Topology of spaces of orderings of groups

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Let G be a group.

A linear order $<$ on G is a binary rel which is transitive ($a < b,\ b < c\ \Rightarrow a < c$) and either $a < b$ or $b < a$ or $a = b$ for every $a, b \in G$.

$<$ is a left order if $a < b\ \Rightarrow ca < cb$, for every $c \in G$

$<$ is a right order if $a < b\ \Rightarrow ac < bc$, for every c.

Let $LO(G), RO(G)$ be the sets of all left and right orderings of G.

Every left ordering $<$ defines a right ordering $<'$ such that $a <' b\ \Leftrightarrow\ b^{-1} < a^{-1}$.

Hence $LO(G) \stackrel{1-1}{\longleftrightarrow} RO(G)$.

$<$ is a <u>bi-order</u> on $G \Leftrightarrow$

$<$ is both left & right order $\Leftrightarrow$

$(a < b\ \Rightarrow cac^{-1} < cbc^{-1})$

Positive Cones

Positive cone of $<$ is $P = \{g : g > e\} \subset G$. It is a semigroup such that

$$P \cup \{e\} \cup P^{-1} = G. \tag{1}$$

Every semi-group $\subset G$ satisfying (1) defines a unique right order $g < h$ iff $h^{-1}g \in P$ and a unique left order $g < h$ iff $g^{-1}h \in P$.

sub-semi-groups of G satisfying (1) $\overset{1-1}{\longleftrightarrow} LO(G) \overset{1-1}{\longleftrightarrow} RO(G)$

$<$ is a bi-order iff $gPg^{-1} = P$ for every $g \in G$.

Classical Question: Which groups have a left order, bi-order?

Free groups, surface groups, free abelian groups are biorderable.
Locally indicable groups are right & left orderable.

My Goal: Analyze $LO(G)$ and $BiO(G)$.

3-**ways to topologize** $L0(G)$

Def 1. Let $U_g = \{h : e < h < g\}$. The topology on $LO(G)$ has basis composed of finite intersections of U_g's.

Def 2. Function: ordering $\to$ positive cone defines $LO(G) \hookrightarrow 2^G = \{f : G \to \{0,1\}\}$ has compact-open topology. Take the induced top on $LO(G)$.

Prop Def 1 and 2 are equivalent.

Def 3. Let $G_0 \subset G_1 \subset G_2 ... \subset G$ be a filtration of G by its subsets such that $\bigcup_i G_i = G$. For $<_1, <_2 \in LO(G)$,

$$\rho(<_1, <_2) = \frac{1}{2^r},$$

where r is the largest number such that $<_1$ and $<_2$ coincide on G_r.
$\rho(<_1, <_2) = 0$ if such r does not exist ($r = \infty$).

ρ is a metric on $LO(G)$.

Prop If filtration is by finite sets, then def 3 is equivalent to the other two. In particular, ρ does not depend on the choice of filtration.

Thm. (S.) For every G, $LO(G)$ is a compact, totally disconnected topological space.

X is totally disconnected $\Leftrightarrow$ $\forall x_1 \neq x_2$ have disjoint open nbhds U_1, U_2 such that
$U_1 \cup U_2 = X$.

Proof of compactness: We need to prove that any $<_1, <_2, ... \in LO(G)$ has a convergent subsequence.
Construction: Since there are only finitely many orderings of elts of G_1, there is an infinite subsequence of $<_1, <_2, ...$ whose elements induce the same order on G_1. Now, pick out of this sequence an infinite subsequence,

which agree on G_2, and so on ad infinitum. In this way, we get a convergent subsequence.

M. Dabkowska, M. Dabkowski, V. Harizanov, J. Przytycki, M. Veve: generalization to semi-groups and other structures.

My motivation was computational algebraic geometry.
Every $I \triangleleft \mathbb{C}[x_1, ..., x_n]$ has a Gröbner basis with respect to a given monomial ordering (i.e. an ordering of semi-group $\mathbb{N}^n$).

Thm Each ideal has a universal Gröbner basis.

Def 1. $<$ is isolated in $LO(G)$ if it uniquely determined by

$$a_1 < b_1 \ \& \ \ldots \ \& \ a_n < b_n$$

for some $a_1, b_1, \ldots., a_n, b_n \in G$.

Prop If the positive cone of $<$ is fin. generated semi-group then $<$ is isolated.

Prop If G has no isolated orderings then $LO(G) =$ the Cantor set or $\emptyset$.

Similarly, if G has no isolated bi-orderings then $BiO(G) =$ the Cantor set or $\emptyset$.

Eg. $\mathbb{Z}^n$ has no isolated orderings, for $n > 1$.

There are examples of groups with infinite countable $Bi(G)$.

We conjectured that free groups have no isolated left orderings.

Thm Free groups have no isolated left orderings. Hence $LO(F_n)$ is Cantor set.

Proofs:
1. Storozhuk-Kopytov that it follows from McCleary'85 work on lattice ordered groups.
2. A. Navas

Conj F_n does not have isolated bi-orderings.

Conrad orderings

Thm (Conrad) Let $<$ be a left order on G.
For any $g, h > e$ TFAE:
1. $(gh)^n > hg$ for some n.
2. if $g < h$ then $gh^n g^{-1} > h$ for some n
3. $g^n h > g$ for some n.

Navas: One can take $n = 2$.

Def $<$ is Conradian iff it satisfies above conds.

Every bi-order is Conradian.

Given $(G, <)$, $H \subset G$ is convex if for every $h_1, h_2 \in H$, and $g \in G$ such that $h_1 < g < h_2$, $g \in H$.

Eg. $\mathbb{Z}^2$

Properties

(1) $\{e\}, G$ are convex subgroups of G

(2) any intersection of convex subgroups is convex

(3) any union of convex subgroups is convex

(4) if H_1, H_2 are convex then $H_1 \subset H_2$ or $H_2 \subset H_1$.

$H' \subset H$ is a jump, denoted by $H' \prec H$, if there is no H'' such that $H' \subset H'' \subset H$.

Thm(Conrad) $<$ is Conradian iff

(5) $H' \triangleleft H$ for all $H' \prec H$ and $H/H' \subset \mathbb{R}$ (as an ordered group).

H/H' is a jump quotient.

Prop
(6) The induced order on H/H' has no proper convex subgroups.

Thm Conrad orderings $\stackrel{1-1}{\longleftrightarrow}$ systems of subgroups of G (with ordered jump quotients) satisfying (1)-(6).

Cor $Co(G) = \emptyset$ or 2^n or Cantor set.

$\{e\} = G_n \triangleleft G_{n-1} \triangleleft \ldots \triangleleft G_0 = G$ is a rational series for G if $G_k/G_{k+1} \subset \mathbb{Q}$.

By Tatarin, if $|Co(G)| = 2^n$ then G has a rational series of length n and that series is unique.

Thm G is Conrad orderable iff G is locally indicable, i.e. every fin. gen subgroup of G has

an epimorphim onto $\mathbb{Z}$.
$\Leftarrow$ Brodskii, $\Rightarrow$ Navas.

Thm(Navas) If $LO(G)$ is finite then,
$LO(G) = Co(G)$.

Cor If $LO(G)$ is finite then $|LO(G)| = 2^n$ or 0.

Conj $|Bi(G)|$ can be any even number.