Limit points in the space of left orderings of a group

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The space $LO(G)$ also comes equipped with a natural G-action by conjugation: A positive cone P is sent to gPg^{-1} by $g \in G$.

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Therefore, for a given group G, we look for isolated points in $LO(G)$; those positive cones P satisfying

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for some finite family of elements $g_i \in G$.
One way of determining that $P \in LO(G)$ is not an isolated point is to show that it is an accumulation point of its conjugates $gPg^{-1} \in LO(G)$.

The braid groups

Recall that for each integer $n \geq 2$, the Artin braid group B_n is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1, \quad \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \text{ if } |i-j| = 1.$$

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Definition

Let w be a word in the generators $\sigma_i, \cdots, \sigma_{n-1}$. Then w is said to be: i-positive if the generator σ_i occurs in W with only positive exponents, i-negative if σ_i occurs with only negative exponents, and i-neutral if σ_i does not occur in w.

The Dehornoy ordering

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The Dehornoy ordering is discrete, with least element σ_{n-1}.

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1. $\alpha P_D \alpha^{-1} \in \bigcap_{i=1}^m U_{\beta_i}$, and
2. $\alpha P_D \alpha^{-1} \neq P_D$.

Proof in the case of B_3

Given a finite family $\beta_1, \cdots, \beta_m$ in B_3 with $P_D \in \bigcap_{i=1}^{n} U_{\beta_i}$, suppose that $\alpha \in B_3$ satisfies $1 < \alpha \leq \beta_i$, whenever β_i is not a power of σ_2. Then we have:

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In other words, such an α satisfies $\alpha P_D \alpha^{-1} \in \bigcap_{i=1}^{m} U_{\beta_i}$.

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If α does not commute with σ_2, then the least element of the Dehornoy ordering is σ_2, while the least element of the ordering determined by the positive cone $\alpha P_D \alpha^{-1}$ is $\alpha \sigma_2 \alpha^{-1}$. Therefore $P_D \neq \alpha P_D \alpha^{-1}$.

Lemma

Suppose that $\beta_1, \cdots, \beta_m$ *is any finite family in* B_3*. Then there exists* α *in* B_3 *that does not commute with* σ_2*, and satisfies* $1 < \alpha \leq \beta_i$ *whenever* β_i *is not a power of* σ_2*.*

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1. If β_1 and σ_2 don't commute, choose $\alpha = \beta_1$.
2. If β_1 and σ_2 commute, then $\beta_1 = (\sigma_1\sigma_2\sigma_1)^{2p}\sigma_2^q$ for $p > 1$, and $\alpha = \sigma_1\sigma_2$ works.

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So P_D is an accumulation point of its conjugates in $LO(B_3)$, Andrés Navas showed that this implies that the theorem holds for all $n > 3$.

Working in an arbitrary group G

The key element to take away from this proof, that works in any group G: If we have a positive cone $P \in \bigcap_{i=1}^{m} U_{g_i}$, and we choose $h \in G$ with $1 < h \leq g_i$ for all i, then $h \leq g_i$ implies $1 \leq h^{-1}g_i$, so that $1 < h^{-1}g_i h$ for all i.

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Thus, if $\{P\} = \bigcap_{i=1}^{m} U_{g_i}$ is an isolated point, then $hPh^{-1} \in \bigcap_{i=1}^{m} U_{g_i}$ implies $hPh^{-1} = P$.
"Small elements are biordered."

Theorem (C)

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This method shows that the Conradian soul of an isolated point in $LO(G)$ is nontrivial, for any group G (not only countable).

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Thus, the ability to choose arbitrarily small elements in P should allow us to approximate P in $LO(G)$.
"Able to choose arbitrarily small elements" happens when P gives a dense ordering of the group G.

Theorem (C)

Let G be a group in which every rank one abelian subgroup is isomorphic to $\mathbb{Z}$, and let $Z \subset LO(G)$ be the set of all dense orderings of G. Then $\bar{Z}$ is a Cantor set, and Z is a dense G_δ set in $\bar{Z}$.