even recover the structure of the model. We develop results on conditions for the existence of automorphisms that fix a given initial segment of a countable recursively saturated model of ZF.

In certain cases an additional axiom $\mathbf{V} = \mathbf{OD}$ will be needed in order to establish analogues of some results for models of Peano Arithmetic. This axiom will provide us with a definable global well-ordering of the model. Models of set theory do not automatically possess such a well-ordering, but a definable well-ordering is already in place for models of Peano Arithmetic, that is, the natural order of the model.

We investigate some finitely presentable groups that arise from topology. These groups are the fundamental groups of certain manifolds and their or-

derability properties have implications for the manifolds they come from. A group $\langle G, \circ \rangle$ is called left-orderable if there is a total order relation $<$ on G that preserves the group operation $\circ$ from the left.

Finitely presentable groups constitute an important class of finitely generated groups and we establish criteria for a finitely presented group to be non-left-orderable. We also investigate the orderability properties for Fibonacci groups and their generalizations.

Chapter 1

Model-theoretic Preliminaries

Let $\mathcal{M}$ be a model in a first-order language $\mathcal{L}$ with universe M. A bijection f on M is an automorphism of $\mathcal{M}$ if f preserves the structure of $\mathcal{M}$. More specifically, if $\mathcal{M} = \langle M, E \rangle$ is a model in the language of set theory, $\{\in\}$, then a bijection $f : M \to M$ is an *automorphism* of $\mathcal{M}$ if for all $x, y \in M$, we have

$$xEy \text{ if and only if } f(x)Ef(y).$$

1.1 The Automorphism Group

Let $\mathcal{M}$ be a model in a first-order language $\mathcal{L}$. The collection of all automorphisms of $\mathcal{M}$, $Aut(\mathcal{M})$, together with the operation of composition of functions forms a group. This is true, because the identity function on M is an automorphism, the composition of two automorphisms of a model is an automorphism, and the inverse of an automorphism is also an automorphism.

Therefore, as is customary in mathematics, one can think of $Aut(\mathcal{M})$ as an invariant of the model $\mathcal{M}$. As is usual about invariants, a main question will be: How much information about a model $\mathcal{M}$ can one extract from its automorphism group? For example, under what circumstances one can say that models with isomorphic automorphism groups are themselves isomorphic?

There have been recent developments in the realm of models of the Quine-Jensen set theory NFU. More specifically, Enayat [E] has refined the work of Holmes [Holm] and Solovay [S] by establishing a close relationship between Mahlo cardinals and automorphisms of models of set theory. In particular, we have the following

Theorem 1.1.1 *Let $\Phi = \{\varphi_n : n \in \omega\}$, where φ_n asserts the existence of a Σ_n-reflecting n-Mahlo cardinal. Suppose T is a completion of ZFC +* **V=OD***. The following are equivalent.*

(1) $T \vdash \Phi$.

(2) Some model $\mathcal{M} \models T$ has an automorphism f such that the set of fixed points of f is a proper initial segment of $\mathcal{M}$.

Remark 1.1.2 *Over ZFC, Φ is equivalent to the scheme which says, for each formula $\varphi(x)$, and each standard natural number n, "If $\varphi(x)$ defines a closed unbounded subset of ordinals, then $\varphi(\kappa)$ holds for some n-Mahlo cardinal κ".*

We should also note that automorphism groups admit a topology in a very natural way. Recall that a subset X of a group G is said to be a *translation* of $Y \subseteq G$, if for some $g \in G$ we have

$$X = gY =_{\text{def}} \{gy : y \in Y\}$$

Now let $\mathcal{M}$ be as above. For a finite sequence $\bar{a} = \langle a_0, a_1, \ldots, a_{n-1} \rangle$ of elements of M we define $U_{\bar{a}}$ to be the set of all automorphisms of $\mathcal{M}$ that fix all a_i, for $i < n$. In symbols,

$$U_{\bar{a}} =_{\text{def}} \{f \in Aut(\mathcal{M}) : \forall i < n(f(a_i) = a_i)\}.$$

A basis for the topology on $Aut(\mathcal{M})$ is then defined as the set of all translations of the sets of the form $U_{\bar{a}}$, where $\bar{a}$ runs over all possible finite sequences in M. This way, the sets $U_{\bar{a}}$ will be the only open subgroups of $Aut(\mathcal{M})$.

Under appropriate assumptions, there are interesting connections between the size of $U_{\bar{a}}$ and the recursion-theoretic properties of the isomorphic type of the structure $\mathcal{M}$ [HKM].

Note that when $\mathcal{M}$ is a finite model, this topology is simply the discrete topology on $Aut(\mathcal{M})$, and therefore of no interest. In any case the topology thus introduced, turns $Aut(\mathcal{M})$ into a topological group. Interestingly, in some occasions, it is possible to recover the topology of $Aut(\mathcal{M})$ merely from the algebraic structure of $\mathcal{M}$.

We say a topological group G has *the small index property* if for all subgroups $H \leq G$, H is open if and only if $[G : H] \leq \aleph_0$. We say a model $\mathcal{M}$ has *the small index property* if so does $Aut(\mathcal{M})$ as a group. For example, the group of of permutations on a set, the rational numbers with their usual ordering $\langle \mathbb{Q}, < \rangle$, and countable atomless Boolean algebras have small index property.

To answer the question of how much information about $\mathcal{M}$ can be derived from $Aut(\mathcal{M})$, we may want to examine special cases. For example, what if $\mathcal{M}$ is *rigid*, that is, the only automorphism of $\mathcal{M}$ is given by the identity map, that is always an automorphism.

Since all definable elements of a model $\mathcal{M}$ are fixed by all its automorphisms a model whose all elements are definable is bound to be rigid. The converse of this, however, is not true. For example, if $\mathcal{A}$ is the model $\langle \mathbb{Q}, <, c_n \rangle_{n \in \omega}$, where $\mathbb{Q}$ is the set of rational numbers, $<$ is the usual ordering of the rationals and c_n is interpreted as $1/n$, then every automorphism of $\mathcal{A}$ should fix 0, while 0 is not $\mathcal{L}_{\omega\omega}$-definable in $\mathcal{A}$.

However, a result by Scott [Sco-2] provides a complete answer to this question when $\mathcal{M}$ is a countable model. In the following definition we think of a language as a set of well-formed formulas.

Definition 1.1.3 *Let $\mathcal{L}$ be a first-order language. By $\mathcal{L}_{\omega_1\omega}$ we denote the smallest set of strings in $\mathcal{L}$'s alphabet that includes all $\mathcal{L}$-formulas, as well as string of symbols of the form $\bigwedge_{n<\omega} \phi_n$ and $\bigvee_{n<\omega} \phi_n$, whenever ϕ_n are given formulas in $\mathcal{L}_{\omega_1\omega}$.*

Definition 1.1.4 *Let $\mathcal{M}$ be a countable model. A* Scott sentence *for $\mathcal{M}$ is a sentence whose countable models are all isomorphic to $\mathcal{M}$.*

Such sentences are called after Dana Scott because of the following result.

Theorem 1.1.5 [Sco-2] *Every countable model in a countable language has a Scott sentence.*

We have the following corollaries.

Corollary 1.1.6 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable models in a countable language $\mathcal{L}$ is a countable language that satisfy the same $\mathcal{L}_{\omega_1\omega}$-sentences. Then $\mathcal{M} \cong \mathcal{N}$.*

By induction on the complexity of $\mathcal{L}_{\omega_1\omega}$ formulas, it can be shown that definable elements of a model $\mathcal{M}$ are fixed by all automorphisms of $\mathcal{M}$. The converse of this, however, is not necessarily true. However, thanks to Scott sentences, we have the following.

Corollary 1.1.7 *Let $\mathcal{M}$ be a countable model in a countable language. Then $a \in M$ is $\mathcal{L}_{\omega_1\omega}$-definable in $\mathcal{M}$ if and only if $f(a) = a$ for all $f \in Aut(\mathcal{M})$.*

Proof. Let $a \in M$ and suppose $f(a) = a$ for all $f \in Aut(\mathcal{M})$. Consider the new countable model $\mathcal{N} = \langle \mathcal{M}, c \rangle$, where c is a new constant to be interpreted in $\mathcal{N}$ as a. Let $\varphi(x)$ be an $\mathcal{L}_{\omega_1\omega}$-formula with one variable in $\mathcal{L}_{\mathcal{M}}$, the language of our original model, such that $\varphi(c)$ is a Scott sentence for $\mathcal{N}$. We claim that φ is defines a.

Suppose $\mathcal{M} \models \varphi(c)[b]$, where $b \in M$. This means $\mathcal{N}' \models \varphi(c)$, where c is interpreted in $\mathcal{N}' = \langle \mathcal{M}, c \rangle$ as b. Since $\varphi(c)$ is a Scott sentence of $\mathcal{N}$ we have $\mathcal{N} \cong \mathcal{N}'$. Let f be one such isomorphism. So we have

$$f(a) = f(c^{\mathcal{N}}) = f(c^{\mathcal{N}'}) = f(b).$$

On the other hand, since $\mathcal{M}$ is a reduct of both $\mathcal{N}$ and $\mathcal{N}'$, the map f is also an automorphism of $\mathcal{M}$, and therefore $f(a) = a$. Therefore $a = b$, that is, φ does define a, as claimed. The other direction is proved by a simple induction on the complexity of formulas. ■

Corollary 1.1.8 *Suppose a countable model $\mathcal{M}$ in a countable language $\mathcal{L}$ is rigid. Then all elements of $\mathcal{M}$ are definable by $\mathcal{L}_{\omega_1\omega}$-formulas.*

1.2 Saturation, Categoricity, and Homogeneity

Definition 1.2.1 *A model $\mathcal{M}$ is* saturated *if for every subset $A \subseteq M$ that is not of the same cardinality of the model $\mathcal{M}$ itself, if we add new constants c_a to our language for all $a \in A$ and interpret each c_a as the corresponding $a \in A$, then the expanded model $\langle \mathcal{M}, c_a \rangle_{a \in A}$ realizes every type $p(\bar{x})$ of the new language which is not inconsistent with $Th(\langle \mathcal{M}, c_a \rangle_{a \in A})$ to begin with.*

Obviously, whenever $\mathcal{M}$ is countable, then it is saturated if and only if every type $p(\bar{x}, \bar{a})$ that is *finitely* realizable in $\mathcal{M}$ is realizable in $\mathcal{M}$. We will be more interested in countable, rather than arbitrarily large, models, for the simple reason that they are easier to handle, not to mention that if we are to gain an understanding of the general situation, then starting with countable models seems to be a natural starting point.

Every consistent theory T with infinite models has a saturated model of arbitrarily large cardinalities. In fact, given a model $\mathcal{M}$ of a theory T, one can use the Compactness Theorem and build an elementary chain of models, starting with $\mathcal{M}$ and thus construct a saturated model $\mathcal{N} \succ \mathcal{M}$. (We will see a similar argument below in the proof for the existence of recursively saturated models in more detail.) However, as it turns out, not all theories (with infinite models) have *countable* saturated models. Nonetheless, an important class of theories do have countable saturated models.

Definition 1.2.2 *Let κ be an infinite cardinal. A theory T in a countable language $\mathcal{L}$ is* κ-categorical *if there is exactly one model $\mathcal{M} \models T$ (up to isomorphism).*

Proposition 1.2.3 *If a theory T in a countable language $\mathcal{L}$ is $\aleph_0$-categorical, then it has a countable saturated model.*

We also have the following

Theorem 1.2.4 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable saturated models and $\mathcal{M} \equiv \mathcal{N}$. Then $\mathcal{M} \cong \mathcal{N}$.*

The proof is a straightforward generalization of Cantor's back-and-forth proof of the $\aleph_0$-categoricity of the theory of dense linear orderings with no endpoint. We enumerate both models and use finite initial parts of the enumerations as parameters for the types of elements in one model that will be realized in the other model. The resulting correspondence between the elements of $\mathcal{M}$ and $\mathcal{N}$ is in fact an isomorphism of models.

Now we introduce and discuss the notion of homogeneity and its rôle in the study of recursively saturated models. In order to motivate the idea of homogeneous models we start by noticing that any order-preserving partial function on the set of rationals with finite domain can be extended to an order-preserving bijection of $\mathbb{Q}$. In general, we have the following definition.

Definition 1.2.5 *A model $\mathcal{M}$ is said to be* ω-homogeneous *if every partial elementary map on M with finite domain can be extended to an automorphism of $\mathcal{M}$. If $\mathcal{M}$ is countable, we simply say $\mathcal{M}$ is* homogenous *if it is ω-homogeneous.*

It turns out that some interesting classes of countable models are homogeneous. In particular we will be interested in countable recursively saturated models, to be defined later, will show that such models are homogeneous, and will point out why that is a useful property.

In the next section we will discuss a slightly different notion of "saturation" that will prove to be extremely useful. We will also offer justifications for why the new notion is important.

1.3 Recursively Saturated Models

Recursively saturated models exhibit interesting structural properties that are worth investigating. For example, although countable saturated models of a given infinite cardinality do not necessarily exist, even when the language under consideration is countable, countable recursively saturated models exist as long as the minimum requirement is satisfied, whenever the language is recursive. For our purposes it suffices to take the language to be finite.

Definition 1.3.1 *Let M be the domain of a model $\mathcal{M}$ in a first-order language*

$\mathcal{L}$. *Let* $p(\bar{x}, \bar{a})$ *be a* type*, that is, a set of formulas in* $\mathcal{L}$ *where* $\bar{x}$ *is a finite tuple of variables and* $\bar{a}$ *is a finite tuple of parameters from* M. *We say the model* $\mathcal{M}$ *realizes the type* p *if for some* $\bar{b} \in M$ *of the same length of* $\bar{x}$ *we have*

$$\mathcal{M} \models \phi(\bar{b}, \bar{a}) \text{ for all } \phi(\bar{x}, \bar{a}) \in p(\bar{x}, \bar{a}).$$

A type p *is said to be* finitely realizable *in* $\mathcal{M}$ *if* $\mathcal{M}$ *realizes all the finite subsets of* p. *We say a type* $p(\bar{x}, \bar{a})$ *is* recursive *if* $\{\ulcorner \phi(\bar{x}, \bar{y}) \urcorner : \phi \in p\}$ *is recursive, where* $\ulcorner . \urcorner$ *is a Gödel numbering of formulas of* $\mathcal{L}$. *The model* $\mathcal{M}$ *is said to be* recursively saturated *if it realized all its finitely realizable recursive types.*

Recall that a subset A of ω is said to be recursive or computable if its characteristic function is computable, that is, if there is an algorithm that decides which natural numbers are in A. In other words, given $n \in \omega$, this algorithm should answer the question "Is $n \in A$?". Even though the above definition of recursively saturated models seems to be strongly dependent on the notion of recursiveness, it turns out that there are equivalent definitions not using the notion of recursive sets.

But first note that, thanks to Craig's trick, it will not make any difference if we require all finitely realizable recursively enumerable types to be realized, because each such type is equivalent to a recursive type, in the following sense.

Proposition 1.3.2 (Craig's trick) *Let* $p(\bar{x})$ *be a recursively enumerable set of formulas in a recursive language* $\mathcal{L}$. *Then there is a recursive set* $q(\bar{x})$ *of* $\mathcal{L}$-*formulas such that for every* $\mathcal{L}$-*structure* $\mathcal{M}$ *and every* $\bar{a} \in M$, *we have*

$$\mathcal{M} \models \varphi(\bar{a}) \text{ for all } \varphi \in p \text{ if and only if } \mathcal{M} \models \varphi(\bar{a}) \text{ for all } \varphi \in q$$

Proof. Enumerate $p(\bar{x}) = \{\varphi_1, \varphi_2, \dots\}$ and set $q(\bar{x}) = \{\psi_1, \psi_2, \dots\}$, where

$$\psi_n =_{\text{def}} \varphi_1 \wedge \varphi_2 \wedge \dots \varphi_n.$$

■

Before going any further, we would like to mention two results that point to the structural "richness" of recursively saturated models [KM].

Theorem 1.3.3 *Let $\mathcal{M}$ be a countable recursively saturated model. Then the group $Aut(\langle \mathbb{Q}, < \rangle)$ embeds in $Aut(\mathcal{M})$. Moreover, the free group with continuum many generators, $F(2^{\aleph_0})$, embeds* densely *in $Aut(\mathcal{M})$.*

Recursively saturated models pop up in various places and admit interesting characterizations. For example, the following is a characterization of these models in the context of arithmetic.

Definition 1.3.4 Presburger arithmetic *is the theory with the following axioms in the language $\{0, 1, +, <\} \cup \{\equiv_n : n \in \omega\}$.*

(i) The axioms of discretely ordered abelian monoids with 0 as the identity element, and 1 as the smallest nonzero element.

(ii) The ordering $<$ is left-invariant, that is, $\forall x, y, z(x < y \rightarrow z + x < z + y)$.

$(iii)_n$ $\forall x \exists y(x = ny \vee x = ny + 1 \vee \dots \vee x = ny + n - 1)$

$(iv)_n$ $\forall x, y[x \equiv_n y \leftrightarrow \exists z(x = y + nz \vee y = x + nz)]$.

Obviously, $\langle \omega, 0, 1, +, <, \equiv_n \rangle_{n \in \omega}$, where $<$ is the natural order on ω and $\equiv_n$ is interpreted as the congruence relation modulo n, is a model of Presburger arithmetic as well as a model of Peano arithmetic. We call it the *standard* model. For other models of Presburger arithmetic, we have the following result.

Theorem 1.3.5 [LN] *If $\mathcal{M} = \langle M, 0, 1, +, <, \equiv_n \rangle$ is a countable non-standard model of Presburger arithmetic, then we can define a multiplication $\cdot$ on M such that $\langle M, 0, 1, +, \cdot \rangle$ is a model of PA if and only if $\mathcal{M}$ is recursively saturated.*

The following is a characterization of recursively saturated models that shows these models can be described without appealing to the notion of recursive sets.

First, we need a fact from Set Theory.

Theorem 1.3.6 *(The Reflection Theorem)* [Ku] *Let $\mathcal{M}$ be a model of ZF and φ a formula in the language of ZF. Then*

$$\forall \alpha \exists \beta > \alpha (R(\beta)^{\mathcal{M}} \models \varphi \leftrightarrow \mathcal{M} \models \varphi).$$

Definition 1.3.7 *A model $\mathcal{M}$ of set theory is called ω-standard if we do not have $\omega^{\mathcal{M}} \cong \omega$.*

Otherwise we say $\mathcal{M}$ is non-ω-standard.

Theorem 1.3.8 *A model $\mathcal{M}$ of ZF is recursively saturated if and only if it is non-ω-standard and there are arbitrarily large $\mathcal{M}$-ordinals α such that $R(\alpha)^{\mathcal{M}}$ is an elementary submodel of $\mathcal{M}$.*

Proof. Suppose $\mathcal{M} \models ZF$ is recursively saturated. For each formula φ, let $\tilde{\varphi}(x)$ be the formula in the language of ZF that states

$$\text{“}x \text{ is an ordinal} \wedge R(x) \prec \mathcal{M}\text{”}.$$

Note that the set $\{\ulcorner \tilde{\varphi}(x) \urcorner : \varphi\}$ is recursive and the corresponding type is finitely satisfiable, because if we take finitely many formulas $\varphi_1, \ldots, \varphi_n$, then we may simply invoke the Reflection Theorem to find an ordinal α in $\mathcal{M}$ such that $R(\alpha) \prec_\varphi \mathcal{M}$, where φ is $\varphi_1 \wedge \cdots \wedge \varphi_n$.

Since $\mathcal{M}$ is assumed to be a recursively saturated model, the type $\{\tilde{\varphi}(x) : \varphi\}$ is realizable by an $\mathcal{M}$-ordinal α. This means $R(\alpha) \prec \mathcal{M}$, because of how we defined $\tilde{\varphi}$. We are now only one step away from having arbitrarily large ordinals α such that $R(\alpha) \prec \mathcal{M}$. We just need to apply the Reflection Theorem. The fact that $\mathcal{M}$ is not ω-standard is a direct consequence of $\mathcal{M}$ being recursively saturated, because the type $\{x \in \omega\} \cup \{x > n : n \in omega\}$, where $n =_{\text{def}} 1 + 1 + \cdots + 1$, is finitely realizable in ω-standard models, but not realizable.

To prove the other direction, suppose $p(x)$ is a finitely realizable *recursive* type with finitely many parameters and $\mathcal{M} \models ZF$ is not ω-standard and, further, for every $\mathcal{M}$-ordinal α, there is an $\mathcal{M}$-ordinal β where $\mathcal{M} \models \beta > \alpha$ and $R(\beta)^{\mathcal{M}} \prec \mathcal{M}$. We claim that $p(x)$ is realizable in $\mathcal{M}$.

Suppose the $\mathcal{M}$-ordinal α is so large that all parameters that occur in p are in $R(\alpha)^{\mathcal{M}}$ and let β be an $\mathcal{M}$-ordinal such that $\mathcal{M} \models \beta > \alpha$ and $R(\beta)^{\mathcal{M}} \prec \mathcal{M}$. Let $\tilde{p}(x)$ be the type in $\mathcal{M}$ that is given by the same algorithm that generates $p(x)$ in the real universe. Now $\tilde{p}(x)$ will be finitely realizable inside $R(\beta)^{\mathcal{M}}$.

Since $\{\varphi_i \in \tilde{p}(x) : i < n\}$ is realizable in $R(\beta)^{\mathcal{M}}$ for all $n \in \omega$, by overspill there exists some *non-standard* $H \in \omega^{\mathcal{M}}$ such that $\{\varphi_i \in \tilde{p}(x) : i < H\}$ is realizable in $R(\beta)^{\mathcal{M}}$. Since

$$p(x) = \{\varphi_n \in \tilde{p}(x) : n \text{ is a standard natural number }\} \subseteq \{\varphi_i \in \tilde{p}(x) : i < H\}.$$

Thus $p(x)$ is realizable in $\mathcal{M}$ and we are done. ■

Corollary 1.3.9 *If $\mathcal{M} \models ZF$ is a recursively saturated model, then all its cofinal elementary extensions are also recursively saturated, but an elementary end extension of $\mathcal{M}$ need not be recursively saturated.*

Obviously, every saturate d model is recursively saturated, but the converse is far from true. In particular, an advantage of recursively saturated models vis-à-vis saturated models comes from the following

Proposition 1.3.10 (Existence of Recursively Saturated Models)*) Let $\mathcal{M}$ be an arbitrary model in a recursive language. Then there is a model $\mathcal{N} \succ \mathcal{M}$ that is recursively saturated and of the same cardinality of $\mathcal{M}$.*

Therefore, thanks to Löwenheim-Skolem theorem every consistent theory in a recursive language $\mathcal{L}$ has a countable recursively saturated model, while there is no such guarantee in the case of saturated models.

On the other hand, while consistent theories can have at most one countable saturated model up to isomorphism, countable recursively saturated models need not be unique. However, in some cases this defect can be somewhat remedied.

Definition 1.3.11 *Let $\mathcal{L}$ be a recursive language. A theory T in $\mathcal{L}$ is* rich *if there is a recursive sequence $\{\varphi_n(\bar{x}) : n \in \omega\}$ of $\mathcal{L}$-formulas such that for all disjoint finite sets $X, Y \subset \omega$*

$$T \vdash \exists \bar{x}(\bigwedge_{n \in X} \varphi_n(\bar{x}) \wedge \bigwedge_{n \in Y} \neg\varphi_n(\bar{x}))$$

A set A is coded *in a model $\mathcal{M}$ of T if there is some $\bar{a} \in M$ such that*

$$A = \{n : \mathcal{M} \models \varphi_n(\bar{a})\}$$

The standard system *of $\mathcal{M}$, denoted $SSy(\mathcal{M})$, is defined to be the set of all coded subsets of ω that are coded in $\mathcal{M}$.*

Note that no $\aleph_0$-categorical theory can be rich, because any rich theory has continuum many n-types for some n.

Lemma 1.3.12 *Let T be a rich theory and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be elementary equivalent countable recursively saturated models of T such that $SSy(\mathcal{M}_1) = SSy(\mathcal{M}_2)$. Then $\mathcal{M}_1 \cong \mathcal{M}_2$.*

Standard systems have a close connection with Scott sets [Sco-1].

Definition 1.3.13 *Let $(t_n)_{n\in\omega}$ be a fixed computable* 1-1 *enumeration of all the elements of $2^{<\omega}$. We shall identify a sequence t_n with its index n. For a set $A \subseteq \omega$ we say T_A is a* tree *if the set of $\{t_n : n \in A\}$ is closed under initial segments in $2^{<\omega}$. A* path *P on a tree T is a maximal linearly ordered set of elements of T. For a tree T, let $[T]$ denote the set of all* infinite *paths on T.*

Definition 1.3.14 *A* Scott set *is a nonempty family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ such that for all $A, B \subseteq \omega$*

(i) $(A \in \mathcal{S} \wedge B \in \mathcal{S}) \rightarrow (A \cap B \in \mathcal{S} \wedge A \cup B \in \mathcal{S})$,

(ii) $(A \in \mathcal{S} \wedge B$ *is Turing-reducible (see below) to* $A) \rightarrow B \in \mathcal{S}$,

(iii) $(A \in \mathcal{S} \wedge T_A$ *an infinite tree* $) \rightarrow (\exists P \in \mathcal{S})(P \in [T_A])$.

Let A and B be subsets of ω. We say A is *Turing-reducible* to B, or A is *recursive in* B, if given an algorithm for deciding what natural numbers are in

B and which ones are not, we can find and algorithm for deciding which natural numbers are in A and which ones are not. (In other words, A is Turing-reducible to B if and only if $\chi_A = \varphi_e{}^{\chi_B}$ for some $e \in \omega$.)

Another way to look at condition (iii) of the previous definition is that a consistent set of axioms in $\mathcal{S}$ has a completion in $\mathcal{S}$. For example, the set of all arithmetical sets is a Scott set.

Theorem 1.3.15 *(Scott* [Sco-1]*) If $\mathcal{M}$ is a model of PA, then $SSy(\mathcal{M})$ is a Scott set. Conversely, for any countable Scott set $\mathcal{S}$, there is a nonstandard model $\mathcal{M}$ of PA such that $\mathcal{S} = SSy(\mathcal{M})$.*

We end this section by yet another fascinating characterization of recursively saturated models.

Theorem 1.3.16 *Let $\mathcal{M}$ be a model in a recursive language. The $\mathcal{M}$ is recursively saturated if and only if there is a non-ω-standard model $\mathcal{N}$ of ZF such that $\mathcal{M} \in \mathcal{N}$.*

Chapter 2

Automorphisms of Models of Set Theory

2.1 Friedman's Theorem

Harvey Friedman [Fr] proved the following result for the countable recursively saturated models of Peano Arithmetic that opened an era of "back-and-forth" results for models of PA and ZF.

Theorem 2.1.1 *Let $\mathcal{M} \models PA$ be countable and recursively saturated. Then a proper initial segment of $\mathcal{M}$ is isomorphic to $\mathcal{M}$.*

In this chapter we show that a similar result can be formulated and proved for models of ZF set theory. In fact, our formulation can be further generalized, but that would substantially complicate the proof. In particular, we do not really need our model to be non-ω-standard.

Theorem 2.1.2 *Let $\mathcal{M} = \langle M, E\rangle \models ZF$ be countable and non-ω-standard. Then there is $A \subset M$ such that*

(i) $\forall x \in M \forall y \in A(\mathcal{M} \models \rho(x) < \rho(y) \to x \in A)$*;*

(ii) $\mathcal{M} \cong \langle A, E \mid_{A\times A}\rangle$*.*

The theorem follows from several lemmas, but first of all, we need some definitions. Note that with some extra effort one can show that the above theorem is true even when $\omega^{\mathcal{M}} \cong \omega$, as long as $\mathcal{M}$ is non-standard. Also Schlipf has proved that if $\mathcal{M}$ is a countable non-standard model of ZF, then there is an $\mathcal{M}$-ordinal α such that not only $R(\alpha)^{\mathcal{M}} \cong \mathcal{M}$, but also $R(\alpha)^{\mathcal{M}} \prec \mathcal{M}$.

Note that PA is a rich theory and we have already proved a special case of this lemma when $T = PA$.

Definition 2.1.3 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of T, where $T = PA$ or $T = ZF$. We write $\mathcal{M} \prec_e \mathcal{N}$ and say $\mathcal{N}$ is an* elementary end extension *of $\mathcal{M}$ if*

(i) $\mathcal{M} \prec \mathcal{N}$*;*

(ii) $(\forall x \in N)(\forall y \in M)[\mathcal{N} \models xRy \to x \in M]$*, where R is the order relation symbol when $T = PA$ and it is the membership symbol $\in$ when $T = ZF$.*

Definition 2.1.4 *Let $\mathcal{M}$ and $\mathcal{N}$ be models of set theory and $\mathcal{M} \prec \mathcal{N}$. Let $\rho_{\mathcal{M}}$ and $\rho_{\mathcal{N}}$ be the rank functions of these models. Since $\mathcal{M} \prec \mathcal{N}$, we will have $\rho_{\mathcal{M}}(x) = \rho_{\mathcal{N}}(x)$ for all $x \in M$. We write $\mathcal{M} \prec_r \mathcal{N}$ to mean the model $\mathcal{N}$ is an elementary* rank extension *of $\mathcal{M}$, if*

$$(\forall x \in N)(\forall y \in M)[\mathcal{N} \models \rho(x) < \rho(y) \rightarrow x \in M].$$

In general, a model $\mathcal{N}$ of ZF can be an elementary rank extension of one of its submodels $\mathcal{M}$, but fail to be an elementary end extension of it. However, we have the following.

Proposition 2.1.5 *Let $\mathcal{M}$ and $\mathcal{N}$ be models of ZF such that $\mathcal{M} \prec_e \mathcal{N}$. Then $\mathcal{M} \prec_r \mathcal{N}$.*

Lemma 2.1.6 *Suppose $\mathcal{M}$ is a countable model of arithmetic or set theory. Then $\mathcal{M}$ has a countable elementary end extension. That is, there is a countable model $\mathcal{N}$ such that $\mathcal{M} \prec_e \mathcal{N}$.*

There are different methods of establishing this result. One is given in [CK].

Recall the following

Lemma 2.1.7 *Let $\mathcal{M}$ be a non-ω-standard model of set theory and $\mathcal{A}$ be a structure living inside $\mathcal{M}$. That is, $\mathcal{A}$ is a structure that is coded by an element of M. Then $\mathcal{A}$ is recursively saturated.*

A converse of the above lemma is also true, but harder to prove and of little interest to us here, as we now have all the ingredients that are needed for proving Friedman's theorem for models of ZF.

Suppose $\mathcal{M}$ is a non-ω-standard countable recursively saturated model of ZF. let $\mathcal{N}$ be a countable recursively saturated elementary end extension of $\mathcal{M}$. Inside $\mathcal{M}$ fix a coding of formulas and note that every structure $\mathcal{A}$ that lives in $\mathcal{M}$ has a theory $Th(\mathcal{A})$ that "is" a subset of $\omega^{\mathcal{M}}$ via this coding.

In particular, for every α, let $R(\alpha)$ denote the set of all sets x such that $\rho(x) < \alpha$. We shall write $R^{\mathcal{M}}(\alpha)$ to denote both the set $R(\alpha)$ in the sense of $\mathcal{M}$ *and* the substructure of $\mathcal{M} = \langle M, E \rangle$ whose domain is $R^{\mathcal{M}}(\alpha)$ (common abuse of notation).

Also, in the universe, for every $\alpha \in \mathbf{Ord}$, define $r(\alpha)$ to be the subset of ω that codes the theory of $\langle R(\alpha), \in \rangle$. Since ω has only continuum many subsets, while **Ord** is too big to be even a set, there must be a real number $r_0 \in \mathcal{P}(\omega)$ such that for unboundedly many ordinals α we have $r(\alpha) = r_0$.

Relativizing this argument to $\mathcal{M}$ yields the following.

$$\exists r_0 \in M[(\mathcal{M} \models r_0 \subseteq \omega) \land \forall \alpha \in \mathbf{Ord}^{\mathcal{M}} \exists \beta \in \mathbf{Ord}^{\mathcal{M}}[\mathcal{M} \models (\beta > \alpha \land r(\beta) = r_0)]]$$

Now let $\mathcal{N}$ be a countable elementary end extension of $\mathcal{M}$. We claim that the set of $\mathcal{N}$-ordinals α for which $r^{\mathcal{N}}(\alpha) = r_0$ is unbounded in $\mathcal{N}$ as well. This is simply due to the fact that $\mathcal{M} \prec \mathcal{N}$ and the following first order formula, $\varphi(r)$, is true in $\mathcal{M}$:

$$(\forall \alpha \in \mathbf{Ord})(\exists \beta \in \mathbf{Ord})[\beta > \alpha \land r(\beta) = r_0],$$

where $(\forall \gamma \in \mathbf{Ord})\psi$ and $(\exists \gamma \in \mathbf{Ord})\psi$ are short-hands for $\forall \gamma(\gamma$ is an ordinal $\rightarrow \psi)$ and $\exists \gamma(\gamma$ is an ordinal $\land\, \psi)$, respectively.

Now pick $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ and $\beta \in \mathrm{Ord}^{\mathcal{N}} \setminus M$ such that $\mathcal{M} \models \alpha > \omega$ and $\mathcal{N} \models r(\alpha) = r(\beta)$. Since $\mathcal{M}$ is not ω-standard, its elementary *end* extension cannot be ω-standard either. In fact, $\omega^{\mathcal{N}} = \omega^{\mathcal{M}}$, because $\mathcal{M} \prec \mathcal{N}$. This implies that both structures $R^{\mathcal{N}}(\alpha)$ and $R^{\mathcal{N}}(\beta)$ are recursively saturated.

The strategy is to show $R^{\mathcal{N}}(\alpha) \equiv R^{\mathcal{N}}(\beta)$ and also $SSy(R^{\mathcal{N}}(\alpha)) = SSy(R^{\mathcal{N}}(\beta))$. Thanks to the above lemmas, these together will ensure that $R^{\mathcal{N}}(\alpha) \cong R^{\mathcal{N}}(\beta)$ in the real world, even though the model $\mathcal{N}$ does not see them even as equinumerous structures.

In order to show that $R^{\mathcal{N}}(\alpha)$ and $R^{\mathcal{N}}(\beta)$ have the same first order theory, we notice that $\mathcal{N}$ sees these structures as elementary equivalent, because $r^{\mathcal{N}}(\alpha) = r^{\mathcal{N}}(\beta)$. This means for every $\mathcal{L}$-formula ψ that is coded in $\mathcal{N}$ we have

$$\mathcal{N} \models \text{“}R(\alpha) \models \psi\text{” if and only if } \mathcal{N} \models \text{“}R(\beta) \models \psi\text{”},$$

where $\mathcal{L}$ denotes the first order language of ZF set theory, $\{\in\}$. But this will automatically ensure

$$R^{\mathcal{N}}(\alpha) \models \psi \text{ if and only if } R^{\mathcal{N}}(\beta) \models \psi$$

for all “real” formulas ψ. Hence, $R^{\mathcal{N}}(\alpha) \equiv R^{\mathcal{N}}(\beta)$.

Now, suppose $\mathcal{A}$ is a structure that lives in the model $\mathcal{M}$. Recall that the standard system of $\mathcal{A}$ is, by definition,

$$SSy(\mathcal{A}) = \{x \cap \omega : \mathcal{N} \models (x \in A \cap \mathcal{P}(\omega)\}.$$

The subtle point here is that the first ω appearing in the above line is the ω in the real world, while the second ω is simply a symbol that is interpreted inside $\mathcal{N}$ as its own set of natural numbers.

Since $\mathcal{N} \models R(\alpha) \subset R(\beta)$, it is obvious that $SSy(R(\alpha)) \subseteq SSy(R(\beta))$, simply because $R(\beta)$ “has more elements”. The other direction, $SSy(R(\beta)) \subseteq SSy(R(\alpha))$, is also true, because by the above lemmas $\mathcal{N}$ is a rank extension of

$\mathcal{M}$ and therefore every element of $R(\beta) \setminus R(\alpha)$ has a higher ordinal rank than any element of $R(\alpha)$.

Now, suppose $t \in SSy(R(\beta))$. Then $t = x \cap \omega$ for some $x \in N$, where $\mathcal{N} \models x \in R(\beta) \cap \mathcal{P}(\omega)$. Let γ be an $\mathcal{M}$-ordinal such that $\mathcal{M} \models \omega \leq \gamma < \alpha$ and let $y = x \cap R(\alpha)$ in the sense of $\mathcal{M}$. Then

$$y \cap \omega = x \cap R(\alpha) \cap \omega = x \cap \omega = t.$$

Since $\mathcal{M} \models y \in R(\alpha) \cap \mathcal{P}(\omega)$, we have $t \in SSy(R(\alpha))$.

To summarize, $R^{\mathcal{N}}(\alpha)$ and $R^{\mathcal{N}}(\beta)$ are elementary equivalent countable recursively saturated structures with the same standard systems. Using the above lemmas, we have $R^{\mathcal{N}}(\alpha) \cong R^{\mathcal{N}}(\beta)$. Let $f : R^{\mathcal{N}}(\beta) \to R^{\mathcal{N}}(\alpha)$ be an isomorphism, that is, a bijective map such that for all x, y in $R^{\mathcal{N}}(\beta)$ we have

$$R^{\mathcal{N}}(\beta) \models x \in y \text{ if and only if } R^{\mathcal{N}}(\alpha) \models f(x) \in f(y).$$

Since $\beta \in N \setminus M$ and $\mathcal{M} \prec_r \mathcal{N}$, we have $M \subseteq R^{\mathcal{N}}(\beta)$. Therefore,

$$f[M] = \{f(x) : x \in M\} \subseteq R^{\mathcal{N}}(\alpha) \subset M$$

is a proper substructure of $\mathcal{M}$ that is isomorphic to $\mathcal{M}$. The proof of our theorem is now complete, but it is also worth noting that $f[M]$ as a subset of M is closed downward under the rank function, because the isomorphism f preserves the ranks as well.

More work shows that for any $\alpha \in \mathbf{Ord}^{\mathcal{M}}$, there is an embedding of $\mathcal{M}$ *onto* an initial segment of itself that fixes all elements of $R(\alpha)^{\mathcal{M}}$.

2.2 Automorphisms that Fix Ordinals Fix a Lot More

In this section, as before, ZF abbreviates the Zermelo-Fraenkel set theory, and ZFC is the theory ZF plus the Axiom of Choice, AC.

Definition 2.2.1 *A set x is called* transitive *if every element of x is a subset of x. In other words, x is transitive, if $\forall y, z(z \in y \wedge y \in x \to z \in x)$.*

Given a set x, the transitive closure *of x, denoted $trcl(x)$, is the "smallest" transitive set that contains x as a subset. Alternatively, $trcl(x) = \bigcup\limits_{n<\omega} x_n$, where $x_0 =_{\text{def}} x$ and $x_{n+1} =_{\text{def}} \bigcup x_n$ for all $n \in \omega$.*

Definition 2.2.2 *A binary relation $R \subseteq x \times x$ is* well-founded on x *if every nonempty subset y of x has an "R-minimal" element z, that is, there is no $t \in y$ for which tRz. R is called* extensional on x *if*

$$\forall y, z \in x[\forall t \in x(tRy \leftrightarrow tRz) \to y = z].$$

Remark 2.2.3 *Note that if $\langle M, E\rangle$ is a model of ZF, then E is extensional but may not be well-founded, even though the model itself believes that it is. Well-founded models of ZF are easily observed to be rigid.*

Proposition 2.2.4 *(Mostowski Collapsing Theorem)* [Ku] *Suppose R is a well-founded, extensional relation on x. Then there is a transitive set T such that $\langle x, R\rangle \cong \langle T, \in\rangle$. Furthermore, such T is unique, and so is the isomorphism between $\langle x, R\rangle$ and $\langle T, \in\rangle$.*

Lemma 2.2.5 *Let $\mathcal{M} = \langle M, E \rangle \models ZF$ and let for $x \in M$, x_E denote the set of all $y \in M$ such that yEx. If for an $f \in Aut(\mathcal{M})$ we have $f(y) = y$ for all $y \in x_E$, then $f(x) = x$.*

To avoid a possible ambiguity, we emphasize that $f(x) = x$ in the conclusion of the above lemma simply means f maps x to itself *as a point*.

Proof. For all $y \in M$ we have yEx if and only if $f(y)Ef(x)$ if and only if $f(y)Ex$. By extensionality of E we are done. ■

Definition 2.2.6 *Given an infinite cardinal κ, $H(\kappa) =_{\text{def}} \{x : |\ trcl(x)| < \kappa\}$.*

Now we can state and prove the following

Theorem 2.2.7 *Let $\mathcal{M} = \langle M, E \rangle \models ZFC$ and κ be an element of M such that $\mathcal{M} \models$ "κ is an infinite cardinal". Suppose $f \in Aut(\mathcal{M})$ fixes all $\mathcal{M}$-ordinals α that are below κ. Then f fixes all $x \in H(\kappa^+)^{\mathcal{M}}$.*

Proof. In this proof we make repetitive use of the fact that if an automorphism of a model $\mathcal{M}$ fixes and element $x \in M$, that is, if $f(x) = x$, then $f(\tau(x)) = \tau(x)$ as well, where τ is a term in the extended language $\mathcal{L} \cup \{x\}$. We argue inside the model. (But note that the automorphism f itself cannot be inside the model as a set, although f is definable inside the model, if it is a trivial automorphism.) For $x \in M$, let $\widehat{x} =_{\text{def}} \text{trcl}(\{x\})$. Since κ is infinite, we have $x \in H(\kappa^+)$ if and only if $|\widehat{x}| \leq \kappa$. Let $g : \widehat{x} \to \alpha$ be a bijection, where $\alpha \leq \kappa$. We use g to induce on α a binary relation R. More precisely, $\beta R \gamma$ if and only if $g^{-1}(\beta) \in g^{-1}(\gamma)$. Note that the relation R is a subset of $\alpha \times \alpha$, and thanks to Gödel, we can code all of $\kappa \times \kappa$ by elements of κ. Therefore R is coded by some ordinal $< \kappa$ and since these ordinals are fixed by f we have $f(R) = R$. Note that by a *coding* between $\kappa \times \kappa$ and κ we simply mean a definable bijection between these two sets.

Note that from the point of view of the model, $\in$ is a well-founded and extensional relation, and therefore so is R. Therefore, we may "collapse" $\langle \alpha, R\rangle$ to get a unique set T such that $\langle T, \in\rangle \cong \langle \alpha, R\rangle \cong \langle \widehat{x}, \in\rangle$, hence $\widehat{x} = T$. This shows that $\widehat{x}$ is definable from R. Since $f(R) = R$, we must have $f(\widehat{x}) = \widehat{x}$, but $x = \bigcup \widehat{x}$, so we have $f(x) = x$. ■

Corollary 2.2.8 *Let $\mathcal{M}$ be a model of ZFC and f be an automorphism of $\mathcal{M}$ that fixes all the ordinals. Then f is the identity map.*

2.3 Cuts in Models of Set Theory

We prove an analogue of a result by Smoryński about countable recursively saturated models of Peano Arithmetic for countable recursively saturated models of Set Theory. More specifically, we define a notion of *cut* for a given model $\mathcal{M}$ of Set Theory and prove necessary and sufficient conditions for a cut to be the largest initial segment of ordinals of a model that is pointwise fixed by an automorphism of $\mathcal{M}$.

In [Smo] Craig Smoryński proved that for a countable recursively saturated model M of Peano arithmetic, PA, and for an initial segment I of M that is closed under exponentiation, there is always an automorphisms f of M such that I is the *largest* initial segment of M that f fixes pointwise. This result has been the main inspiration for the attempt to discover other model-theoretic contexts where an analogous result can be formulated *and* proved. Because of the affinity the models of PA demonstrate with models of Set Theory, one may attempt to see what it means for a Smoryński-like result to be true for the latter as well as to see when it will be in fact true.

Once properly formulated, it is almost immediate that an analogue for countable recursively saturated models of Finite Set Theory (hereafter shortened to FST [BF]) can be proved by direct appeal to Smoryński's theorem and by using a certain well-known coding using dyadic representations. FST is axiomatized by replacing the axiom of infinity in the standard axioms of ZF by its negation. It is a simple, but elegant, observation that models of FST can be re-interpreted as models of PA and vice versa, hence the "affinity". It should come as no surprise, then, that the analogue of Smoryński's result is true for models of FST, but before getting our hands dirty with the details, let us introduce a terminology that is applicable to all different models of reasonable "set theories".

Definition 2.3.1 *Let $\mathcal{M} = \langle M, E\rangle$ be a model of Kripke-Platek set theory, KP. A set I of ordinals of $\mathcal{M}$ is an* initial segment *if it is closed downwards under the order relation induced on it by E. An initial segment I is a* cut *if there is no ordinal α of $\mathcal{M}$ for which we have*

$$\beta \in I \textit{ if and only if } \mathcal{M} \models \text{“}\beta \textit{ is an ordinal smaller than } \alpha\text{”}.$$

It is worth mentioning that cuts are defined slightly differently for models of

PA [Ka]. A cut of a model of arithmetic is an initial segment of that model with no "supremum" inside the model. Note that in the case of models of PA, any initial segment that is closed under addition, let alone under exponentiation, is automatically a cut. However, there are easy examples of initial segments of models of Set Theory that are closed under cardinal exponentiation but fail to be cuts. For instance, the set of natural numbers of the model is closed under exponentiation, but $\omega^{\mathcal{M}}$ is the supremum of itself in $\mathcal{M}$.

It is therefore important to keep in mind what the context is when we use the terms "initial segment" and "cut". In the models of PA a linear order of all elements of the model is automatically present, but for models of ZFC no such natural linear order exists (unless ZFC is augmented with the axiom $\mathbf{V} = \mathbf{OD}$). It turns out that having a linear order on the elements of our model that is recognizable by the model itself, especially if the model can be fooled into believing that that order is a well-ordering, proves quite useful in establishing similar results.

It is worth mentioning that the converse of Smoryński's theorem holds true [Ka]. Similarly, we have the following

Proposition 2.3.2 *Let $\mathcal{M} \models \mathrm{ZF} + \mathbf{V{=}OD}$ or $\mathcal{M} \models \mathrm{FST}$. Then the largest initial segment of (ordinals of) $\mathcal{M}$ that is fixed pointwise by a given automorphism of $\mathcal{M}$ is closed under cardinal exponentiation.*

Proof. Assume $\mathcal{M} \models \mathrm{ZF} + \mathbf{V{=}OD}$ and f is an automorphism of $\mathcal{M}$. (The other case, when $\mathcal{M} \models \mathrm{FST}$, can be dealt with similarly.) Let I be the largest initial segment of $\mathrm{Ord}^{\mathcal{M}}$ that is pointwise fixed by f. Note that if a set X in $\mathcal{M}$

is a subset of some $\alpha \in I$, then by extensionality $f(X) = X$. Suppose $\kappa \in I$ is a cardinal in the sense of $\mathcal{M}$. There is a bijection between $\mathcal{P}(\kappa)$ and 2^κ. Since $\mathcal{M} \models \mathbf{V} = \mathbf{OD}$, we have a global well-ordering of the universe and the "least" such bijection, m, is thus definable in terms of κ. Since $f(\kappa) = \kappa$, we have

$$f(m) = f(\tau(\kappa)) = \tau(f(\kappa)) = \tau(\kappa) = m.$$

Recall that our goal is to show that $2^\kappa \subseteq I$. To that end, let $\beta < 2^\kappa$ be an arbitrary ordinal. There is an $X \subseteq \kappa$ such that $m(X) = \beta$. Therefore,

$$f(\beta) = f(m(X)) = [f(m)](X) = m(X) = \beta.$$

■

We show that the converse of the above proposition is also true. The following theorem is obtained jointly with Ali Enayat [T] .

Theorem 2.3.3 *Let $\mathcal{M} \models$ ZF + **V=OD** be countable and recursively saturated. Then every cut of $\mathcal{M}$ which is closed under cardinal exponentiation can be realized as the* largest *initial segment of $\mathcal{M}$ that is fixed pointwise by an automorphism of $\mathcal{M}$.*

The theorem will follow from the following three lemmas. The first lemma below guarantees the existence of a bijection that will be needed in the proofs of the subsequent pair of lemmas.

Lemma 2.3.4 *Suppose $\mathcal{M} \models$ ZF + **V=OD**. Then there is a bijection h definable inside $\mathcal{M}$ between sets of ordinals and $\mathrm{Ord}^{\mathcal{M}}$ such that for all $\alpha \in \mathrm{Ord}^{\mathcal{M}}$ and all $x \subseteq \alpha$ we have $h(x) < 2^{|\alpha|}$.*

Proof. We shall build h, piece by piece, from the bottom up. First, it is not hard to define an appropriate bijection between all finite subsets of ω and the ordinal ω itself. For example, one can recursively define $h_0(x) =_{\text{def}} \sum_{y \in x} 2^{h_0(y)}$.

We provide a sketch of how to extend h_0 to a definable mapping from *all* subsets of ω (and more) to $\text{Ord}^{\mathcal{M}}$. We leave out the details of how to do so all the way up the ladder of ordinals of $\mathcal{M}$. Once carried out properly, the transfinite construction gives rise to a *definable* bijection h with the desired property.

Define $C =_{\text{def}} \{\kappa : \mathcal{M} \models \text{“}\kappa \text{ is a cardinal” and } 2^\kappa = 2^{\aleph_0}\}$ and let $\kappa_0, \kappa_1, ...$ be an increasing enumeration of C. We shall argue inside $\mathcal{M}$. Note that since $C \subset 2^{\aleph_0}$, we have $|C| \leq 2^{\aleph_0}$. Now partition $2^{\aleph_0} \backslash \omega$ into $|C|$-many cells, $A_0, A_1, ...$ each of cardinality $2^{\aleph_0}$. Thanks to the axiom of choice such a partition exists, and since $\mathcal{M} \models$ **V=OD** we can pick the “first” such partition in the global well-ordering of the universe.

Now map all *infinite* subsets of ω onto A_0, all subsets of κ_1 that are not subsets of $\kappa_0 = \omega$ onto A_1, etc. This way all subsets of members of C will be eventually mapped to some ordinal less than $2^{\aleph_0}$. In order to have this mapping definable, we will take the “least” such bijection that does the job. Next we look at the first cardinal $\lambda \notin C$ and define $D =_{\text{def}} \{\kappa : \mathcal{M} \models \text{“}\kappa \text{ is a cardinal”}$ and $2^\kappa = 2^\lambda\}$. ■

Lemma 2.3.5 (back-and-forth lemma) *Suppose* $\mathcal{M} \models \text{ZF} +$ **V=OD** *is recursively saturated and* $\alpha \in \text{Ord}^{\mathcal{M}}$ *and* $\bar{a}, \bar{b} \in \mathcal{M}$, *and suppose for all formulas* φ

$$\mathcal{M} \models \forall x \in 2^{2^{|\alpha|}} (\varphi(x, \bar{a}) \leftrightarrow \varphi(x, \bar{b})).$$

Then for every $a \in \mathcal{M}$ there is some $b \in \mathcal{M}$ such that for all formulas φ of $\mathcal{L}_{\mathrm{ZF}}$ we have

$$\mathcal{M} \models \forall x \in \alpha(\varphi(y, \bar{a}, a) \leftrightarrow \varphi(x, \bar{b}, b)).$$

Proof. Let $\Gamma(x) = \{\forall y \in \alpha(\varphi(y, \bar{a}, a) \leftrightarrow \varphi(y, \bar{b}, x)) : \varphi \in \mathcal{L}_{\mathrm{ZF}}\}$. We wish to verify that $\Gamma(x)$ is realized in $\mathcal{M}$. Since Γ is recursive we only need to show that it is finitely satisfiable.

Let $\varphi_0, ..., \varphi_{k-1}$ be formulas in the language of ZF. Inside $\mathcal{M}$ define for $i \in k$

$$s_i =_{\mathrm{def}} \{y \in \alpha : \varphi_i(y, \bar{a}, a)\}.$$

If h is the bijection given by the previous lemma, we have $h(s_i) < 2^{|\alpha|}$ for all $i \in k$. Since inside $\mathcal{M}$ we have $2^{|\alpha|} \cdot i + h(s_i) < 2^{|\alpha|} \cdot k$ for all $i \in k$, where $\cdot$ and $+$ denote ordinal multiplication and ordinal addition respectively, we have

$$h(\{2^{|\alpha|} \cdot i + h(s_i) : i < k\}) < 2^{2^{|\alpha|}}.$$

Now let $\varphi(x, \bar{u})$ be the formula

$$\exists v \bigwedge_{i<k} \forall y \in \alpha \exists t((2^{|\alpha|} \cdot i + t \in h^{-1}(x) \wedge y \in h^{-1}(t)) \leftrightarrow \varphi_i(y, \bar{u}, v)).$$

If we fix $x = h(\{2^{|\alpha|} \cdot i + h(s_i) : i < k\}$, then $\mathcal{M} \models \varphi\ (x, \bar{a})$. Since $x < 2^{2^{|\alpha|}}$, by our assumption we also have $\mathcal{M} \models \varphi\ (x, \bar{b})$. That is,

$$\mathcal{M} \models \exists v \bigwedge_{i<k} \forall y \in \alpha[(\exists t(2^{|\alpha|} \cdot i + t \in h^{-1}(x) \wedge y \in h^{-1}(t)) \leftrightarrow \varphi_i(y, \bar{b}, v)].$$

Note that for $y \in \alpha$ the formula $\exists t(2^{|\alpha|} \cdot i + t \in h^{-1}(x) \wedge y \in h^{-1}(t))$ is simply the same as saying $y \in s_i$. Therefore $\mathcal{M} \models \forall y \in \alpha(\text{“}y \in s_i\text{”} \leftrightarrow \varphi_i(y, \bar{b}, b))$ which implies $\mathcal{M} \models \forall y < \alpha(\varphi_i(y, \bar{a}, a) \leftrightarrow \varphi_i(y, \bar{b}, b))$, because of the way s_i are defined.

■

Lemma 2.3.6 *Suppose $\mathcal{M}$ is as in the back-and-forth lemma and $\alpha \in \mathrm{Ord}^{\mathcal{M}} \setminus \omega^{\mathcal{M}}$ and $\bar{a}, \bar{b}$ are tuples of elements of $\mathcal{M}$ of the same finite length such that for all formulas φ of $\mathcal{L}_{\mathrm{ZF}}$ we have*

$$\mathcal{M} \models \forall x \in 2^{2^{|\alpha|}}(\varphi(x, \bar{a}) \leftrightarrow \varphi(x, \bar{b})).$$

Then there are $a \in \mathrm{Ord}^{\mathcal{M}}$ and $b \in \mathcal{M}$ such that $a \neq b$, $\mathcal{M} \models a \in 2^{2^{|\alpha|}}$ and for all formulas φ of $\mathcal{L}_{\mathrm{ZF}}$ we have

$$\mathcal{M} \models \forall x \in \alpha(\varphi(x, \bar{a}, a) \leftrightarrow \varphi(x, \bar{b}, b)).$$

Before we start the proof, note that the case where the given cut is contained in, rather than contains, $\omega^{\mathcal{M}}$ can be dealt with similar to Smoryński's original proof for models of PA.

Proof. We just need to realize the following recursive type:

$$\{\forall x \in \alpha(\varphi(x, \bar{a}, y) \leftrightarrow \varphi(x, \bar{b}, z)) : \varphi \in \mathcal{L}_{\mathrm{ZF}}\} \cup \{y \neq z \wedge y \in 2^{2^{|\alpha|}}\}.$$

As before, since $\mathcal{M}$ is recursively saturated we shall show that the above type is finitely satisfiable.

Consider formulas $\varphi_0, ..., \varphi_{k-1}$ from $\mathcal{L}_{\mathbf{ZF}}$ with the proper number of free variables. If $\bar{u}$ and v are fixed variables, then there are at most $k|\alpha| = |\alpha|$ formulas $\varphi_i(x, \bar{u}, v)$ such that $i < k$ and $\mathcal{M} \models x \in \alpha$.

So there are at most $2^{|\alpha|}$ different sets of such formulas satisfied by $\bar{u} = \bar{a}$ and v being some element a of $\mathcal{M}$ with $\mathcal{M} \models x \in \alpha$.

Since $2^{2^{|\alpha|}} > 2^{|\alpha|}$, at least one such set is satisfied by some a_0, a_1 where $a_0 \in a_1 \in 2^{2^{|\alpha|}}$ in the sense of $\mathcal{M}$.

Let A denote the set

$$\{(i,x) : i < k \text{ and } \mathcal{M} \models (x < \alpha \wedge \varphi_i(x, \bar{a}, a_0))\}$$
$$[= \{(i,x) : i < k \text{ and } \mathcal{M} \models (x < \alpha \wedge \varphi_i(x, \bar{a}, a_1))\}].$$

Now for $X \subseteq k \times \alpha$ let $\widehat{X}$ be the following ordinal defined inside $\mathcal{M}$, where h is a fixed definable bijection given by our first lemma.

$$\widehat{X} =_{\mathbf{def}} h(\bigcup_{i<k} \{(|\alpha|^+ \cdot i) + \beta : (i, \beta) \in X\}),$$

where $\cdot$ and $+$ are ordinal multiplication and ordinal addition respectively. Note that $h^{-1}(\widehat{X}) \subseteq |\alpha|^+ \cdot k$ for all $X \subseteq k \times \alpha$. We therefore have

$$\widehat{X} = h(h^{-1}(\widehat{X})) < 2^{||\alpha|^+ \cdot k|} = 2^{|\alpha|^+} \leq 2^{2^{|\alpha|}}$$

for all such X.

In particular, $\widehat{A} < 2^{2^{|\alpha|}}$. Now define $\theta(w, \bar{u})$ to be the following formula

$$w \in \mathbf{Ord} \wedge \exists v \in 2^{2^{|\alpha|}} \bigwedge_{i<k} (\forall x \in \alpha((|\alpha|^+ \cdot i + x) \in h^{-1}(w) \leftrightarrow \varphi_i(x, \bar{u}, v))).$$

Using either a_0 or a_1 as witness, it becomes clear that $\mathcal{M} \models \theta(\widehat{A}, \bar{a})$. Since $\widehat{A}$ is an ordinal $< 2^{2^{|\alpha|}}$, by our assumption we have $\mathcal{M} \models \theta(\widehat{A}, \bar{b})$. This means there is a $b \in \mathcal{M}$ which is $< 2^{2^{|\alpha|}}$ and for all $i < k$

$$\mathcal{M} \models \forall x \in \alpha(\varphi_i(x, \bar{a}, a_0)\ [\leftrightarrow \varphi_i(x, \bar{a}, a_1)] \leftrightarrow \varphi_i(x, \bar{b}, b)).$$

Since a_0 and a_1 are distinct, b must be distinct from at least one of them. This establishes the realizability of the type we set out to realize. ∎

Theorem 2.3.3 has an interesting corollary.

Corollary 2.3.7 *Let $\mathcal{M}$ be a model of ZF +* **V=OD** *of any cardinality and I be a cut of $\mathbf{Ord}^{\mathcal{M}}$ which is closed under exponentiation. Suppose an element $c \in M$ is definable within the structure $\langle \mathcal{M}, I \rangle$. Then c is definable in the structure $\langle \mathcal{M}, a \rangle$, for some $a \in I$.*

Not that when I is the standard cut, this shows that if c is definable within $\langle \mathcal{M}, \omega \rangle$, then c is already definable with no parameters. Vladimir Kanovei first proved this for models of PA, and later Bamber and Kossak expanded his result [BK].

Finally, it is worth mentioning that if ZF + Global Choice is the theory in the language $\{\in, f\}$ consisting of the axioms of ZF(f), that is, we declare that formulas using f can be used in the replacement scheme, plus the axiom "f is a global choice function", then every countable model $\mathcal{M}$ of ZFC expands to a model $(\mathcal{M}, f)$ of ZF + Global Choice (see [F] for a forcing proof or [G] for a proof "from scratch").

2.4 Open Problems

It is only natural to explore the behaviorial similarity that models of Set Theory bear with models of PA. It seems that, as a rule of thumb, the presence of a definable global well-ordering in $\mathcal{M}$, or in other words, when $\mathcal{M} \models \mathbf{V} = \mathbf{OD}$, helps. We close this chapter by mentioning some of the rather intriguing results of Richard Kaye [Ka] and asking the natural question: Can analogous results be proved for models of $ZF + \mathbf{V} = \mathbf{OD}$?

Definition 2.4.1 *An initial segment I of a countable recursively saturated models $\mathcal{M}$ of PA is called* invariant *if $I = sup(I \cap M_0))$ or $I = inf((M \setminus I) \cap M_0)$, where M_0 is the set of all definable elements of $\mathcal{M}$.*

Proposition 2.4.2 *The set of invariant initial segments I of a non-standard model $\mathcal{M}$ of PA that are closed under exponentiation is order-isomorphic to one of the following:*

1. *A linearly ordered set with 2 elements;*
2. $\xi + 1$;
3. $1 + \xi + 1$;

where ξ is the order-type of the Cantor set. The first possibility occurs if and only if there are no definable non-standard elements in $\mathcal{M}$. Both other possibilities also occur.

Theorem 2.4.3 *Let $\mathcal{M}$ be a countable recursively saturated model of PA and consider $G = Aut(\mathcal{M})$ as a topological group with the usual topology. Then there is a one-to-one correspondence between closed normal subgroups of $Aut(\mathcal{M})$ and the invariant initial segments of $\mathcal{M}$ that are closed under exponentiation. Moreover, this correspondence reverses the inclusion order of invariant initial segments and closed normal subgroups.*

Corollary 2.4.4 *The closed normal subgroups of the topological group $Aut(\mathcal{M})$ constitute a chain, whenever $\mathcal{M}$ is a countable recursively saturated model of PA.*

Is the same true for countable recursively saturated models of $ZF + \mathbf{V} = \mathbf{OD}$?

Chapter 3

(Left-)orderability of Groups

In this chapter we will introduce the basic properties of (left-)orderable groups. The history of the subject goes as far back as the beginning of the 20th century [Hold]. Aside from connections inside algebra, there has been recent interest in left-orderability of groups in topology and their implications [BRW]. Moreover, D. Reed Solomon discovered interesting connections with recursion theory [Sol-2, Sol-3].

3.1 (Left-)ordered Groups and (Left-)orderability

First, a few definitions.

Definition 3.1.1 *A set* $\langle L, < \rangle$ *is a* partially ordered set*, where* $<$ *is a binary* *relation on* L*, if* $<$ *is irreflexive and transitive. A partially ordered set* $\langle L, < \rangle$ *is a* linearly ordered set *if, in addition, for all* $x, y \in L$ *we have either* $x < y$ *or* $x = y$ *or* $y < x$*. We always write* $x > y$ *to mean* $y < x$*. We call* $<$ *a partial or a linear order on* L *depending on what the case may be.*

Definition 3.1.2 *A structure* $\langle G, \circ, < \rangle$ *is called a* left-ordered group *if* $<$ *is a linear order on the set* G *and* $\langle G, \circ \rangle$ *is a group and the multiplication from the left in* G *respects the order* $<$*. That is,*

$$\forall x, y, z \in G(x < y \to z \circ x < z \circ y).$$

If $\langle G, \circ, < \rangle$ *is left-ordered, we say* $<$ *is* left-invariant*. A group* $\langle G, \circ \rangle$ *is called* left-orderable *if there is a linear order* $<$ *on* G *such that* $\langle G, \circ, < \rangle$ *is a left-ordered group.*

Throughout the chapter, we will follow the common practice and write $\langle G, < \rangle$ and xy rather than the more formal and more cumbersome $\langle G, \circ, < \rangle$ and $x \circ y$. Right-ordered and right-orderable groups and right-invariant orders are defined similarly. It can be easily verified that if a group $\langle G, < \rangle$ is left-ordered, then the group $\langle G, R \rangle$ is right ordered, if we simply define xRy iff $x^{-1} < y^{-1}$. This shows that a group is left-orderable if and only if it is right-orderable, so nothing will be lost by focussing our attention on one or the other. This said, a left-invariant order on a group G may not be right-invariant, as we will see below. If $<$ is both a left- and right-invariant order on G, then we say $\langle G, < \rangle$ is *orderable.*

Obviously, any abelian group is left-orderable if and only if it is orderable. In fact, all orderable abelian groups are easily characterized. (See below.)

There are many example of orderable groups, such as the additive group of reals and all its subgroups, and one can form new left-ordered groups by forming Cartesian products and defining lexicographic orders. (When forming an infinite product, we will need to well-order the index set.) We note that the only finite left-orderable group if the trivial group, because of the following observation.

Proposition 3.1.3 *Let G be a left-orderable group. Then G is torsion-free.*

Proof. Let $\langle G, <\rangle$ be a left-ordered group and $x \in G$ and let $n > 1$ be an integer and e denote the identity element of G. If $x \neq e$, then either $x > e$ or $x < e$, because $<$ is a total order. If $x < e$, then we have $x^2 = xx < xe = x < e$. If we keep multiplying from the left, we will have $x^n < x^{n-1} < \cdots < x < e$. Therefore, $x^n \neq e$. The other case can be handled similarly, or one simply notices that if $e < x$, then $x^{-1} < e$. ■

Example 3.1.4 A left-invariant order $<$ on group G is called *Archimedean* if for all $x, y \in G$ that satisfy $e < x < y$, there exists an $n \in \mathbb{Z}$ such that $y < x^n$. Groups that admit a left-invariant order with Archimedean property are characterized by the following theorem.

Theorem 3.1.5 *(Conrad 1959, Hölder 1902) If $\langle G, <\rangle$ is a left-ordered Archimedean group, then the ordering $<$ is also a right-invariant order on G and $\langle G, < \rangle$ is isomorphic (as an ordered group) to an additive subgroup of the real numbers with the usual ordering of $\mathbb{R}$. In particular, G must be an abelian group.*

The theorem above states that if the ordering relation is Archimedean then the group has no "interesting" structure. Another reason why being Archimedean is not that interesting is the Cartesian product of two non-trivial ordered groups with Archimedean orderings, if ordered lexicographically, is not Archimedean. In particular, if we order the additive group of complex numbers $\langle \mathbb{C}, + \rangle$, with ordering $\prec$ that is defined as: for $a_1 + b_1 i$, $a_2 + b_2 i \in \mathbb{C}$, set $a_1 + b_1 i \prec a_2 + b_2 i$ if and only if $a_1 < a_2$ or $a_1 = a_2$ and $b_1 < b_2$, then $\langle \mathbb{C}, +, \prec \rangle$ is not Archimedean.

There are other ways of looking at the left-invariant orders. For example, each left-invariant order $<$ on a group G can be identified with the subsemigroup $P_+ =_{\text{def}} \{x \in G : x > e\}$, where e is the identity element of G. We call P_+ the *positive cone* of G associated with the ordering $<$. Let $P_- =_{\text{def}} \{x \in G : x < e\} = \{x \in G : x^{-1} \in P_+\}$. We call P_- a *negative cone* of G.

The following properties of cones are simple consequences of the properties of the left-invariant order $<$.

Proposition 3.1.6 *Let P_+, P_- be positive and negative cones of an ordered group $\langle G, < \rangle$. Then P_+, P_-, and $\{e\}$ form a partition of G into three of its subsemigroups.*

In other words, P_+ and P_- are maximal subsemigroups of G that do not have the identity element. Conversely, it is easy to verify that if P is a subsemigroup of G that together with P^{-1} and $\{e\}$ partition G, then one can define a left-invariant order on G as follows: For $x, y \in G$ define ($x <_P y$ if and only if $x^{-1}y \in P$). Moreover, if we have $\forall x \in G$ ($x^{-1}Px \subseteq P$), then $<_P$ is both left-and right-invariant.

So a group is (left-)orderable if it has a subsemigroup with certain properties. However, there are more interesting, and perhaps more useful, characterizations of left-orderable groups in terms of their subsemigroups. The following is an example of such a result.

For $x_1,\ x_2, \ldots x_n \in G$, define $sgr(\{x_1,\ x_2,\ ...,\ x_n\})$ to be the minimal subsemigroup of G containing $\{x_1,\ x_2,\ ...,\ x_n\}$.

Theorem 3.1.7 *(Conrad* [Co]*) A group G admits a left-invariant order if and only if for any finite set $x_1, x_2, \ldots, x_n$ of non-identity elements of G, there are $\epsilon_1, \epsilon_2, \ldots \epsilon_n \in \{+1, -1\}$ such that $e \notin sgr(\{x_1^{\epsilon_1}, x_2^{\epsilon_2}, \ldots, x_n^{\epsilon_n}\})$.*

The following corollary is oddly reminiscent of the Compactness Theorem for first order logic.

Corollary 3.1.8 *If every finitely generated subgroup of a group G admits a left-invariant order, then the group G admits a left-invariant order too.*

The above gives us the necessary and sufficient conditions for left-orderability of groups. A different condition for the left-orderability problem is given by the following theorem.

Theorem 3.1.9 *(Burns-Hale* [BH]*) A group G is left-orderable if and only if every finitely generated subgroup of G has a non-trivial quotient which is left-orderable.*

Interesting examples of (countable) groups arise from topological spaces and the question of their left-orderability has consequences for the structure of the corresponding spaces. In particular, for every path-connected topological space X one associates a unique (up to the isomorphism) group called the *fundamental group of X*. Usually, these groups are given by their finite presentations and we are going to discuss many examples of them in the next chapter.

3.2 Characterizations and Connections with Logic

We open this section by a beautiful result of James Schmerl [Schm] that links the two parts of this work by means of an unexpected connection.

Theorem 3.2.1 *A group G is left-orderable if and only if $G \cong Aut(\mathcal{M})$ for some model $\mathcal{M}$ of Peano Arithmetic.*

One direction of Schmerl's theorem is fairly easy to establish (see below). The hard direction is to construct a model $\mathcal{M}$ of PA for a given left-ordered group $\langle G, < \rangle$ such that $G \cong Aut(\mathcal{M})$. Given the close connection of PA with ZF, one naturally wonders

Question 3.2.2 *Can the same be said about $Aut(\mathcal{M})$, where $\mathcal{M}$ is a model of set theory? More precisely, for what axiomatization of set theory, if any at all, the automorphism groups of models of that "set theory" are exactly the same, up to group isomorphism, as the automorphism groups of models of PA?*

We conjecture that ZF + **V=OD** should be one such axiomatization.

Before introducing another characterization theorem, we may ask if the notion of left-orderability can be captured by a "nice" set of first-order axioms *in* *the language of groups*. The following result provides a complete answer to this question.

Theorem 3.2.3 *There is a recursive set of formulas Φ in the language of group theory such that for every group G we have $\Phi \models G$ if and only if G is left-orderable. Moreover, such a set Φ cannot be finite.*

Proof. Let $\mathcal{L}$ denote the language of group theory and $\mathcal{L}'$ denote the language of left-ordered groups. Let LOG denote the theory of left-ordered groups (in $\mathcal{L}'$) and let $\Phi = \{\varphi : \varphi$ is a sentence in $\mathcal{L}$ and $LOG \vdash \varphi\}$. Obviously Φ is recursively enumerable, because LOG is finite, and therefore, thanks to Craig's trick (see Chapter 2) it is equivalent to a recursive set.

Now, suppose there is a *finite* set Φ of $\mathcal{L}$-sentences such that $G \models \Phi$ iff G is left-orderable. Let ϕ be the conjuction of all sentences in Φ. Since finite groups are not left-orderable, we have $\langle \mathbb{Z}_m, + \rangle \models \neg\phi$, where m is any integer > 1. For $n \in \omega$, let ϕ_n be the $\mathcal{L}$-sentence $\forall x \exists y (y^n = x)$, or, in additive notation $\forall x \exists y (ny = x)$. Now $\Psi =_{\text{def}} \{\phi_n : n \in \omega \setminus \{0\}\} \cup \{\forall x, y (x \circ y = y \circ x), \neg\phi\}$ is a consistent set of formulas, because every finite subset of Ψ is satisfiable by $\langle \mathbb{Z}_p, + \rangle$, whenever p is a prime bigger than all the n with ϕ_n in the given finite subset of Ψ. But every group G that is a model of Ψ is an abelian divisible group, which means it is a vector space over $\mathbb{Q}$, and therefore orderable (See chapter 2), so $G \models \phi$. This contradiction completes our proof. ■

Assuming AC, the Axiom of Choice, it is not hard to show that if $\mathcal{M} = \langle L, < \rangle$ is a linearly ordered set, then $Aut(\mathcal{M})$ is left-orderable. The proof is part of the proof of a more general result by Paul Conrad.

Definition 3.2.4 *We say that a group G* acts from the left on a set X *if there is a group homomorphism $\psi : G \to Sym(X)$, where $Sym(X)$ denotes the group of all bijections of X, the operation being the composition of maps. We say that G acts* effectively *on X if* $\ker(\psi) = \{e\}$.

Theorem 3.2.5 *(*[Co]*) A group G is left-orderable if and only if it acts effectively on a linearly ordered set $\langle L, < \rangle$ by order-preserving bijection on L.*

Proof. Suppose the linear order $<$ on G is left-invariant. Since G acts effectively on itself by right multiplication one can take $L = G$. Since the linear order $<$ on G is left-invariant, the statement follows in one direction.

Suppose G acts effectively on a linearly ordered set $\langle L, < \rangle$ by order-preserving bijections, that is, for all $g, x, y \in G$ whenever $x < y$ one has $\psi(g)(x) < \psi(g)(y)$. Now let $\prec$ be a fixed well-ordering of the elements of L. Recall that all sets can be well-ordered if and only if we accept AC as an axiom. For $x, y \in G$ define xRy if and only if $x \neq y$ and $\psi(x)(a) \prec \psi(y)(a)$, where a is the $\prec$-least element of L for which $\psi(x)(a) \neq \psi(y)(a)$. It is easy to check that R on G is left-invariant. ■

Not surprisingly, we have the following characterization. (See [Schm] and [L].)

Theorem 3.2.6 *A group G is left-orderable if and only if it can be embedded in $Aut(\langle L, < \rangle)$ for some linearly ordered set $\langle L, < \rangle$. Moreover, if G is countable, L can be taken to be either the set of real numbers with their usual ordering, or even the set of rationals with their usual ordering.*

3.3 The Abelian Case

As we mentioned earlier, while a necessary condition, being torsion-free is not always a sufficient condition for left-orderability of a group. However, for abelian groups the two notions coincide. That is, an abelian group is torsion-free if and only if it is left-orderable (and therefore, orderable). The following theorem sheds light on why this is the case.

Theorem 3.3.1 *An abelian group G is torsion-free if and only if $G \cong \bigoplus_{i \in I} G_i$, where each G_i is a subgroup of the additive group $\langle \mathbb{Q}, + \rangle$.*

In other words, every torsion-free abelian group can be embedded in the additive group $\langle V, + \rangle$ of a vector space V over the rationals. If we well-order the index set I and then order the elements of $\bigoplus_{i \in I} G_i$ lexicographically by $<$, then $\langle \bigoplus_{i \in I} G_i, < \rangle$ is an ordered group.

Theorem 3.3.2 *Let G be a non-trivial, torsion-free, abelian group. Then there is a set I and group homomorphisms $f : \bigoplus_{i \in I} \mathbb{Z} \to G$ and $g : G \to \bigoplus_{i \in I} \mathbb{Q}$ such $g \circ f$ is the inclusion map $i : \bigoplus_{i \in I} \mathbb{Z} \to \bigoplus_{i \in I} \mathbb{Q}$. Moreover, there is a one-to-one correspondence between orders on G and orders on $\bigoplus_{i \in I} \mathbb{Z}$ via f.*

In other words, every order R on $\bigoplus_{i \in I} \mathbb{Z}$ extends uniquely to an order S on $\bigoplus_{i \in I} \mathbb{Q}$. If $\bigoplus_{i \in I} \mathbb{Z} \leq G \leq \bigoplus_{i \in I} \mathbb{Q}$, then every order $<$ on G can be recovered from the restriction of that order on $\bigoplus_{i \in I} \mathbb{Z}$.

Proof. Suppose $\langle G, +_G \rangle$ is a non-trivial abelian torsion-free group. Define on $G \setminus \{e\}$ an equivalence relation by declaring $x \sim y$ if and only if there are

$n, m \in \mathbb{Z} \setminus \{0\}$ such that $mx = ny$. Let $I = \{[x] : x \in G\}$ be the collection of $\sim$-equivalence classes on $G \setminus \{e\}$. It is easy to verify that $G_x =_{\text{def}} [x] \cup \{e\}$ is a subgroup of $\langle G, +_G \rangle$ that can be embedded in $\langle \mathbb{Q}, + \rangle$. Let $\prec$ be a well-ordering of I, and let

$$J = \{[x] \in I : x \neq x_0 + x_1 + \cdots x_{n-1}, \text{ where } [x_i] \prec [x] \text{ for all } i < n, n \in \omega\}.$$

Now, the mapping $f : G \to \bigoplus_{[x] \in J} G_x$ defined by $f(x) : [y] \mapsto \chi_{[y]}(x) \cdot x$ for all $x \in G \setminus \{e\}$ is a group homomorphism. ■

These theorems reduce the study of (left-)orderable abelian groups to the study of the orders on the additive groups of the vector spaces, or, to the study of the groups of the form $\bigoplus_{i \in I} \mathbb{Z}$.

3.4 Left-orderability of Non-abelian Groups

Abelian groups provide examples of left-orderable groups that are automatically orderable, but there are left-orderable groups that fail to be orderable. An example of such a group is given by the fundamental group of the Klein bottle, given by $\pi_1(K)$ (See below).

In general, in dealing with the non-abelian case, it makes sense to look at presentations of the given group G and search for criteria based on which one can tell whether G is left-orderable or not. In fact, we have a criterion for left-orderability of a large class of finitely represented groups. First, we should quickly review some algebraic facts.

Definition 3.4.1 *Let X be a subset of a group F. Then F is a* free group with basis X *if for any function* $\phi : X \to G$*, where* G *is any group, there exists a unique extension of* ϕ *to a group homomorphism* $\phi^* : F \to G$.

Remark 3.4.2 *The requirement that homomorphism* $\phi^* : F \to G$ *to be* unique *for* $\phi : X \to G$ *is equivalent to saying that* F *is* generated *by* X*. Moreover, one can easily prove that all bases of a given free group* F *have the same cardinality called the* rank *of* F*. The free group of rank* 0 *is the trivial group.*

Every group G is isomorphic to the quotient of a free group. In particular, one has the following

Proposition 3.4.3 *Let* κ *be a cardinal number. If a group is generated by a set of* κ *elements, then it is a quotient of a free group of rank* κ*.*

Given a nonempty set X. it is not hard to construct a free group with basis X by forming finite strings of elements of X and "X^{-1}" and defining an equivalence relation on these strings. Each equivalence class will then have a canonical representative, called the *reduced* form of words inside that class.

Now, let $F(X)$ be a free group with basis X and let $R \subseteq F(X)$. Let $\langle\langle R \rangle\rangle =_{\text{def}} \bigcap\{N : R \subseteq N \trianglelefteq F(X)\}$ denote the smallest *normal* subgroup of $F(X)$ containing R. Also, denote by $\langle X \mid R\rangle$ the group obtained as the quotient of $F(X)$ by the normal subgroup $\langle\langle R \rangle\rangle$, that is, $\langle X \mid R\rangle =_{\text{def}} F(X)/\langle\langle R \rangle\rangle$. We call $\langle X \mid R\rangle$ a *presentation* for any group that is isomorphic to $F(X)/\langle\langle R \rangle\rangle$ and we refer to elements of X as the *generators* and to the elements of R as *relators* of $\langle X|R\rangle$. Note that in order to describe any group G it is sufficient to provide the set X of generators of $F(X)$ and the subset R of $F(X)$ such that

$G \cong F(X)/\langle\langle R\rangle\rangle$. It seems natural to start by looking at the case where both X and $R \subseteq F(X)$ are finite sets. Such groups, $\langle x_1, x_2, ..., x_n \mid r_1, r_2, ..., r_m\rangle$, where $r_i \in F(\{x_1, x_2, ..., x_n\})$, $1 \leq i \leq n$, are called *finitely presented.*

Example 3.4.4 *The group G given by the presentation $\langle x, y \mid xyx^{-1}y\rangle$ is isomorphic to fundamental group of Klein bottle (see next chapter), is left-orderable but not orderable. To see this, one observes that since $xy = y^{-1}x$. Hence every element of G can be writen in the form $x^a y^b$ where $a, b \in \mathbb{Z}$. One defines $P_+ = \{x^a y^b \in G \mid a \in \mathbb{Z}_+$ or $(a = 0$ and $b \in \mathbb{Z}_+)\}$. It is easy to verify that P_+ is a positive cone. Moreover, since $xyx^{-1} = y^{-1}$, for any bi-invariant order $<$ on G if $y > e$ then $xy > x$, since $<$ is left-invariant. Also, $e = xx^{-1} < xyx^{-1} = y^{-1}$, since $<$ is right-invariant. Thus $e < y^{-1}$ which means $y < e$–a contradiction.*

Example 3.4.5 *The group G given by the presentation $\langle x, y \mid x^2yx^2y^{-1}, y^2xy^2x^{-1}\rangle$ is not left-orderable. The group described by this presentation is isomorphic to the fundamental group of the triple fold branched covering (see next chapter) of S^3 along the figure eight knot (Figure 3.1), $\pi_1(M^{(3)}_{4_1})$,*

Figure 3.1

(or double branched covering of S^3 along the Borromean rings (Figure 3.2),

Figure 3.2

$\pi_1(M_{BR}^{(2)})$). This group could also be viewed as the Fibonacci group $F(2,6)$. The proof will bne given in the next chapter in a more general setting.

The theory of one relator groups, that is groups with presentations of the form $\langle x_1, x_2, ..., x_n \mid r \rangle$ where $r \in F(\{x_1, x_2, ..., x_n\})$, is quite well understood. Many decision problems, such as word problem or conjugacy problem, in the theory of one relator groups are decidable. Therefore, one can hope to understand the problem of orderability of one relator groups.

Let $r \in F_n$, where $F_n = \langle x_1, x_2, ..., x_n \mid \ \rangle$ denotes the free group of rank $n \in \mathbb{Z}_+$. Then r can be uniquely written in the form $r = s^m$ for some $m \in \mathbb{Z}_+$, $s \in F_n$ such that m is the maximum positive integer with this property. We call s a *root* of r. If $m = 1$, we say that r is not a *proper power*.

Theorem 3.4.6 *(W. Magnus, A. Karrass, W. Solitar* [KMS]*)*

If $G = \langle x_1, x_2, ..., x_n \mid r \rangle$ and r is not a proper power, then G is torsion free.

In order to prove the next result, we need to introduce some terminology first. The following definition is due to Higman.

Definition 3.4.7 *A group G is* locally-indicable *if every nontrivial finitely-generated subgroup has $\mathbb{Z}$ as a quotient.*

The next theorem establishes the connection between local-indicability and orderability of groups [Co, BH].

Theorem 3.4.8 *If G is an orderable group, then G is locally-indicable. If G is locally-indicable then G is left-orderable*

Interestingly, torsion-free groups with one relator in their presentation are also left-orderable which follows from the following theorem.

Theorem 3.4.9 *(S. D. Brodskiĭ* [Br]*) Any torsion-free subgroup of one relator group is locally indicable.*

Since every locally indicable group is left-orderable, one has a very easy criterion for orderability of one relator groups. The most well-known family of one relator groups is the family of fundamental groups of closed, oriented and connected 2-manifolds (surfaces). Let S_g be the closed, oriented and connected surface of genus $g \geq 1$. The fundamental group of S_g is given by $\pi_1(S_g) = \langle x_1, y_1, x_2, y_2, ..., x_g, y_g \mid \Pi_{i=1}^{g}[x_i, y_i] \rangle$, where $[x_i, y_i] =_{\text{def}} x_i^{-1} y_i^{-1} x_i y_i$, $1 \leq i \leq g$.

Corollary 3.4.10 *Fundamental groups of connected, closed, oriented, 2-manifolds are left-orderable.*

The following is a more general result.

Theorem 3.4.11 *(D. Rolfsen, B. Wiest* [RW]*) If N is any connected surface other than the projective plane $\mathbb{R}P^2$ or the Klein bottle, then $\pi_1(N)$ is orderable.*

Of course, one may ask a similar question for fundamental groups of 3-manifolds. However one cannot expect a simple answer[BRW, DPT]. We end this chapter by stating some further results.

Theorem 3.4.12 *(D. S. Passman* [Pa]*)If the group G is left-orderable, then the group ring $\mathbb{Z}G$ has no zero divisors, and its units are only of the form ng where n is a unit in $\mathbb{Z}$ and $g \in G$. The same is true for any integral domain R replacing $\mathbb{Z}$.*

Theorem 3.4.13 *(A. I. Mal'cev* [M]*, B. Neumann* [N]*)If a group G is orderable, then the group ring $\mathbb{Z}G$ can be embedded in a division algebra.*

Theorem 3.4.14 *(LaGrange, Rhemtulla) Let G and H be groups and suppose G is left-orderable. Then G and H are isomorphic (as groups) if and only if their group rings $\mathbb{Z}G$ and $\mathbb{Z}H$ are isomorphic as rings.*

Chapter 4

Non-left-orderability of Fundamental Groups of 3-Manifolds

4.1 Topological Preliminaries

Let M be a topological space. (In particular, and for our purposes, M can be taken to be a manifold, hence the choice of the letter M.) A continuous map $\gamma : [0, 1] \to M$ is called a *path from* $\gamma(0)$ *to* $\gamma(1)$. M is *path-connected* if for all $x, y \in M$ there is a path from x to y. Given $x, y \in M$, if $\gamma_1, \gamma_2 : [0, 1] \to M$ are both paths from x to y that can be continuously deformed to each other, they are said to be *homotopic* paths. Being homotopic to each other is an equivalence relation and we can form equivalence classes.

Given a path-connected space M and a base point $b \in M$ the *fundamental group of* M, denoted by $\pi_1(M, b)$, is defined as follows. The ground set is the set of all$[\gamma]$ where γ is a path from b to b and $[\gamma]$ denotes the set of all paths δ from b to b that are homotopic to γ. As for the operation of the group, $[\gamma_1] \circ [\gamma_2]$ is the "concatenation" of the paths $[\gamma_1]$ and $[\gamma_2]$. (See [Hat] for details.)

It is worth mentioning that the structure of the group $\pi_1(M, b)$ is independent of the choice of the base point b, as long as M is path-connected. So we simply write $\pi_1(M)$ to denote the fundamental group of M, whenever we only care about the isomorphism type of $\pi_1(M, b)$.

Definition 4.1.1 *Let M and $\tilde{M}$ be topological spaces, M be path-connected and $p : \tilde{M} \to M$ be a continuous map such that for every $x \in M$ there is an open neighborhood U of x such that $p^{-1}(U) = \bigcup_{i \in I} V_i$ such that the sets V_i are mutually disjoint and $p|V_i : V_i \to U$ is a homeomorphism for all $i \in I$. Then we say $(\tilde{M}, p)$ (or simply $\tilde{M}$, if it is clear what p is) is a* covering *of M.*

Theorem 4.1.2 *If $\tilde{M}$ is a path-connected covering of a manifold M, then $\pi_1(M)$ can be embedded in $\pi_1(\tilde{M})$. Conversely, if $H \leq \pi_1(M)$, then there is a covering $\tilde{M}$ of M such that $\pi_1(\tilde{M}) \cong H$.*

This implies that if $f : \pi_1(M) \to G$ is an epimorphism (of groups) for some path-connected space M and some group G, then there is a unique (up to homeomorphism) covering $\tilde{M}$ of M such that $\pi_1(\tilde{M}) \cong Ker(f)$. In particular, if G is the group $\langle \mathbb{Z}_n, + \rangle$, then we say $\tilde{M}$ is the *n-fold cyclic covering* of M.

Definition 4.1.3 *A* link *is the homeomorphic image of the disjoint union of finitely many circles,* $\bigsqcup_{i<n} S^1$*, in the Euclidean space* $\mathbb{R}^3$*. A* knot *is a connected link.*

Recall that the *abelianization* of a group G is defined to be G/N, where N is the smallest normal subgroup of G containing the set $\{xyx^{-1}y^{-1} : x, y \in G\}$.

Theorem 4.1.4 *Let* $K \subset \mathbb{R}^3$ *be a knot. Then the abelianization of* $\pi_1(\mathbb{R}^3 \setminus K)$ *is isomorphic to the cyclic group* $\langle \mathbb{Z}, + \rangle$*.*

This theorem shows that $\langle \mathbb{Z}_n, + \rangle$ is a homomorphic image of $\pi_1(\mathbb{R}^3 \setminus K)$, where K is an arbitrary knot. Let H be the kernel of a the epimorphism $\psi : \pi_1(\mathbb{R}^3 \setminus K) \to \mathbb{Z}_n$. Then there is a covering $\tilde{M}$ of M for which $\pi_1(\tilde{M}) \cong H$.

Now, let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the unit disc and $K \subset \mathbb{R}^3$ be a knot. If $T \supset K$ is a closed subset of $\mathbb{R}^3$ that is homeomorphic to $K \times D^2$, via a homeomorphism $\varphi : T \to K \times D^2$ such that $\varphi(x) = (x, 0)$ for all $x \in K$. T is called a *regular neighborhood of* K. Suppose there is a continuous map $p_1 : T \to T$ such that $\varphi \circ p_1 \circ \varphi^{-1}(x, z) = (x, z^n)$ for all $x \in K$ and $z \in D^2$, and there is an n-fold cyclic covering $(\tilde{M}, p_2)$ of $\mathbb{R}^3 \setminus T$, such that p_1 and p_2 can be "glued" together to get a continuous map $p : \tilde{M} \cup T \to \mathbb{R}^3$, where $p|T = p_1$ and $p|\tilde{M} = p_2$. Then we say p is an *n-fold branched covering of* $\mathbb{R}^3$ *along the knot* K.

4.2 Non-left-orderable 3-Manifold Groups

This part is the result of joint work with Mietek Dabkowski and Józef Przytycki [DPT]. In [BRW] examples of large classes of left-orderable fundamental groups have been explored. Here, we would like to contrast those examples with fundamental groups of spaces that we found to be *non*-left-orderable. Our spaces will typically be 3-dimensional manifolds that are obtained by taking n-fold branched coverings along various hyperbolic 2-bridge knots.

We first list links and multiplicity of branched coverings along them from which we obtain manifolds with non-left-orderable fundamentals groups. Thereby, we obtain an abundance of examples of groups that while being torsion-free are nevertheless non-left-orderable. Then we describe presentations of these groups in a way which allows the proof of non-left-orderability in a uniform way. The Main Lemma is the algebraic underpinning of our method and the non-left-orderability follows easily from it in almost all cases. Moreover we prove the non-left-orderability of a family of 3-manifold groups to which the Main Lemma does not apply. These groups, known as generalized Fibonacci groups $F(n-1,n)$, arise as groups of double coverings of S^3 branched along pretzel links of type $(2,2,...,2,-1)$. We end the chapter with some open questions.

It is known that groups of compact, P^2-irreducible 3-manifolds with non-trivial first Betti number are left-orderable [BRW, HS]. On the other hand, the theorem below lists various classes of 3-manifolds with non-left-orderable groups. Non-left-orderability of 3-manifold groups has interesting consequences for the geometry of the corresponding manifolds [CD, RSS].

Theorem 4.2.1 *Let $M_L^{(n)}$ denote the n-fold branched covering of S^3 along the link L, where $n > 1$. Then the fundamental group, $\pi_1(M_L^{(n)})$, is not left-orderable in the following cases:*

(a) $L = T_{(2',2k)}$ is the torus link of the type $(2, 2k)$ with the anti-parallel orientation of strings, and n is arbitrary (Figure 4.1).

(b) $L = P(n_1, n_2, ..., n_k)$ is the pretzel link of the type $(n_1, n_2, ..., n_k)$, $k > 2$, where either (i) $n_1, n_2, ..., n_k > 0$, or (ii) $n_1 = n_2 = \cdots = n_{k-1} = 2$ and $n_k = -1$ (Figure 4.2). The multiplicity of the covering is $n = 2$.

(c) $L = L_{[2k,2m]}$ is a 2-bridge knot of the type $\frac{p}{q} = 2m + \frac{1}{2k} = [2k, 2m]$, where $k, m > 0$, and n is arbitrary (Figure 4.4).

(d) $L = L_{[n_1,1,n_3]}$ is the 2-bridge knot of the type $\frac{p}{q} = n_3 + \frac{1}{1+\frac{1}{n_1}} = [n_1, 1, n_3]$, where n_1 and n_3 are odd positive numbers. The multiplicity of the covering is $n \leq 3$.

Figure 4.1

The manifolds described in parts (a), (b), and also for $n \leq 3$ and the figure eight knot, $L = L_{[2,2]} = 4_1$, in part (c) are Seifert fibered manifolds. The non-left-orderability of their groups follows from the general characterization of Seifert fibered manifolds with a left-orderable group [BRW]. Part (c) for

Figure 4.2

the figure eight knot when $n = 3$ is of historical interest because it was the first known example of a non-left-orderable torsion-free 3-manifold group [R]. This Euclidean manifold was first considered by Hantzsche and Wendt [HW]. J. Conway has proposed to call this manifold *didicosm*. It can be also described as the 2-fold branched covering over S^3 branched along the Borromean rings (Figure 3.2).

Part (c) for the figure eight knot when $n > 3$ gives rise to hyperbolic manifolds that are related to examples discussed in [RSS], as they are Dehn fillings of punctured-torus bundles over S^1. The manifolds obtained in parts (c) and (d), when $n > 2$ (except $M_{4_1}^{(3)}$), are all hyperbolic manifolds as well It follows from the Orbifold Theorem that branched n-fold coverings ($n > 2$) of S^3 branched along hyperbolic 2-bridge knots and links or along the Borromean rings are hyperbolic, except for $M_{4_1}^{(3)}$ which is the Euclidean manifold known as didicosm [Bo, HJM, Hod, Th].

The case $\frac{p}{q} = \frac{7}{4} = 1 + \frac{1}{1+\frac{1}{3}} = [3, 1, 1]$, that is, the branching set being the

5_2 knot, is of special interest since $M^{(3)}_{5_2}$ is conjectured to be the hyperbolic 3-manifold with the smallest volume [Ki]. The fact that $\pi_1(M^{(3)}_{5_2})$ is not left-orderable was observed in [CD, RSS]. The non-left-orderability in other cases is proved in [DPT] for the first time.

The special form of the presentations of the groups listed in our theorem above, allows us to conclude the theorem in most cases, using our Main Lemma.

Proposition 4.2.2 *The groups listed in Theorem 4.2.1 have the following presentations:*

(a) $\pi_1(M^{(n)}_{T_{(2',2k)}}) = \langle x_1, x_2, ..., x_n | \;\; x_1^k x_2^{-k},\; x_2^k x_3^{-k}, ..., x_n^k x_1^{-k},\; x_1 x_2 \cdots x_n\rangle$

(b) (i) $\pi_1(M^{(2)}_{P_{(n_1,n_2,...,n_k)}}) =$

$\langle x_1, x_2, ..., x_k | \;\; x_1^{n_1} x_2^{-n_2}, x_2^{n_2} x_3^{-n_3}, ..., x_k^{n_k} x_1^{-n_1},\; x_1 x_2 \cdots x_k\rangle$

(ii) $\pi_1(M^{(2)}_{P_{(2,2,...,2,-1)}}) = \langle x_1, x_2, ..., x_k | \;\; x_1^2 = x_2^2 = \cdots = x_k^2 = x_1 x_2 \cdots x_k\rangle$

(c) $\pi_1(M^{(n)}_{L_{[2k,2m]}}) =$

$\langle z_1, z_2, \ldots, z_{2n} | \;\; z_{2i+1} = z_{2i}^{-k} z_{2i+2}^{k},\; z_{2i} = z_{2i-1}^{-m} z_{2i+1}^{m},\; z_2 z_4 \ldots z_{2n} = e\rangle$ *where* $i = 1, 2, \ldots, n$ *and subscripts are taken modulo* $2n$*.*

(d) $\pi_1(M^{(n)}_{L_{[2k+1,1,2l+1]}}) = \langle x_1, \cdots, x_n | \;\; r_1, \cdots, r_n,\; x_1 x_2 \cdots x_n\rangle$, *where* $k \geq 0$, $l \geq 0$,

$$r_i = x_i^{-1}(x_i^{-k} x_{i+1}^{k+1} x_i^{-1})^l x_i^{-k} x_{i+1}^{k+1}((x_{i+1}^{-k} x_{i+2}^{k+1} x_{i+1}^{-1})^l x_{i+1}^{-k} x_{i+2}^{k+1})^{-1},$$

and subscripts are taken modulo n*.*

Proof. Since the presentations for all the manifolds from Theorem 4.2.1 are obtained by similar calculations, we shall only provide full details for the case (c) (See [MV]). Let T_1 denote the 2-tangle in Figure 4.3(a), and let T_2 denote the 2-tangle in Figure 4.3(b). Let us assume that the arcs of T_1 and T_2 are oriented in the way shown in Figure 4.3.

$T_1 = -[2k] \quad T_2 = [2m]$

Figure 4.3

Let $F_2 = \langle a, b \mid \ \rangle$ be the free group generated by a and b. Assign to the initial arcs of T_1 the generators a and b. By successive use of Wirtinger relations, progressing from left to right in the diagram, one finally decorates the terminal arcs by $\bar{u} = (ba^{-1})^k a (ab^{-1})^k$ and $u = (ba^{-1})^k b (ab^{-1})^k$, respectively (see Figure 4.3(a)). Analogously, assigning to initial arcs of the tangle $T_2 = [2m]$ (Figure 4.3(b)) the elements b and u of F_2 and using Wirtinger relations successively one obtains terminal arcs decorated by $w = (u^{-1}b)^m b (b^{-1}u)^m$ and $\bar{w} = (u^{-1}b)^m u (b^{-1}u)^m$, respectively. Combining these calculations in the fashion illustrated in Figure 4.4, we obtain the relation $((ba^{-1})^k b^{-1} (ab^{-1})^k b)^m b = a((ba^{-1})^k b^{-1} (ab^{-1})^k b)^m$ and the presentation

$$\pi_1(S^3 \backslash L_{[2k,2m]}) = \langle a, b |\ r = ((ba^{-1})^k b^{-1} (ab^{-1})^k b)^m b ((ba^{-1})^k b^{-1} (ab^{-1})^k b)^{-m} a^{-1} \rangle.$$

Figure 4.4; The 2-bridge knot $[2k, 2m]$

Using Fox non-commutative calculus [Cr], as explained in [Pr, PR], we compute a presentation of $\pi_1(M^{(n)}_{L_{[2k,2m]}})$ by "lifting" the generators a and b as well as the defining relation r of $\pi_1(S^3 \setminus L_{[2k,2m]})$.

We illustrate this by first computing a presentation of the fundamental group of the n-fold cyclic *unbranched* covering of $S^3 \setminus L_{[2k,2m]}$. Since $\pi_1(S^3 \setminus L_{[2k,2m]})$ has 2 generators, a and b, the covering space will have $n+1$ generators, that is, $y = ab^{-1}, \tau(y), \tau^2(y)..., \tau^{n-1}(y)$ and b^n, where τ is the inner automorphism of F_2, given by $w \mapsto bwb^{-1}$ (see Figure 4.5).

The relation r will also be lifted to n relations $\tilde{r}, \tau(\tilde{r}), \tau^2(\tilde{r}), ..., \tau^{n-1}(\tilde{r})$, in

Figure 4.5

the group of the n-fold cyclic covering, where

$$\tilde{r} = ((y^{-1})^k(\tau^{-1}(y))^k)^m((\tau(y^{-1}))^k(y)^k)^{-m}y^{-1}.$$

When dealing with the branched case the relations $a^n = e$ and $b^n = e$ should also be added in principle, but since the relation $a^n = e$ follows from the relation $b^n = e$ and the relations $\tau^i(\tilde{r})$, we can write the word a^n in terms of the new generators as $y\tau(y)...\tau^{n-1}(y)$. In order to simplify the presentation of $\pi_1(M^{(n)}_{L_{[2k,2m]}})$ we put $x_1 = \tau^{-1}(y)$, $x_2 = y$, $x_3 = \tau(y)$, $\ldots$, $x_n = \tau^{n-2}(y)$. Thus

$$\pi_1(M^{(n)}_{L_{[2k,2m]}}) = \langle x_1,\, x_2, \ldots,\, x_n |\ \ x_i^{-1}(x_i^{-k}x_{i-1}^k)^m(x_{i+1}^{-k}x_i^k)^{-m},\ x_1x_2\cdots x_n\rangle,$$

where $i = 1,\, 2,\, \ldots,\, n$ and subscripts are taken modulo n.

To change this presentation to the one described in part (c) of the above proposition we change variables by putting $z_{2i} = x_i$ and $z_{2i+1} = x_i^{-k}x_{i+1}^k$. In new variables the presentation has the desired form $\pi_1(M^{(n)}_{L_{[2k,2m]}}) =$ $\langle z_1, z_2, \ldots, z_{2n} |\ \ z_{2i+1} = z_{2i}^{-k}z_{2i+2}^k, z_{2i} = z_{2i-1}^{-m}z_{2i+1}^m, z_2z_4\cdots z_{2n} = e\rangle$, where

$i = 1, 2, ..., n$ and subscripts are taken modulo $2n$. (In the special case of $k = m = 1$ we obtain the classical Fibonacci group $F(2, 2n)$ already known to be the fundamental group of $M_{4_1}^{(n)}$). ■

It is worth mentioning that the case (c) that we singled out for illustrating the proof of the proposition involves a step that the proofs for other cases do not require. More specifically, all of the presentations given in the statement of this proposition, except for the case (c), are results of straightforward calculations and we do not need to change the variables in any way in those cases in order to obtain the desired presentation.

The following definition and Main Lemma capture the algebraic properties of listed groups.

Definition 4.2.3 *(i) Given a finite sequence $\langle \epsilon_1, \epsilon_2, ..., \epsilon_n \rangle$, $\epsilon_i \in \{-1, 1\}$, for all $i = 1, 2, ..., n$ and a nonempty reduced word $w = x_{a_1}^{b_1} x_{a_2}^{b_2} ... x_{a_m}^{b_m}$ of the free group $F_n = \langle x_1, x_2, ..., x_n \mid \rangle$, we say w* blocks *the sequence $\langle \epsilon_1, \epsilon_2, ..., \epsilon_n \rangle$ if either $\epsilon_{a_j} b_j > 0$ for all j or $\epsilon_{a_j} b_j < 0$ for all $j = 1, 2, ..., m$.*

(ii) A set W of reduced words of F_n is complete *if for any given sequence $\epsilon_1, \epsilon_2, ..., \epsilon_n$, $\epsilon_i \in \{-1, 1\}$, for $i = 1, 2, ..., n$, there is a word $w \in W$ that blocks $\epsilon_1, \epsilon_2, ..., \epsilon_n$.*

(iii) A presentation $\langle x_1, x_2, ..., x_n \mid W \rangle$ of a group G is called complete *if the set W of relations is complete.*

Lemma 4.2.4 (Main Lemma) *If a nontrivial group G admits a complete presentation, then it is not left-orderable.*

Proof. Suppose, on the contrary, that $<$ is a left-ordering on G and let $G = \langle x_1, x_2, ..., x_n \mid W \rangle$ be a complete presentation of G. Let E be the set of all $\langle \epsilon_1, \epsilon_2, ..., \epsilon_n \rangle$ such that $x_i^{\epsilon_i} \leq e$ in the group $\langle G, < \rangle$, where $\epsilon_i \in \{-1, 1\}$, $i = 1, 2, ..., n$. Since W is complete, each sequence $\langle \epsilon_1, \epsilon_2, ..., \epsilon_n \rangle \in E$ is blocked by some word $w \in W$. Since w is a relator, this is impossible, because the product of a number of "positive" elements in a left-orderable group will be "positive", not the identity, where by "positive" elements we simply mean elements that are in the positive cone given by that order. This contradiction completes the proof. ■

Our theorem now follows easily from the Main Lemma and the above proposition in all cases except for part (b)(ii) which we deal with separately in the following lemma.

Lemma 4.2.5 *Let* $F(n-1, n) =$

$\langle x_1, \cdots, x_n \mid x_1 x_2 \cdots x_{n-1} = x_n,\ x_2 x_3 \cdots x_n = x_1,\ \cdots, x_n x_1 \cdots x_{n-2} = x_{n-1} \rangle$.

If $n > 2$, *then* $F(n-1, n)$ *is not left-orderable.*

Proof. $F(2, 3)$ is finite (it is the quaternion group with eight elements), hence it is not left-orderable. Let us assume, then, that $n > 3$. First of all, note that the mapping $x_i \mapsto g : F(n-1, n) \to \{g \mid g^{n-2} = e\} = \mathbb{Z}_{n-2}$ defines an epimorphism, and since $n - 2 > 1$ our group is not the trivial group.

It is not hard to see that in $F(n-1, n)$ we have $x_1^2 = x_2^2 = \cdots = x_n^2 = x_1 x_2 \cdots x_n$. Let $t = x_i^2 = x_1 x_2 \cdots x_n$ for any i. Suppose $<$ is a left-ordering on $F(n-1, n)$. Since $F(n-1, n)$ is not the trivial group, we have $t \neq e$ unless our group has a torsion, which cannot be the case, since our group is assumed to be left-orderable. We will consider the case $t < e$. The case $t > e$ can be dealt with similarly.

Since $t = x_i^2$, we must have $x_i < e$ for all i. In particular, $x_i \neq e$ for all i. This makes $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_1$ impossible, because if $x_1 = x_2 = \cdots = x_n \neq e$, then $x_1^2 = t = x_1 x_2 \cdots x_n = x_1^n$ implies $x_1^{n-2} = e$, which in turn makes $F(n-1, n)$ a torsion group and thus non-left-orderable.

Therefore, $x_{i+1} < x_i$ for some i modulo n. Assume, without loss of generality, that $x_n < x_{n-1}$. Multiplying from the left by $x_1 x_2 \cdots x_{n-1}$ one obtains

$$t = x_1 x_2 \cdots x_{n-1} x_n < x_1 x_2 \cdots x_{n-2} x_{n-1} x_{n-1} = x_1 x_2 \cdots x_{n-2} t = t x_1 x_2 \cdots x_{n-2}.$$

The last equality holds because $t = x_i^2$ commutes will all the x_i. Multiplying both sides from the left by t^{-1} gives $e < x_1 x_2 \cdots x_{n-2}$, contradicting the fact that $x_i < e$ for all i. ■

Recall that the left-orderability of a countable group G is equivalent to G being isomorphic to a subgroup of $Aut(\langle \mathbb{R}, < \rangle) = Homeo_+(\mathbb{R})$ (See [BRW]). Calegari and Dunfield related left-orderability of the group of a 3-manifold M with foliations on M. We have the following

Corollary 4.2.6 *(i) The groups of manifolds described in Theorem 4.2.1 do not admit a faithful representation to* $Homeo_+(\mathbb{R})$.

(ii) Manifolds described in Theorem 4.2.1 do not admit a co-orientable $\mathbb{R}$-*covered foliation* [CD].

Thurston proved that if an atoroidal 3-manifold M has a taut foliation then there exists a faithful action of $\pi_1(M)$ on S^1 [CD]. Exploring the fact that the group of the manifold of the smallest known volume, $M^{(3)}_{5_2}$, (together with some of its subgroups) is not left-orderable, Calegari and Dunfield showed that $\pi_1(M^{(3)}_{5_2})$ does not admit a faithful action of $\pi_1(M)$ on S^1 and therefore $M^{(3)}_{5_2}$ does not admit a taut foliation [CD]. The connection between faithful actions of $\pi_1(M)$ on S^1 and on $\mathbb{R}$ needs to be explored further.

We would like to contrast our non-left-orderability results with some examples of left-orderable 3-manifold groups.

It is known that if $M^{(n)}_K$ is irreducible (as is always the case for a hyperbolic knot K) and the group $H_1(M^{(n)}_K)$ is infinite, then the group $\pi_1(M^{(n)}_K)$ is left-orderable [BRW, HS]. There are several examples of 2-bridge knots with infinite homology groups of cyclic branched coverings along them. For the trefoil knot 3_1 we have $H_1(M^{(6k)}_{3_1}) = \mathbb{Z} \oplus \mathbb{Z}$. For hyperbolic 2-bridge knots $9_6 = K_{[2,2,5]}$ and $10_{21} = K_{[3,4,1,2]}$ the groups $H_1(M^{(6)}_{9_6})$ and $H_1(M^{(10)}_{10_{21}})$ are also infinite.

To see why $H_1(M^{(n)}_K)$ is infinite one can use a result by Fox which says $H_1(M^{(n)}_K)$ is infinite if and only if the Alexander polynomial, $\Delta_K(t)$, is equal to zero when evaluated at some n-th root of unity. To test the last condition for small knots one can use tables of knots with $\Delta_K(t)$ decomposed into irreducible factors [BZ]. One may check, for example, that $\Delta_K(e^{\pi i/3}) = 0$ for hyperbolic 2-bridge knots $K = 8_{11}, 9_6, 9_{23}, 10_5, 10_9, 10_{32}$ and 10_{40}. Note also that Casson and Gordon proved that p^k-fold cyclic branched coverings along a knot, where p is prime, are rational homology spheres.

4.3 Open Problems

We end the chapter with some questions about possible generalizations of our results.

Problem 4.3.1 *(i) Are the groups* $\pi_1(M_{5_2}^{(n)})$ *non-left-orderable for* $n > 3$*?*

(ii) Are the groups $\pi_1(M_K^{(n)})$ *of hyperbolic 2-bridge knots* K *with finite* $H_1(M_K^{(n)})$ *non-left-orderable?*

(iii) Are the groups $\pi_1(M_K^{(n)})$ *of hyperbolic knots* K *with finite* $H_1(M_K^{(n)})$ *non-left-orderable?*

(iv) In general, for which links L *and multiplicities of covering* n*, is the group* $\pi_1(M_L^{(n)})$ *non-left-orderable?*

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