Simple Structures with Complex Symmetry

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November 12, 2009

Abstract

We define the *automorphism spectrum* of a computable structure $\mathcal{M}$, a measurement of the complexity of the symmetries of $\mathcal{M}$, and prove that certain sets of Turing degrees can be realized as automorphism spectra, while certain others cannot.

1 Introduction

Throughout the history of mathematics, the study of symmetry has been central to the subject. It is an intrinsic part of geometry, with wide-ranging aesthetic and philosophical connotations. Moreover, it has come to have meaning not merely for geometric forms, but for abstract mathematical objects as well. Since the introduction of the *Erlanger Programm* by Felix Klein in 1872, it has become widely accepted that one studies the symmetries of any mathematical object by examining its automorphisms: those bijective maps from the object onto itself which preserve the essential properties of the object. For geometric forms, this is intuitively natural, but the definition lends

[*]The first author was partially supported by the NSF grants DMS-0704256 and DMS-0904101 and by the GWU REF Award. All three authors were partially supported by the NSF binational grant DMS-0554841. The authors wish to acknowledge useful conversations with Denis Hirschfeldt, Jim Schmerl, and Steve Simpson. Also, an anonymous referee gave a nice simplification of our original proof of Proposition 3.7.

[†]The second author was partially supported by grants numbered 60095-34-35, 67182-00-36, 68470-00 37, 69723-00 38, 61467-00 39, and 80209-04-12 from The City University of New York PSC-CUNY Research Award Program, and by Grant #13397 from the Templeton Foundation.

itself to almost all structures one meets in mathematics. In model theory, for example, the essential properties of a structure are its functions, relations, and constants, as named by the signature in which the structure is built. An automorphism of the structure gives an abstract way of "reflecting" the structure onto itself, so that the reflected image is indistinguishable from the original structure. If the integers $\mathbb{Z}$ are regarded as a linear order, i.e. with only the $<$ relation in the signature, they have countably many symmetries, whereas in the signature with only the addition function (i.e. as an additive group) they have just two symmetries, including the trivial one, and in the signature with both these symbols – that is, as an ordered additive group – they have only the trivial symmetry of the identity map.

In this paper we introduce a notion of complexity for the symmetries of a computable structure. (Below we remind the reader of this and other basic notions from computable model theory.) The automorphisms of the structure will all be functions from its domain ω, the set of natural numbers, onto ω, so it is natural to consider the Turing degree of an automorphism. A computable automorphism suggests a relatively simple symmetry, from the point of view of complexity: an algorithm given by a finite program can determine the symmetry defined by that automorphism. Automorphisms of larger Turing degree, on the other hand, suggest symmetries of greater complexity. For example, a computable finite-dimensional rational vector space will have only computable automorphisms: its symmetries are all readily presented, even though there are infinitely many of them. On the other hand, an infinite-dimensional space, assuming that it has a computable basis, would have an automorphism (hence a symmetry) of each Turing degree. Although we will not address vector spaces in this paper, the reader may find it illuminating also to consider the case of a computable rational vector space with no computable basis, and to ask whether such a space will still have symmetries of all possible degrees. (Such spaces are discussed in [1] and [13].)

For these reasons we consider our definition of the *automorphism spectrum* both natural and compelling.

Definition 1.1 Let $\mathcal{M}$ be any computable structure. The *automorphism spectrum of* $\mathcal{M}$ is the set

$$\mathrm{AutSp}^*(\mathcal{M}) = \{\deg(f) : f \in \mathrm{Aut}(\mathcal{M}) \ \& \ (\exists x \in \mathcal{M})[f(x) \neq x]\}$$

where $\mathrm{Aut}(\mathcal{M})$ is the group of all automorphisms of $\mathcal{M}$.

Of course, the identity function on $\mathcal{M}$ is always an automorphism, with Turing degree $\mathbf{0}$. We exclude it from consideration, on grounds that under this exclusion, $\text{AutSp}^*(\mathcal{M})$ provides more information about the symmetries of $\mathcal{M}$: it may still contain $\mathbf{0}$, if there is a computable automorphism of $\mathcal{M}$ besides the identity, or it may not. Readers who feel that $(\mathbb{Z}, +)$ has just one symmetry, not two (as stated above), should approve of this exclusion. The star in our notation $\text{AutSp}^*(\mathcal{M})$ is intended to denote the exclusion, much as $\mathbb{Q}^*$ often denotes the set of nonzero rational numbers.

One could equally well apply the definition of automorphism spectrum to a noncomputable structure $\mathcal{M}$ with domain ω. However, this paper is devoted mainly to the following question.

Question 1.2 *Which sets of Turing degrees can be the automorphism spectrum of a computable structure?*

Recall that in a computable language, an infinite structure is *computable* if its domain is ω, and its atomic diagram is computable. This is equivalent to requiring that all the functions and relations in the structure be computable, when viewed as functions and relations on ω. (If the language is infinite, it also requires a uniformity in the presentation of the functions, relations, and constants of the structure.) For example, the three examples in our first paragraph are all readily seen to be isomorphic to computable structures, although technically, having domain $\mathbb{Z}$, they are not themselves computable. For finite structures, one applies the same definition, allowing a finite subset of ω as the domain. However, all symmetries of such a structure are computable, so we will have no interest in finite structures.

The use of the term *spectrum* is intended to connect this topic to two related notions: the spectrum of a structure, and the spectrum of a relation. For a countable infinite structure $\mathcal{S}$, the spectrum $\text{Spec}(\mathcal{S})$ is the set of all Turing degrees of structures (with domain ω) isomorphic to $\mathcal{S}$. Thus $\text{Spec}(\mathcal{S})$ measures both the complexity intrinsic to the isomorphism type of $\mathcal{S}$, by excluding degrees which cannot compute a copy of $\mathcal{S}$, and the possibility of encoding additional complexity into a copy of $\mathcal{S}$, by including higher degrees in $\text{Spec}(\mathcal{S})$. Likewise, for an additional relation R on a computable structure $\mathcal{A}$, the spectrum $\text{DgSp}_{\mathcal{A}}(R)$ is the set of all Turing degrees of images of R under isomorphisms from $\mathcal{A}$ onto other computable structures; again, this measures both the relation's intrinsic complexity and its capacity to encode additional complexity. In posing Question 1.2, we exclude noncomputable structures for the same reason that they are excluded from the definition of the spectrum of a relation: we wish to measure purely the complexity of the symmetries, without interference from any particular complexity built into the structure. Many of our results would carry over to noncomputable structures if one relativized the results to a specific Turing degree and allowed the structure to be computable in that degree.

In Sections 2, 3, and 4, we begin considering Question 1.2 by trying to realize finite sets of arithmetical Turing degrees as automorphism spectra. For singleton sets, we prove that the known property of containing a Π^0_1-function singleton is equivalent to being the unique degree in the automorphism spectrum of some computable structure. For certain sets of degrees we also have uniformity results, which lead to a theorem allowing us to realize many infinite sets as well, including the set of all c.e. degrees, the set of all Σ^0_{n+1} degrees which compute $\mathbf{0}^{(n)}$, and the union of those sets for all n. It also yields a structure $\mathcal{A}$ with $\mathrm{AutSp}^*(\mathcal{A}) = \{\mathbf{0}^{(\omega)}\}$, showing that one can move beyond arithmetical degrees.

The structures built in these sections are not readily recognizable to mathematicians, since their specific purpose is to realize particular automorphism spectra. However, in Section 5, we cite a construction from [6] to show that for every computable structure $\mathcal{M}$, there is a computable graph $\mathcal{G}$ with the same automorphism spectrum as $\mathcal{M}$. This result includes the case where $\mathcal{M}$ has infinite computable signature.

The set of automorphisms of $\mathcal{M}$ carries a natural group structure under composition, and the composition of two automorphisms is always computable from the join of their Turing degrees. However, the degree of the composition may be strictly below that join, so it is not clear that the automorphism spectrum need be closed under the join operation. In Section 6 we show that it is possible for an automorphism spectrum to consist of exactly three degrees, pairwise incomparable with each other. However, if an automorphism spectrum contains exactly two degrees, we show that those two degrees must be comparable.

Finally in Section 7, we show how to take a given computable $\mathcal{M}$ and build a computable $\mathcal{S}$ such that $\mathrm{AutSp}^*(\mathcal{S})$ is the upward closure of $\mathrm{AutSp}^*(\mathcal{M})$ under Turing reducibility. This allows us to build automorphism spectra which are the unions of the upper cones above the members of various (finite or countable) antichains of degrees. It remains unknown whether such unions can be spectra of structures.

Our computability-theoretic notation is standard and follows [20]. We use the usual notation $\langle x_1, \ldots, x_n \rangle$ to denote a computable coding of tuples from $\omega^{<\omega}$ by elements of ω.

2 Singleton Automorphism Spectra

The goal of Sections 2, 3, and 4 is to prove that various well-known sets, finite and infinite, of arithmetical degrees can be realized as the automorphism spectra of computable structures. Indeed, a few of our results will pass beyond the arithmetical degrees. Along the way we notice that it is common for two isomorphic computable (but not computably isomorphic) structures to have distinct automorphism spectra.

We begin with singleton automorphism spectra, by considering the structure $\mathcal{A}_1$ consisting of two disjoint copies of $(\omega, <)$. We build $\mathcal{A}_1$ itself to be the disjoint union of two standard copies of this structure. (For simplicity, each chain will have domain ω.) By definition, the *standard copy* of the ordinal ω^n is given by the lexicographic order on the set ω^n, under a computable bijection from this set onto ω. Thus the immediate-successor relation and the set of limit points (and the set of limit points of limit points, etc.) are computable.

Now fix any infinite c.e. set C, and let $\mathcal{B}_1$ be the isomorphic copy of $\mathcal{A}_1$ which we build as follows. One string in $\mathcal{B}_1$ is a standard copy of ω, and for the other, which we call $\mathcal{B}_C$, we start by defining

$$0 \prec 2 \prec 4 \prec 6 \prec \cdots .$$

Call these the *key numbers* $k_n = 2n$. Then, every time an element n enters C, we add to $\mathcal{B}_C$ the least remaining odd number, placing it between k_n and k_{n+1}. Since C is infinite, $\mathcal{B}_C$ has domain ω, and clearly the disjoint union of these two strings forms a computable copy $\mathcal{B}_1$ of $\mathcal{A}_1$.

Now $\mathcal{A}_1$ has a unique nontrivial automorphism, which is computable. For $\mathcal{B}_1$, however, the nontrivial automorphism f has the same Turing degree as C: if one knows f, then one can compute whether $n \in C$ by checking whether $f(k_n)+1 = f(k_{n+1})$; and if one can compute C, then it is clear how to compute f. (Notice that $f = f^{-1}$, since f has order 2.) This proves:

Proposition 2.1 *There exists a computable structure $\mathcal{A}_1$ such that for every c.e. degree $\boldsymbol{d}$, some computable copy of $\mathcal{A}_1$ has automorphism spectrum $\{\boldsymbol{d}\}$.* ∎

Corollary 2.2 *There exists a structure whose computable copies have infinitely many distinct automorphism spectra.* ■

We give a name to the technique used here.

Definition 2.3 For any computable structures $\mathcal{M}$ and $\mathcal{N}$ in a relational signature, the *disjoint sum* of $\mathcal{M}$ and $\mathcal{N}$ is the computable structure $\mathcal{C}$ whose domain is the disjoint union of the domains of $\mathcal{M}$ and $\mathcal{N}$. For each relation symbol P, $P^{\mathcal{C}} = P^{\mathcal{M}} \cup P^{\mathcal{N}}$, but the language of $\mathcal{C}$ has an additional equivalence relation $R(x, y)$, which holds in $\mathcal{C}$ for all pairs (x, y) from $\mathcal{M}^2$ and all pairs from $\mathcal{N}^2$, but does not hold for any $x \in \mathcal{M}$ and $y \in \mathcal{N}$ or *vice versa.*

If, as with $\mathcal{A}_1$ in Proposition 2.1, $\mathcal{C}$ is the disjoint sum of two copies $\mathcal{M}$ and $\mathcal{N}$ of a rigid structure, then the only possible automorphisms of $\mathcal{C}$ map the $\mathcal{M}$ part either entirely onto itself or entirely onto the $\mathcal{N}$ part. (This is the role of the relation R.) By rigidity, the first of these is the identity and the second interchanges $\mathcal{M}$ with $\mathcal{N}$, using the unique isomorphism between them. So the only nontrivial automorphism of $\mathcal{C}$ has precisely the same degree as that isomorphism.

We give one further example in this vein. Let $\mathcal{A}_2$ be the disjoint sum of two standard copies of the linear order ω^2. Fix a set C which is c.e. in $\emptyset'$ and $\geq_T \emptyset'$; we say for short that C is *c.e.a. in* $\emptyset'$. Then there exists a 1-1 computable function g such that $C \leq_1 \text{Fin}$ via g. (Recall that $\text{Fin} = \{e : W_e \text{ is finite}\}$ is a Σ_2-complete set.) We define $\mathcal{B}_2$ to be the disjoint sum of two computable ω^2-chains built as follows. Again the first ω^2-chain is the standard copy. For the second chain $\mathcal{B}_C$, we start with half the numbers, in order:

$$0 \prec 2 \prec 4 \prec \cdots .$$

Again we refer to these numbers as the *key numbers* $k_n = 2n$. At stage $s+1$, for each n such that $W_{g(n)}$ acquires a new element at stage s, we add the least available odd number as the immediate predecessor of k_n in the second chain. Let f be the unique nontrivial automorphism of $\mathcal{B}_2$. Then $n \in C$ iff $g(n) \in \text{Fin}$, iff k_n has an immediate predecessor in the structure $\mathcal{B}_C$, iff $f(k_n)$ is a pair $\langle j_1, j_2 \rangle$ in the standard copy with $j_2 > 0$. Hence $C \leq_T f$. On the other hand, given a C-oracle, we can build f. First, for each $n \notin C$, $f(k_n) = \langle |\overline{C} \cap \{0, \ldots, n\}|, 0 \rangle$, which computes f on all limit points in each chain. For each $n \in C$, we can use the C-oracle to determine the size of $W_{g(n)}$, since $C \geq_T \emptyset'$ and $W_{g(n)}$ is finite, and thus we can determine $f(k_n)$

for all those n as well. To compute f on any remaining element j in the second chain, we just run the construction until j appears in $\mathcal{B}_2$, and then find $f(k_n)$ for the greatest key number $k_n \prec j$. Since no elements $> j$ will ever be placed between j and this k_n, we can now compute $f(j)$. This proves:

Proposition 2.4 *There exists a computable structure $\mathcal{A}_2$ such that for every Σ^0_2 degree $\boldsymbol{d} \geq_T \mathbf{0}'$, some computable copy of $\mathcal{A}_2$ has automorphism spectrum $\{\boldsymbol{d}\}$.* ∎

It is possible to continue with similar constructions for all Σ^0_{n+1} degrees above $\mathbf{0}^{(n)}$, for each $n \in \omega$, either for individual n or by induction on n. In fact, Theorem 2.6 will prove similar results for a significantly larger class of Turing degrees, but only for individual degrees: for two distinct degrees, the corresponding computable structures will not be isomorphic. Here we wish to extend Proposition 2.4 to the Σ^0_n degrees, preserving the uniformity. To do so, we apply a technique of Marker and others, as given in [12] and described in [2]. For simplicity, we will assume Corollary 5.2 for the moment, so that our structures may be taken to have finite relational signature. Given a structure $\mathcal{A}$ of relational signature, Marker's technique builds structures $\mathcal{A}_\forall$ and $\mathcal{A}_\exists$. For each n-ary predicate P in the signature of $\mathcal{A}$ such that $P^\mathcal{A}$ is infinite and coinfinite, the signature of $\mathcal{A}_\exists$ contains one $(n+1)$-ary predicate $P_\exists$ and one unary predicate X_P. (The predicate symbol P itself is not in the signature of $\mathcal{A}_\exists$, unless $P^\mathcal{A}$ was finite or cofinite, in which case $P^{\mathcal{A}_\exists} = P^\mathcal{A}$.)

Each X_P defines an infinite set $X_P^{\mathcal{A}_\exists}$ of *witness elements* for P in $\mathcal{A}_\exists$, and the domain of $\mathcal{A}_\exists$ contains all these disjoint sets $X_P^{\mathcal{A}_\exists}$, along with the original domain of $\mathcal{A}$. If $P(\vec{a})$ holds in $\mathcal{A}$, then $P_\exists(\vec{a}, x)$ holds for exactly one $x \in \mathcal{A}_\exists$, and that witness x lies in X_P; if not, then $P_\exists(\vec{a}, x)$ holds for no x in $\mathcal{A}_\exists$. Every $x \in X_P$ serves as witness for exactly one tuple $\vec{a}$.

The model $\mathcal{A}_\forall$ is dual, with the same signature as $\mathcal{A}_\exists$. Each $x \in X_P$ now serves as a negative witness for exactly one $\vec{a}$, with $P(\vec{a})$ holding iff $P_\forall(\vec{a}, x)$ fails for exactly one $x \in X_P$; when $P(\vec{a})$ is false, then $P_\forall(\vec{a}, x)$ holds iff $x \in X_P$. (Again the original predicate symbol P is in the language of $\mathcal{A}_\forall$ iff $P^\mathcal{A}$ was finite or cofinite.)

The usefulness of these new structures for us follows from Proposition 5.7 of [2], which we summarize here, with $\mathcal{A}_{\forall\exists} = (\mathcal{A}_\forall)_\exists$ built by applying both of these operations in turn.

Theorem 2.5 (various authors; see [2]) *In a finite signature, a structure $\mathcal{A}$ is $\emptyset^{(n+1)}$-presentable iff the structure $\mathcal{A}_{\forall\exists}$ is $\emptyset^{(n)}$-presentable. More-*

over, isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ correspond bijectively with isomorphisms between $\mathcal{A}_{\forall\exists}$ and $\mathcal{B}_{\forall\exists}$, and if $\mathcal{A}_{\forall\exists}$ and $\mathcal{B}_{\forall\exists}$ are built in the natural way (as suggested by the description above), then this correspondence preserves the Turing degree of any isomorphism of degree $\geq deg(\mathcal{A})$. ■

The construction in Proposition 2.1 built isomorphic computable structures for all infinite c.e. sets. Since $\mathcal{A}_\forall \cong \mathcal{B}_\forall$ and $\mathcal{A}_\exists \cong \mathcal{B}_\exists$ whenever $\mathcal{A} \cong \mathcal{B}$, we retain uniformity in the case of Σ^0_{n+1} degrees: the structures we get from Theorem 2.5 are all isomorphic (for any single n), and their construction is uniform in the oracle set.

Theorem 2.6 *For each $n \in \omega$, there exists a rigid computable structure $\mathcal{C}_n$ such that for every Σ^0_{n+1} degree $\boldsymbol{d} \geq_T \mathbf{0}^{(n)}$, some computable $\mathcal{D}_{\boldsymbol{d}} \cong \mathcal{C}_n$ has an isomorphism onto $\mathcal{C}_n$ of degree $\boldsymbol{d}$. Consequently, the disjoint sum $\mathcal{B}_{\boldsymbol{d}}$ of $\mathcal{C}_n$ and $\mathcal{D}_{\boldsymbol{d}}$ has automorphism spectrum $\{\boldsymbol{d}\}$, and its isomorphism type depends only on n. Indeed, $\mathcal{D}_{\boldsymbol{d}}$ and $\mathcal{B}_{\boldsymbol{d}}$ may be computed uniformly in any index e such that the set $W_e^{\emptyset^{(n)}}$ is infinite and of degree $\boldsymbol{d}$.* ■

The *standard copy* of $\mathcal{A}_n$ is the structure $(\mathcal{A}_0)_{\forall\exists\cdots\exists}$, with the $\forall\exists$-technique applied n times to the structure $\mathcal{A}_0$ from Proposition 2.1, which consisted of two disjoint chains $(\omega, <)$. We view it as the disjoint sum of two copies of $\mathcal{C}_n$. Using techniques from [10], we can show that one could also have taken this $\mathcal{A}_n$ to be the disjoint union of two standard copies of the linear order ω^{m+1}, where $n = 2m$ or $n + 1 = 2m$.

One naturally asks whether the results in this section for Σ^0_1 degrees can be extended to the Δ^0_2 degrees. An affirmative answer for individual Δ^0_2 degrees has been proven, by Hirschfeldt [5] and independently by Schmerl [19], and Theorem 2.5 then lifts their results to the set of Δ^0_{n+2} degrees above $\mathbf{0}^{(n)}$. However, both of their proofs are nonuniform: given $\boldsymbol{d} \leq_T \mathbf{0}'$, they construct a computable $\mathcal{B}_{\boldsymbol{d}}$ with $\mathrm{AutSp}^*(\mathcal{B}_{\boldsymbol{d}}) = \{\boldsymbol{d}\}$, but for $\boldsymbol{d}_0 \neq \boldsymbol{d}_1$, one may have $\mathcal{B}_{\boldsymbol{d}_0} \not\cong \mathcal{B}_{\boldsymbol{d}_1}$. Since these results follow from Theorem 3.4 below, we omit Hirschfeldt's and Schmerl's proofs, even though the structures they build help illustrate the challenge of trying to uniformize their results. It remains open whether there exists a single computable structure $\mathcal{A}$ such that $\{\mathrm{AutSp}^*(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\ \&\ \mathcal{B} \text{ computable}\}$ contains every singleton $\{\boldsymbol{d}\}$ with $\boldsymbol{d} \leq_T \mathbf{0}'$. Likewise, the results about Σ^0_n degrees in Section 4 depend on uniformity, and therefore the question of extending those results to the Δ^0_n degrees also remains open.

3 Π^0_2-Singletons

Now we investigate singleton automorphism spectra in more generality. For this purpose, it is often useful to view automorphisms as paths in the tree $\omega^{<\omega}$, or in subtrees of this tree.

Definition 3.1 A total function $f : \omega \to \omega$ is said to be a Π^0_1-*function singleton* if there exists a computable tree $T \subseteq \omega^{<\omega}$ through which f is the unique (infinite) path. (To be a *computable tree*, this T must be a computable subset of $\omega^{<\omega}$ closed under initial segments.)

Theorem 3.2 *Let $\mathcal{M}$ be a computable structure. Then the following statements are equivalent:*

1. *The set AutSp*$^*(\mathcal{M})$ *is at most countable.*
2. *The automorphism group of $\mathcal{M}$ is at most countable.*
3. *Every degree in AutSp*$^*(\mathcal{M})$ *contains a Π^0_1-function singleton.*

Proof. ($1 \iff 2$) follows from the fact that each Turing degree is countable, and ($3 \implies 1$) is immediate, since there are countably many computable trees. So we prove ($2 \implies 3$). By Kueker's theorem [11], there is a tuple $\bar{p} = p_1, \ldots, p_n \in \mathcal{M}$ such that $\langle \mathcal{M}, \bar{p} \rangle$ has no nontrivial automorphisms. Hence each automorphism $\varphi \in \mathrm{Aut}(\mathcal{M})$ is uniquely defined by the tuple

$$\bar{q} = \langle q_1, \ldots, q_n \rangle = \langle \varphi(p_1), \ldots, \varphi(p_n) \rangle = \varphi(\bar{p}),$$

i.e., φ is the only automorphism taking $\bar{p}$ to $\bar{q}$. To prove that such a φ is a Π^0_1-function singleton, we let f_0 be the finite function $\{\langle p_i, q_i \rangle \mid i = 1, \ldots, n\}$. Consider a tree T constructed as follows.

Let σ be the signature of $\mathcal{M}$; we may assume σ to be relational. Denote by σ_n the finite part of σ formed by its first n symbols. Let $\mathcal{M}\restriction\sigma_n$ be the reduct of $\mathcal{M}$ to the language σ_n, i.e., the structure whose universe is the same as that for $\mathcal{M}$ and whose operations are exactly those whose names are contained in σ_n.

The vertices of our tree T are pairs $\langle n, f \rangle$ such that:

1. $n \in \omega$, $f \in \omega^{<\omega}$, $f \supseteq f_0$; and
2. f is a partial automorphism of $\mathcal{M}\restriction\sigma_n$; and

3. $n \subseteq \mathtt{dom}\,(f) \cap \mathtt{ran}\,(f)$; and

4. among functions satisfying the properties (1) — (3), f is minimal under inclusion.

If $\langle m, g\rangle, \langle n, h\rangle \in T$, we say that $\langle m, g\rangle$ is a *successor of* $\langle n, h\rangle$ *in* T if $n < m$ and $h \subseteq g$. Identifying the vertices of T with their Gödel numbers, we see that T is computable.

One can easily check that if $\xi = (\langle i, h_i\rangle)_{i<\omega}$ is an infinite branch of T then $h_\xi = \cup_{i<\omega} h_i$ is an automorphism of $\mathcal{M}$ taking $\bar{p}$ to $\bar{q}$. On the other hand, if ψ is an automorphism of $\mathcal{M}$ taking $\bar{p}$ to $\bar{q}$, then the family $\xi_\psi = (\langle i, h_i\rangle)_{i<\omega}$, where

$$h_i = \psi{\restriction}(\{0, \ldots, i-1\} \cup \bar{p} \cup \psi^{-1}(\{0, \ldots, i-1\})),$$

is an infinite branch of T. Note that the mappings $\xi \mapsto h_\xi$ and $\psi \mapsto \xi_\psi$ are mutually inverse. Hence T has a unique infinite branch. It is readily checked that the above mappings preserve Turing degrees. Thus, the unique automorphism φ taking $\bar{p}$ to $\bar{q}$ is a Π^0_1-function singleton. ■

Our next equivalence combines Theorem 3.2 with a theorem of Jockusch and McLaughlin.

Proposition 3.3 *For a Turing degree* $\boldsymbol{d}$*, the following are equivalent.*

1. $\boldsymbol{d}$ *contains a* Π^0_1*-function singleton.*

2. $\boldsymbol{d}$ *contains a* Π^0_2*-set singleton. By definition, this means that* $\boldsymbol{d}$ *contains a set* A *for which there is a* Π^0_2 *formula* φ *with a free set variable* X *to which* A *is the unique solution in* 2^ω*:*

$$X = A \iff X \models \varphi.$$

3. $\{\boldsymbol{d}\}$ *is the automorphism spectrum of some computable structure* $\mathcal{A}$*.*

Proof. ($1 \iff 2$) follows from results of Jockusch and McLaughlin in [7]; see also [16, XII.2.14(d)]. The forward implication ($1 \implies 3$) was proven in Chapter 8 of [15], and its converse is immediate from Theorem 3.2. ■

These equivalences, along with known facts about Π^0_2-set singletons, yield a substantial number of results on singleton automorphism spectra. Most of the known facts have been gathered together by Odifreddi [16]; Sacks [18] also includes some of them.

Theorem 3.4 *For all Turing degrees $\boldsymbol{d}$ satisfying any of the following conditions, there exists a computable structure $\mathcal{A}$ with automorphism spectrum $\{\boldsymbol{d}\}$:*

1. $\boldsymbol{c} \leq_T \boldsymbol{d} \leq_T \boldsymbol{c}'$, *where there exists a computable $\mathcal{B}$ with AutSp$^*(\mathcal{B}) = \{\boldsymbol{c}\}$.*
2. $\mathbf{0}^{(n)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(n+1)}$, *for any $n \in \omega$.*
3. $\boldsymbol{d} = \mathbf{0}^{(\omega)}$.
4. $\boldsymbol{d} = \mathbf{0}^{(\alpha)}$, *for any computable ordinal α.*
5. $\mathbf{0}^{(\alpha)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(\alpha+1)}$, *for any computable ordinal α.*
6. $\boldsymbol{d}$ *is the degree of some ω-c.e.a. set. (This notion is defined using the ω-hop operator; see [16, XIII.2].) However, not all singleton automorphism spectra contain degrees of ω-c.e.a. sets.*

Moreover, every nonempty countable Σ^0_3 class of sets contains a set of some degree $\boldsymbol{d}$ such that $\{\boldsymbol{d}\}$ is an automorphism spectrum.

Proof. The first item follows from [16, XII.2.15 (d)], and the third from [16, XII.2.19] (first proven by Hilbert and Bernays in [4]), using Proposition 3.3. The fourth is an immediate consequence of the existence of Π^0_1-function singletons of these degrees (see [16, p. 797]), and indeed has already been proven in [14] and [15]. These in turn imply the second and fifth items. Notice that the fifth item shows that singleton automorphism spectra are cofinal in the Δ^1_1 degrees. The sixth item follows from the result [16, XIII.2.7], proven in [8] by Jockusch and Shore.

Finally, the remark about Σ^0_3 classes follows from [16, XII.2.20], which is a basis theorem proven by Tanaka. ■

In 1976 Harrington proved that there exists a Π^0_2-set singleton (containing an ω-c.e.a. set, in fact) whose degree $\boldsymbol{d}$ is not arithmetical, but also satisfies $\forall n(\mathbf{0}^{(\omega)} \not\leq_T \boldsymbol{d}^{(n)})$. (See [16, XIII.3] for a sketch of the proof, which Harrington never published.) This can be viewed as a much stronger version of our Corollary 3.8 below. Strictly speaking, it does not generalize that corollary, but this $\boldsymbol{d}$ lies well outside the "bubbles" defined by item 5 in Theorem 3.4.

Theorem 3.5 *There exists a nonarithmetical singleton automorphism spectrum $\{\boldsymbol{d}\}$ such that for all $n \in \omega$, $\mathbf{0}^{(\omega)} \not\leq_T \boldsymbol{d}^{(n)}$.* ■

Finally, we also have some restrictions on the degrees which can form singleton automorphism spectra. First, if $\{\boldsymbol{d}\}$ is an automorphism spectrum, then $\boldsymbol{d}$ must be Δ^1_1, by [16, XII.2.16]. Also, if such a $\boldsymbol{d}$ is a minimal degree (i.e. minimal under Turing reducibility among all nonzero Turing degrees), then $\boldsymbol{d} \leq_T \mathbf{0}'$, by [16, XII.2.15(f)]. This result was first established by Jockusch and McLaughlin in [7]. Since there do exist minimal Turing degrees which are arithmetical but not Δ^0_2, this yields the following.

Theorem 3.6 *If $\{\boldsymbol{d}\}$ is an automorphism spectrum, then $\boldsymbol{d}$ is a hyperarithmetical degree. However, the converse fails, and indeed there exists an arithmetical Turing degree $\boldsymbol{d}$ such that no computable structure has automorphism spectrum $\{\boldsymbol{d}\}$.* ∎

For completeness, we now exhibit singleton automorphism spectra containing arithmetical degrees outside the union of the intervals $[\mathbf{0}^{(n)}, \mathbf{0}^{(n+1)}]$.

Proposition 3.7 *There exists a computable structure with singleton automorphism spectrum containing a degree $\boldsymbol{d} \leq_T \mathbf{0}''$ incomparable with $\mathbf{0}'$.* ∎

Proof of Proposition 3.7. Let $A <_T \emptyset'$ be any nonlow c.e. set. By relativizing to A a theorem of Yates from [21] (see [20, Thm. VIII.4.3]), we get a set $B \geq_T A$ which is c.e. in A, hence computable in $\emptyset''$, but Turing-incomparable to $\emptyset'$. The chain $\emptyset \leq_T A \leq_T B$ shows that this B is 2-c.e.a., as defined in [16, Section XI.5], since each set in the chain is c.e. in the previous one. However, every n-c.e.a. degree forms a singleton automorphism spectrum, by n applications of part (1) of Theorem 3.4. ∎

Corollary 3.8 *For every $n \in \omega$, there exists a computable structure $\mathcal{A}_n$ and a Turing degree $\boldsymbol{d}$ with $\mathbf{0}^{(n)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(n+2)}$ such that $\boldsymbol{d}$ is incomparable with $\mathbf{0}^{(n+1)}$ and $AutSp^*(\mathcal{A}_n) = \{\boldsymbol{d}\}$.*

Proof. One proof builds an $(n+1)$-c.e.a. set B with $\emptyset^{(n)} <_T B \mid_T \emptyset^{(n+1)}$. A second proof relativizes the constructions of Proposition 3.7 to $\mathbf{0}^{(n)}$, replacing A by $A \oplus \emptyset^{(n)}$ before choosing B. This yields a $\mathbf{0}^{(n)}$-computable structure $\mathcal{B}_n$ whose unique nontrivial automorphism has degree $\boldsymbol{d} \mid_T \mathbf{0}^{(n+1)}$ with $\boldsymbol{d} = \deg(B \oplus \emptyset^{(n)}) \geq_T \mathbf{0}^{(n)}$, and Theorem 2.5 allows us to work down to a computable structure $\mathcal{A}_n$ with $\text{AutSp}^*(\mathcal{A}_n) = \{\boldsymbol{d}\}$. ∎

4 Building Larger Automorphism Spectra

Our next goal is to be able to collect together the structures we have built with singleton automorphism spectra, and use them to form larger automorphism spectra. For finite sets of degrees, this is simple, except for concerns about the join. Indeed, the result may be stated for any finite collection of computable structures.

Lemma 4.1 *Let $\mathcal{A}_0, \ldots, \mathcal{A}_n$ be computable structures, each in some computable language. Then there is a computable structure $\mathcal{B}$ whose automorphism spectrum consists of all finite joins of the form $\bigcup_{i \in I} \boldsymbol{d}_i$, where $\emptyset \neq I \subseteq \{0, \ldots, n\}$ and, for each $i \in I$, $\boldsymbol{d}_i \in AutSp^*(\mathcal{A}_i)$.*

If all $\mathrm{AutSp}^*(\mathcal{A}_i)$ are closed under join, then of course so is $\mathrm{AutSp}^*(\mathcal{B})$. Indeed, if every $\mathrm{AutSp}^*(\mathcal{A}_i)$ is finite, we can build $\mathcal{A}$ with $\mathrm{AutSp}^*(\mathcal{A})$ being the closure of $\mathrm{AutSp}^*(\mathcal{B})$ under join, simply by applying Lemma 4.1 to the finite set $\{\mathcal{A}_i^j : i \leq n \ \& \ j < j_i\}$ in which each $\mathcal{A}_i^j$ is just $\mathcal{A}_i$ itself and $j_i = |\mathrm{AutSp}^*(\mathcal{A}_i)|$.

Proof of Lemma 4.1. We may assume that the language of each $\mathcal{A}_i$ is purely relational, since this does not change the automorphisms of $\mathcal{A}_i$. The language of $\mathcal{B}$ will be the disjoint union of the languages of each $\mathcal{A}_i$, along with unary predicates $R_0, \ldots, R_n$. We partition ω into $(n+1)$ disjoint computable subsets, which will be $R_0^{\mathcal{B}}, \ldots, R_n^{\mathcal{B}}$. Then we make each $R_i^{\mathcal{B}}$ identical to $\mathcal{A}_i$, with the predicates of the language of $\mathcal{A}_i$ holding for no tuples except those from $R_i^{\mathcal{A}}$.

Then any nontrivial automorphism f of $\mathcal{B}$ restricts to automorphisms of the substructures $R_i^{\mathcal{B}}$, hence yields an automorphism f_i of each $\mathcal{A}_i$. Since f is nontrivial, so is at least one f_i, and the degree of f is the join of the degrees of those f_i which are not the identity, hence is of the desired form. Conversely, given a nonempty join $\cup_{i \in I} \boldsymbol{d}_i$ as in the lemma, we have a nontrivial automorphism f_i of $\mathcal{A}_i$ for each $i \in I$, and we can regard each as an automorphism of $R_i^{\mathcal{B}}$ of the same degree as f_i. The union of these, along with the identity on those $\mathcal{A}_j$ with $j \notin I$, gives a nontrivial automorphism f of $\mathcal{B}$ Turing equivalent to $\cup_{i \in I} \boldsymbol{d}_i$. ∎

Lemma 4.1 allows us to create an automorphism spectrum of any finite cardinality, generalizing a result from [14].

Corollary 4.2 *If $\{\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n\}$ is a set of Turing degrees such that each singleton $\{\boldsymbol{d}_i\}$ is an automorphism spectrum, then there exists a computable structure $\mathcal{A}$ whose automorphism spectrum is the closure of $\{\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n\}$ under joins. In the particular case where $\boldsymbol{d}_0 \leq_T \cdots \leq_T \boldsymbol{d}_n$ is a chain of such degrees, $\{\boldsymbol{d}_i : i \leq n\}$ forms an automorphism spectrum.* ■

To collect infinitely many automorphism spectra together into a single automorphism spectrum, we need some uniformity in the language, of course. Indeed, we usually require uniformity in the construction of the computable structures, as in Proposition 2.1, for example. Theorem 4.3, our main tool for building larger automorphism spectra, does not use arbitrary automorphism spectra, but requires specific rigid computable structures. It applies nicely to the computable structures $\mathcal{D}_{\boldsymbol{d}}$ which we built to prove Theorem 2.6.

Theorem 4.3 *Let $(\mathcal{M}_i)_{i\in\omega}$ be a computable sequence of computable structures, in a single computable signature, such that*

1. *$\mathcal{M}_i \cong \mathcal{M}_j$ for all $i, j \in \omega$; and*
2. *each $\mathcal{M}_i$, $i \in \omega$, is rigid.*

It follows that for each $i \in \omega$, there exists a unique isomorphism $f_i : \mathcal{M}_0 \xrightarrow{onto} \mathcal{M}_i$. Let $\boldsymbol{d}_i = deg(f_i)$. Then there exists a computable structure $\mathcal{M}$ in a computable signature, such that

$$AutSp^*(\mathcal{M}) = \{\oplus_{i\in J}\boldsymbol{d}_i \mid J \text{ is a finite nonempty subset of } \omega\}.$$

Proof. The following result will be useful in understanding the idea of the proof:

Theorem 4.4 (G. Birkhof) *Let G be a group. Define the operations ℓ_g, $g \in G$ as follows: $\ell_g(x) = gx$. Then the set of all automorphisms of the structure $\langle G; \ell_g\rangle_{g\in G}$ consists of precisely the functions $\varphi_a(x) = xa$, $a \in G$.*

The general idea of the proof is that we consider the free group $F = F(x_0, x_1, \ldots)$ of rank ω with the operations ℓ_g in the predicate form and attach to each element of F a corresponding isomorphic copy of the structure $\mathcal{M}_i$, so that the isomorphisms between these attached structures will give us the desired Turing degrees of automorphisms.

Without loss of generality we may assume that $|\mathcal{M}_0| = \omega$. The structure $\mathcal{M}_0$ will be called the *base copy.* Its signature consists of:

1. a countable set of binary predicate symbols ℓ_w, where w is an arbitrary element of F;

2. all symbols from the signature of $\mathcal{M}_0$;

3. a binary predicate symbol P.

The domain of $\mathcal{M}$ is the disjoint union of the set F of all elements of a free group $F(x_0, x_1, \ldots)$ of countable rank and of the set $F \times \omega$. The identity element of F will be denoted by e.

Define the binary predicates ℓ_w, for all $w \in F$, as follows:

1. $\ell_w(x, y) \Leftrightarrow (x, y \in F \wedge y = wx)$;

2. for any n-ary predicate symbol Q in the signature of $\mathcal{M}_0$, $\mathcal{M} \models Q(x_1, \ldots, x_n)$ if and only if there exists some $w \in F$ such that $x_1 = \langle w, m_1 \rangle, x_2 = \langle w, m_2 \rangle, \ldots, x_n = \langle w, m_n \rangle$, and $\mathcal{M}_{k+1} \models Q(m_1, \ldots, m_n)$, where the normal form of w is $x_k^{\varepsilon} \cdot \ldots$ for $\varepsilon \neq 0$, or $w = e$ and $\mathcal{M}_0 \models Q(m_1, \ldots, m_n)$;

3. $\mathcal{M} \models P(x, y)$ if and only if there exists $n \in \omega$ such that $y = \langle x, n \rangle$.

Note that the structure $\mathcal{M}$ is computable. In $\mathcal{M}$, the set F is computable since $F = \{x \mid \exists y\, P(x, y)\}$ and $\bar{F} = \{x \mid \exists y\, P(y, x)\}$.

For $w \in F$, let $\mathcal{M}_w$ be the substructure of $\mathcal{M}$ with domain $\{\langle w, m \rangle : m \in \omega\}$. It is easily seen that each $\mathcal{M}_w$ is isomorphic to $\mathcal{M}_0$.

Note that for each automorphism φ of $\mathcal{M}$ there exists a unique $w \in F$ such that for all $v \in F$, $\varphi(v) = vw$, and φ isomorphically maps each $\mathcal{M}_v$ onto $\mathcal{M}_{vw}$. Denote such an automorphism by φ_w. Note also that such an automorphism φ_w exists for each $w \in F$.

Lemma 4.5 *Under this construction, $\varphi_{x_i} \equiv_T f_i$. Moreover,*

1. *there exists a uniform procedure that, given i and the oracle for f_i, computes φ_{x_i}.*

2. *there exists a uniform procedure that, given an oracle φ_{x_i}, computes i and f_i.*

Proof. First we prove that $\varphi_{x_i} \leq_T f_i$. The automorphism φ_{x_i} is computable on F. To compute an isomorphism $\mathcal{M}_v \to \mathcal{M}_{vx_i}$, note that

(i) when $v \neq x_i^{-1}$ or $v \neq e$, this is just the computable mapping $\langle v, m\rangle \mapsto \langle vx_i^{\varepsilon}, m\rangle$;

(ii) when $v = x_i^{-1}$, this is the mapping $\langle x_i^{-1}, m\rangle \mapsto \langle e, f_i^{-1}(m)\rangle$; and

(iii) when $v = e$, this is the mapping $\langle e, m\rangle \mapsto \langle x_i, f_i(m)\rangle$.

Thus, $\varphi_{x_i} \leq_T f_i$.

Now we prove that $f_i \leq_T \varphi_{x_i}$. To compute i, it suffices to look at the element $\varphi_{x_i}(e) = x_i$. The automorphism φ_{x_i} is an isomorphism from $\mathcal{M}_e$ onto $\mathcal{M}_{x_i}$. Therefore, if $\varphi_{x_i}(\langle e, m\rangle) = \langle x_i, n\rangle$ then $f_i(m) = n$, and the statement follows.

Items (1) and (2) follow immediately from the above remarks, completing the proof of Lemma 4.5. ∎

Lemma 4.6 $\varphi_{vw} = \varphi_v \cdot \varphi_w$. ∎

Lemma 4.7 *For any* $i_1, i_2, \ldots, i_k \in \omega$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \{1, -1\}$, *we have*

$$\varphi_{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_k}^{\varepsilon_k}} \equiv_T f_{i_1} \oplus f_{i_2} \oplus \cdots \oplus f_{i_k}.$$

Proof. The part $\leq_T$ follows from Lemmas 4.5 and 4.6. Since $\varphi_{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdot \ldots \cdot x_{i_k}^{\varepsilon_k}}$ is an isomorphism from $\mathcal{M}_e$ onto $\mathcal{M}_{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdot \ldots \cdot x_{i_k}^{\varepsilon_k}}$ which maps second coordinates in the same way as f_{i_1} does, we can compute f_{i_1}. Using this f_{i_1}, by Lemma 4.5, we can compute $\varphi_{x_{i_1}^{\varepsilon_1}}$. Then we can compute $\varphi_{x_{i_2}^{\varepsilon_2} \cdot \ldots \cdot x_{i_k}^{\varepsilon_k}} = (\varphi_{x_{i_1}^{\varepsilon_1}})^{-1} \varphi_{x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdot \ldots \cdot x_{i_k}^{\varepsilon_k}}$. Using the same procedure, we can compute f_{i_2}, etc. This proves Lemma 4.7, and Theorem 4.3 then follows directly from the preceding description of the automorphisms. ∎

Corollary 4.8 *There exists a computable structure* $\mathcal{M}$ *such that* $AutSp^*(\mathcal{M})$ *consists of all c.e. degrees.*

Proof. Apply Theorem 4.3 to the structures $\mathcal{B}_C$ built in Proposition 2.1. ∎

Proposition 4.9 *Let* $(A_i)_{i\in\omega}$ *be a uniformly c.e. sequence of sets of natural numbers. Then there exists a computable family of structures* $\mathcal{M}_i$ *such that*

$$\{deg(f) \mid f : \mathcal{M}_0 \xrightarrow{\cong} \mathcal{M}_n, n \in \omega\} = \{deg(A_i) \mid i \in \omega\}.$$

Consequently, there exists a computable structure $\mathcal{M}$ *such that* $AutSp^*(\mathcal{M})$ *consists of all finite joins of degrees* $deg(A_i)$, $i \in \omega$.

Proof. The construction of $\mathcal{B}_C$ in Proposition 2.1 is uniform in the enumeration of C, so we simply build $\mathcal{M}_i = \mathcal{B}_{A_i}$ for each i. Theorem 4.3 then yields the structure $\mathcal{M}$. ∎

Corollary 4.10 *For each $\alpha \leq \omega$, there exists a structure $\mathcal{M}$ for which AutSp*$^*(\mathcal{M})$ *consists precisely of α-many pairwise incomparable Turing degrees and all finite joins of those degrees.*

Proof. This follows from Proposition 4.9, since it is straightforward to build a uniformly c.e. sequence of α-many pairwise incomparable sets. (For instance, see [20, Section VII.2].) ∎

Continuing beyond the Σ^0_1 degrees requires a bit more care, because Theorem 2.6 only applies to Σ^0_{n+1} degrees $\boldsymbol{d} \geq_T \mathbf{0}^{(n)}$. Once seen, however, the trick is easy.

Corollary 4.11 *There exists a computable structure $\mathcal{M}$ such that*

$$\mathit{AutSp}^*(\mathcal{M}) = \{\boldsymbol{d} \in \Sigma^0_{n+1} : \boldsymbol{d} \geq_T \mathbf{0}^{(n)}\}.$$

Proof. We can enumerate all Σ^0_{n+1}-formulas $\theta(x)$, of course. For each such $\theta(x)$, let $\psi(x)$ be a formula defining $\{x : \theta(x)\} \oplus \emptyset^{(n)}$. Then every $\psi(x)$ defines a Σ^0_{n+1} set which computes $\emptyset^{(n)}$. Conversely, if $\boldsymbol{d}$ is a Σ^0_{n+1} degree above $\mathbf{0}^{(n)}$, with some $D \in \boldsymbol{d}$ defined by some Σ^0_{n+1} formula θ, then the corresponding ψ also defines a set of degree $\boldsymbol{d}$. Thus, when we apply Theorem 4.3 to the structures built by Theorem 2.6 for all such formulas ψ, with a standard copy of that structure as the base copy, we get the desired $\mathcal{M}$. ∎

Lemma 4.1 then allows us to make finite unions of these sets of degrees into automorphism spectra as well. However, we can do more.

Theorem 4.12 *There exists a computable structure $\mathcal{M}$ with*

$$\mathit{AutSp}^*(\mathcal{M}) = \{\boldsymbol{d} : (\exists n < \omega)[\boldsymbol{d} \in \Sigma^0_{n+1} \ \&\ \boldsymbol{d} \geq_T \mathbf{0}^{(n)}]\}.$$

Proof. For this we will again appeal to Theorem 4.3. Each index i describes a particular formula $\psi_i(x)$ which is Σ^0_{n+1} for some n computable from i, with every such formula (for every n) corresponding to at least one i. We recall that in Theorem 2.6 we built structures $\mathcal{C}_n$ for each n, and a structure $\mathcal{D}_i = \mathcal{D}_{\deg(S_i)}$ for the set $S_i = \{x : \psi_i(x)\} \oplus \emptyset^{(n)}$, uniformly in i and n. For any fixed i and the corresponding n, the structure $\mathcal{M}_i$ is the cardinal sum of all structures $\mathcal{C}_m$ with $m \neq n$, each one identified by the predicate $P_m^{\mathcal{M}_i}$, along with one copy of the structure $\mathcal{D}_i$. The predicate $P_n^{\mathcal{M}_i}$ refers to this $\mathcal{D}_i$, of course, and the entire structure $\mathcal{M}_i$ is computable uniformly in i. Moreover, for every i, $\mathcal{M}_i$ is isomorphic to the cardinal sum $\mathcal{C}_\omega$ of all $\mathcal{C}_m$ (including $m = n$), hence is rigid, and in each case the unique isomorphism f_i from $\mathcal{M}_i$ onto $\mathcal{C}_\omega$ maps $\mathcal{D}_i$ onto $\mathcal{C}_n$, hence computes the set S_i. On the other hand, S_i computes the isomorphism from $\mathcal{D}_i$ onto $\mathcal{C}_n$, and f_i is the identity on all the rest of $\mathcal{M}_i$. Thus f_i is Turing-equivalent to S_i, and Theorem 4.3 completes the proof. ∎

The same process applies to any c.e. collection of arithmetical formulas, as long as we note that we might change the degrees of the sets defined by these formulas as we pass from φ_i to the corresponding set S_i above $\mathbf{0}^{(n)}$.

Of course, we can apply Lemma 4.1 to Theorem 3.4 to add the degree $\mathbf{0}^{(\omega)}$, or others, to the spectrum of the $\mathcal{M}$ in Theorem 4.12. Also, it is now easy to build a computable structure with automorphism spectrum $\{\mathbf{0}^{(\alpha)} : \alpha \leq \omega\}$.

5 Transfer of Automorphism Spectra

In this section we describe methods of transferring automorphism spectra: given a structure $\mathcal{A}$, we wish to build a nicer structure with the same automorphism spectrum. In particular, we may wish to transfer an automorphism spectrum to a model of a more recognizable theory. The cardinal sums and other structures constructed in Section 2 were built specifically for the purpose of realizing various automorphism spectra, and are not structures one ordinarily meets in mainstream mathematics. However, the well-known construction given in [6] by Hirschfeldt, Khoussainov, Shore, and Slinko enables us to transfer our new automorphism spectra to standard mathematical structures. The construction fails only when $\mathcal{A}$ is *automorphically trivial*, i.e. possesses a finite subset F such that every permutation of the domain of $\mathcal{A}$ which fixes F pointwise is an automorphism of $\mathcal{A}$.

Theorem 5.1 *Let $\mathcal{A}$ be any computable structure in any computable signature, and assume $\mathcal{A}$ is not automorphically trivial. Then the symmetric irreflexive graph $\mathcal{G}$ built in [6] from $\mathcal{A}$ has the same automorphism spectrum as $\mathcal{A}$.*

Proof. This is clear upon inspection of the constructions in Appendix A and Section 3.1 of [6], in which first a computable directed graph and then a computable symmetric irreflexive graph $\mathcal{G}$ are built from $\mathcal{A}$. The automorphisms of $\mathcal{A}$ (excluding the identity) correspond precisely to those of $\mathcal{G}$, and from any automorphism of either $\mathcal{A}$ or $\mathcal{G}$ we can compute the corresponding automorphism of the other, so their automorphism spectra must be equal. ■

Corollary 5.2 *Every automorphism spectrum of any computable structure is also the automorphism spectrum of a computable structure in a finite relational signature (specifically, of a graph).*

Proof. Given Theorem 5.1, we need only consider automorphically trivial structures $\mathcal{A}$. If $\mathcal{A}$ is finite, then of course $\mathrm{AutSp}^*(\mathcal{A})$ is either $\{\mathbf{0}\}$ or $\emptyset$, which are also the possible automorphism spectra of finite graphs. Otherwise, every permutation of ω fixing a finite set is an automorphism of $\mathcal{A}$, so clearly $\mathrm{AutSp}^*(\mathcal{A})$ contains all Turing degrees, hence equals the automorphism spectrum of the complete graph on domain ω (which is also a trivial structure). ■

Moreover, the methods used by the authors of [6] for transferring spectra and other properties from graphs to models of another theory T work for automorphism spectra as well. They list four requisite properties, which we repeat below in simplified form and use to extend their results to automorphism spectra.

Proposition 5.3 *Let $\mathcal{G}$ be a computable nontrivial graph with edge relation E, and $\mathcal{A}$ a structure with invariant relations $D(x)$ and $R(x,y)$ on its domain, such that:*

(P0) $\mathcal{A}$ is computable; and

(P1) there is a computable bijective function g mapping D onto $\mathcal{G}$ such that for all $x, y \in D$
$$R(x,y) \iff E(g(x), g(y));$$
and

(P2) every permutation of D respecting R extends to an automorphism of $\mathcal{A}$; and

(P3) there exists a computable defining family for $(\mathcal{A}, b)_{b \in D}$, as defined in [6].

Then $AutSp^*(\mathcal{A}) = AutSp^*(\mathcal{D})$.

The properties given in [6] are stronger than these, since there one needs to consider all copies of $\mathcal{A}$ together, not just a single computable structure $\mathcal{A}$. For automorphism spectra, however, these suffice.

Proof. Any automorphism f of $\mathcal{G}$, of arbitrary Turing degree, gives rise to a permutation $f_{\mathcal{A}} = g^{-1} \circ f \circ g$ of D (using the map g from (P1)). But for $x, y \in D$, (P1) then yields:

$$\begin{aligned} R(x,y) &\iff E(g(x), g(y)) \iff E(f(g(x)), f(g(y))) \\ &\iff R(g^{-1}(f(g(x))), g^{-1}(f(g(y)))) \iff R(f_{\mathcal{A}}(x), f_{\mathcal{A}}(y)) \end{aligned}$$

so that $f_{\mathcal{A}}$ on D respects R. By (P2), $f_{\mathcal{A}}$ therefore extends to an automorphism of all of $\mathcal{A}$, and (P3) allows us to compute the value of $f_{\mathcal{A}}(x)$ for any $x \in \mathcal{A}$, using an f-oracle to compute $f_{\mathcal{A}}$ on D. The defining family shows that $f_{\mathcal{A}}$ is the identity iff f is.

On the other hand, any automorphism $f_{\mathcal{A}}$ of $\mathcal{A}$ respects R and restricts to a permutation of D (since D and R are invariant), which, according to (P3), is the identity iff $f_{\mathcal{A}}$ is. Therefore the function $f_{\mathcal{G}} = g \circ f_{\mathcal{A}} \circ g^{-1}$ is a permutation of $\mathcal{G}$, and by (P1) again, this $f_{\mathcal{G}}$ is actually an automorphism of $\mathcal{G}$, which is the identity iff $f_{\mathcal{A}}$ was. Moreover, $f_{\mathcal{G}}$ is computable from an $f_{\mathcal{A}}$ oracle, and if $f_{\mathcal{A}}$ was built from an automorphism f of $\mathcal{G}$ as in the preceding paragraph, then

$$f_{\mathcal{G}} = g \circ (g^{-1} \circ f \circ g) \circ g^{-1} = f \geq_T f_{\mathcal{A}} \geq_T f_{\mathcal{G}}.$$

This proves that $\mathcal{A}$ and $\mathcal{G}$ have the same automorphism spectrum. ∎

Corollary 5.4 *For every computable structure $\mathcal{A}$ (in any computable signature), there exist a computable partial order and a computable lattice, each with the same automorphism spectrum as $\mathcal{A}$.*

Proof. Section 3 of [6] shows that, for both lattices and partial orders, we can find a model $\mathcal{A}$ satisfying properties (P0)-(P3) for any given nontrivial computable graph $\mathcal{G}$. If $\mathcal{G}$ is trivial and finite, its automorphism spectrum $\{\mathbf{0}\}$ is easily realized by finite models. For infinite trivial $\mathcal{G}$, we need an automorphism spectrum containing all Turing degrees. Consider the partial order with a least element 0, a greatest element 1, and an infinite antichain of elements in between. This becomes a computable lattice under the obvious meet and join operators, and any permutation of the elements of the antichain is an automorphism. ■

6 Spectra of Incomparable Degrees

In Section 2 we saw many examples where the automorphism spectrum contains a single degree. Proposition 6.1 and Theorem 6.7 consider the same question for pairs and triples of incomparable degrees.

Proposition 6.1 *Let $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ be incomparable Turing degrees. Then no computable structure $\mathcal{M}$ has either AutSp$^*(\mathcal{M}) = \{\boldsymbol{d}_0, \boldsymbol{d}_1\}$ or AutSp$^*(\mathcal{M}) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1\}$.*

Proof. Assume for a contradiction that such a model $\mathcal{M}$ exists. Fix $f_0, f_1 \in \mathrm{Aut}(\mathcal{M})$ so that $\deg(f_0) = \boldsymbol{d}_0$ and $\deg(f_1) = \boldsymbol{d}_1$. Now if $\deg(f_0 f_1) \leq_T \boldsymbol{d}_1$, then $f_0 = (f_0 f_1) \circ f_1^{-1}$ would be computable in $\boldsymbol{d}_1$. Likewise, if $\deg(f_0 f_1) \leq_T \boldsymbol{d}_0$, then $f_1 = f_0^{-1} \circ (f_0 f_1)$ would be computable in $\boldsymbol{d}_0$. ■

Theorem 6.2 *There exist $f_0, f_1 \in \mathtt{Sym}(\omega)$ such that $f_0, f_1 \leq_T \emptyset'$ and the Turing degrees of $f_0 f_1$ and $f_1 f_0$ are incomparable.*

Proof. We use a $\emptyset'$-oracle to construct finite one–to–one mappings $f_i^0 \subseteq f_i^1 \subseteq \cdots \subseteq f_i^k \subseteq \cdots$ so that the unions $f_i = \cup_{s\in\omega} f_i^s$ will have the desired properties, for $i = 0, 1$.

We need to satisfy the following requirements, for every $n \in \omega$:

R_n**:** $\Phi_n^{f_1 f_0} \neq f_0 f_1$;

Q_n**:** $\Phi_n^{f_0 f_1} \neq f_1 f_0$;

T**:** $f_0, f_1 \in \mathtt{Sym}(\omega)$.

If f is a partial function then we use a notation $\Phi_n^{[f]}(x) = y$ to denote the fact that nth Turing machine with oracle f and argument x gives the result y and in the process of computation, questions "what is the value of $f(z)$?" are asked for $z \in \mathtt{dom}\,(f)$ only.

We describe the strategy to satisfy the requirement R_n at stage s. Assume we have already constructed finite one–to–one mappings f_0^s and f_1^s. Take the least finite one–to–one extensions $f_i' \supseteq f_i^s$, $i = 0, 1$, so that $\mathtt{ran}\,(f_0') = \mathtt{dom}\,(f_1')$. If there exist an $a \in \omega \setminus \mathtt{ran}\,(f_0')$ and a one–to–one finite extension $f^* \supseteq f_1' f_0'$ such that $\Phi_n^{[f^*]}(a)\downarrow$, select some extensions $f_i'' \supseteq f_i'$, $i = 0, 1$, so that $f_1'' f_0'' = f^*$ and $a \notin \mathtt{dom}\,(f_1'') \cup \mathtt{ran}\,(f_0'')$. (Since $a \notin \mathtt{ran}\,(f_0') = \mathtt{dom}\,(f_1')$, we have enough freedom to choose new values for f_0'' and f_1'' to do so.) We will have

$$\Phi_n^{[f_1'' f_0'']}(a)\downarrow = \Phi_n^{[f^*]}(a).$$

Then extend the mappings f_i'', $i = 0, 1$, to one–to–one mappings f_i^{s+1}, $i = 0, 1$, respectively, so that $f_0^{s+1} f_1^{s+1}(a)\downarrow \neq \Phi_n^{[f^*]}(a)$. If such a and f^* do not exist, the requirement R_n will be satisfied anyway, and we may set $f_i^{s+1} = f_i^s$.

One can easily see that this construction can be executed with a $\emptyset'$-oracle and that for any extensions $g_0 \supseteq f_0^{s+1}$, $g_1 \supseteq f_1^{s+1}$, $g_0, g_1 \in \mathtt{Sym}\,(\omega)$ we have $g_0 g_1(a) \neq \{n\}^{g_1 g_0}(a)$.

A symmetric strategy could be used to satisfy a requirement Q_n.

To satisfy the requirement T, we just need to extend f_i^s, for $i = 0, 1$, to one–to–one mappings f_i^{s+1}, $i = 0, 1$, so that we satisfy the requirement

$$\{0, \ldots, s\} \subseteq \mathtt{dom}\,(f_0^{s+1}) \cap \mathtt{dom}\,(f_1^{s+1}) \cap \mathtt{ran}\,(f_0^{s+1}) \cap \mathtt{ran}\,(f_1^{s+1}).$$

Construction.

Let $f_0^s = f_1^s = \emptyset$.

Stage 3n. Execute the strategy to satisfy R_n;

Stage 3n+1. Execute the strategy to satisfy Q_n;

Stage 3n+2. Execute the strategy to satisfy T. ∎

Theorem 6.3 *Let $(A_i)_{i\in\omega} \in (2^\omega)^\omega$. There exists a permutation $f \in \mathtt{Sym}\,(\omega)$ such that for all $n \in \omega$ we have $f^{2^n} \equiv_T \bigoplus_{i \geq n} A_i$.*

Proof. Without loss of generality we may assume all sets A_i to be infinite. Fix a family $(R_i)_{i\in\omega}$ of pairwise disjoint infinite subsets of ω such that the relation $x \in R_i$ is computable. Let $R_i = \{r_i^0 < r_i^1 < \cdots < r_i^s < \cdots\}$. The permutation f is defined as follows: Let $A_i = \{a_i^0 < a_i^1 < \cdots\}$. Partition each A_i into convex parts of length 2^i each, as follows:

$$A_i \;=\; \left\{a_i^0, a_i^1, \ldots, a_i^{2^i-1}\right\} \cup \cdots \cup \left\{a_i^{k\cdot 2^i}, a_i^{k\cdot 2^i+1}, \ldots, a_i^{(k+1)\cdot 2^i-1}\right\} \cup \cdots .$$

Define a permutation p_i as the infinite product of cycles

$$p_i = \prod_{k\in\omega} \left(a_i^{k\cdot 2^i}, a_i^{k\cdot 2^i+1}, \ldots, a_i^{(k+1)\cdot 2^i-1} \right).$$

Define now bijections $g_i : \omega \to R_i$ as $g_i(j) = r_i^j$, for $i, j \in \omega$; and finally let $f = \prod_{i\in\omega} g_i p_i g_i^{-1}$.

One can easily verify that f satisfies the above conditions. ■

Corollary 6.4 *Let $\boldsymbol{d}$ be a Turing degree and $(A_i)_{i\in\omega}$ be a family of sets such that $A_i \leq_T A_j$ for $i > j$ uniformly in A_0 and such that the relation $x \in A_i$ is A_0–computable. Then there exists an A_0–computable permutation f such that if we define a sequence $(f_n)_{n\in\omega}$ by $f_0 = f$ and $f_{n+1} = f_n^2$, then*

$$deg(f_i) = deg(A_i)$$

holds for all $i \in \omega$.

On the other hand, for each sequence $(f_i)_{i\in\omega}$ defined from f as above, there exist an appropriate sequence of A_i's. ■

Theorem 6.5 *Let A, B, C be arbitrary pairwise disjoint subsets of ω. Then there exist permutations f_0 and f_1 such that*

$$\begin{aligned} f_0 &\equiv_T A \oplus B && (1)\\ f_1 &\equiv_T B \oplus C && (2)\\ f_0 f_1 &\equiv_T (A \cup B) \oplus C && (3)\\ f_1 f_0 &\equiv_T A \oplus (B \cup C). && (4) \end{aligned}$$

Proof. For an arbitrary set $S \subset \omega$, define

$$\begin{aligned} f_S^+ &= \prod_{i\in S} (3i, 3i+1, 3i+2),\\ f_S^- &= \prod_{i\in S} (3i+2, 3i+1, 3i),\\ g_S^+ &= \prod_{i\in S} (3i+1, 3i+2),\\ g_S^- &= \prod_{i\in S} (3i, 3i+1). \end{aligned}$$

It is immediate that $g_S^+ g_S^- = f_S^+$ and $g_S^- g_S^+ = f_S^-$, for any S. Parts (1) and (2) are likewise easy to check.

Let $f_0 = f_A^+ g_B^+$ and $f_1 = f_C^- g_B^-$. One can easily see that $f_0 f_1 = f_{A\cup B}^+ f_C^- \equiv_T (A\cup B)\oplus C$ and $f_1 f_0 = f_A^+ f_{B\cup C}^- \equiv_T A\oplus(B\cup C)$, which implies the conclusion. ∎

Theorem 6.5 gives another proof of the existence of $f_0 f_1$ and $f_1 f_0$ whose degrees are incomparable. Indeed, it gives the existence of such an f_0 and f_1 for any pair of incomparable degrees.

Theorem 6.6 *Let $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ be Turing degrees. Then there exist permutations f_0 and f_1 such that $deg(f_0) = deg(f_1) = \boldsymbol{d}_0 \oplus \boldsymbol{d}_1$, $deg(f_0 f_1) = \boldsymbol{d}_0$, and $deg(f_1 f_0) = \boldsymbol{d}_1$.*

Proof. Assume $\deg(A)$ and $\deg(C)$ are arbitrary degrees. Without loss of generality we may assume $A \cap C = \emptyset$. Let $B = \omega \setminus (A \cup C)$. The result follows from $A \oplus B \equiv_T A \oplus C$, $B \oplus C \equiv_T A \oplus C$, $(A \cup B) \oplus C = \bar{C} \oplus C \equiv_T C$, and $A \oplus (B \cup C) = A \oplus \bar{A} \equiv_T A$. ∎

A question remains open here: what are the possible Turing degrees of f_0, f_1 such that $f_0 f_1 \equiv_T \boldsymbol{d}_0$ and $f_1 f_0 \equiv_T \boldsymbol{d}_1$?

Now we move to sets of three incomparable degrees. Here, in contrast to Proposition 6.1, we have the following result.

Theorem 6.7 *There exist pairwise incomparable Δ_2^0 Turing degrees $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, and computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $AutSp^*(\mathcal{A}) = \{\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$ and $AutSp^*(\mathcal{B}) = \{\mathbf{0}, \boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2\}$.*

Proof. We begin with a lemma which constructs sets A and B such that the Turing degrees of the sets A, B, and $(B \setminus A)$ will be the degrees $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, and $\boldsymbol{d}_2$ that we need.

Lemma 6.8 *There exist c.e. sets A and B such that $A \subset B$ and the degrees $deg(A)$, $deg(B \setminus A)$, $deg(\overline{B})$ $(= deg(B))$ are pairwise incomparable.*

Proof. We enumerate A and B using standard Friedberg-Muchnik requirements:

$$\begin{aligned} \mathcal{R}_{3e}: &\quad \Phi_e^A \neq B \\ \mathcal{R}_{3e+1}: &\quad \Phi_e^{B\setminus A} \neq A \\ \mathcal{R}_{3e+2}: &\quad \Phi_e^B \neq B \setminus A. \end{aligned}$$

The requirements ensure that none of the sets A, $(B \setminus A)$, and $\overline{B}$ computes both of the other two. Since these sets will form a partition of ω, this suffices to prove the lemma. (If $C_1 \sqcup C_2 \sqcup C_3 = \omega$ and $C_1 \leq_T C_2$, then $C_3 = \overline{(C_1 \cup C_2)} \leq_T C_2$ as well.)

Start with $A_0 = B_0 = \emptyset$. At stage $s+1$, find the least $i \leq s$ for which $\mathcal{R}_i$ is currently not satisfied and one of the conditions below holds, and follow the instruction for satisfying $\mathcal{R}_i$:

- If $i = 3e$ and $\Phi_{e,s}^{A_s}(x_{i,s}) \downarrow = 0$, enumerate $x_{i,s}$ into B_{s+1}.
- If $i = 3e+1$ and $\Phi_{e,s}^{(B_s \setminus A_s)}(x_{i,s}) \downarrow = 0$, enumerate $x_{i,s}$ into A_{s+1} and into B_{s+1}. (Thus $B_{s+1} \setminus A_{s+1} = B_s \setminus A_s$, since by our construction $x_{i,s} \notin B_s$.)
- If $i = 3e+2$ and $\Phi_{e,s}^{B_s}(x_{i,s}) \downarrow = 1$, enumerate $x_{i,s}$ into A_{s+1}. (By our construction $x_{i,s} \in B_s$ already, so this action removes $x_{i,s}$ from $(B \setminus A)$.)

In each of these cases, we then make all witnesses $x_{j,s+1}$ with $j > i$ undefined, declare all corresponding $\mathcal{R}_j$ unsatisfied, and end the stage. If there is no $i \leq s$ to which these conditions apply, then for the least i for which $x_{i,s}$ is undefined, we choose a new number $x_{i,s+1}$ bigger than any number yet seen in the construction. If this i is of the form $3e+2$, then enumerate the new $x_{i,s+1}$ into B_{s+1}; otherwise, leave it out of both A_{s+1} and B_{s+1}. This completes stage $s+1$.

We show by induction that each $\mathcal{R}_i$ acts only finitely often and is satisfied. Once all $\mathcal{R}_j$ with $j < i$ have finished acting, $x_i = \lim_s x_{i,s}$ will be chosen as a large element (not already in A or B) and will remain permanently defined. In the first two cases, if (and only if) the relevant functional ever converges to 0, we satisfy $\mathcal{R}_i$ by putting x_i into the appropriate set (A or B) as required, without changing the oracle. In the third case, with $i = 3e+2$, x_i enters B as soon as it is chosen, and if $\Phi_e^B(x_i)$ ever converges to 1, we act by putting x_i into A. This removes x_i from $B \setminus A$ without changing the oracle B, thus making Φ_e^B and $(B \setminus A)$ disagree on x_i. Making all x_j with $j > i$ undefined at this stage ensures that the convergence of the functional is preserved, so $\mathcal{R}_i$ is satisfied and never acts again. Finally, notice that whenever an element x_i is enumerated into A, either it is enumerated into B at the same stage (if $i = 3e+1$) or else it was already in B (if $i = 3e+2$). Thus $A \subset B$, proving the lemma. ∎

Lemma 6.9 *Let $\mathcal{M}$ be a structure of finite predicate signature whose universe is the set of all natural numbers and whose predicates are all either c.e.*

or co–c.e. Then there exists a computable structure $\mathcal{M}'$ whose automorphism degree spectrum coincides with automorphism degree spectrum for $\mathcal{M}$.

Proof. Without loss of generality, we may assume that all our predicates are c.e. and the universe of our structure is the set of even numbers. The proof is really just Marker's construction of $\mathcal{M}_\exists$ (cf. Theorem 2.5). For each predicate in the signature, we fix its enumeration. Simultaneously enumerate all the signature predicates and for each n-ary predicate P, enumerate a new $(n+1)$–ary predicate X_P as follows: each time a new n–tuple $\langle x_1, \dots, x_n \rangle$ is enumerated into P at step t, we add to the structure the least unused odd number a_t^P and enumerate the tuple $\langle a_t^P, x_1, \dots, x_n \rangle$ into X_P.

One can easily verify that this new structure satisfies the conditions of the lemma. ■

Now we prove the theorem. Fix c.e. sets A and B as in Lemma 6.8.

The basic set of our model will be a family of pairwise distinct elements $S = \{a_i^k \mid i < \omega, k \in \{0,1\}\}$. Consider a natural ordering $\prec$ on S defined as

$$a_i^k \prec a_j^m \;\Leftrightarrow\; (k = m) \wedge (i < j).$$

The ordering $\prec$ will be used to define the basic predicates of our model but will be not contained among them.

Let $A_k = \{a_i^k \mid i \in A\}$, $B_k = \{a_i^k \mid i \in B\}$, for $k = 0, 1$.

Our model will have the following three predicates:

$$\begin{aligned} R_A &= \prec\restriction (A_0 \cup A_1), \text{ i.e., the restriction of } \prec \text{ to } (A_0 \cup A_1), \\ R_B &= \prec\restriction (\overline{B_0 \cup B_1}), \\ R &= [\prec\restriction (B_0 \cup B_1)] \cup [(B_0 \cup B_1) \times (A_0 \cup A_1)]\,. \end{aligned}$$

Let now $\mathcal{B} = \langle \mathcal{S}; \mathcal{R_A}, \mathcal{R_B}, \mathcal{R} \rangle$. By the definition, each predicate of $\mathcal{B}$ is either c.e. or co–c.e.; and the automorphism group of $\mathcal{B}$ is generated by the automorphisms φ_H, $H \in \{A,\ (B\backslash A),\ \overline{B}\}$ completely defined by the following conditions:

$$\varphi_H(a_i^0) = \begin{cases} a_i^1, & \text{if } i \in H \\ a_i^0, & \text{if } i \notin H \end{cases} \qquad\qquad \varphi_H(a_i^1) = \begin{cases} a_i^0, & \text{if } i \in H \\ a_i^1, & \text{if } i \notin H. \end{cases}$$

An immediate check shows that the group $\mathrm{Aut}(\mathcal{B})$ consists of the eight elements 1, φ_A, $\varphi_{B\backslash A}$, $\varphi_{\overline{B}}$, $\varphi_A\varphi_{B\backslash A}$, $\varphi_{B\backslash A}\varphi_{\overline{B}}$, $\varphi_A\varphi_{\overline{B}}$, and $\varphi_A\varphi_{B\backslash A}\varphi_{\overline{B}}$, and that

$$\begin{array}{l} \deg(1) = \deg(\varphi_A \varphi_{B\setminus A} \varphi_{\overline{B}}) = \mathbf{0}; \\ \deg(\varphi_A) = \deg(\varphi_{B\setminus A} \varphi_{\overline{B}}) = \deg(A); \\ \deg(\varphi_{B\setminus A}) = \deg(\varphi_A \varphi_{\overline{B}}) = \deg(B \setminus A); \\ \deg(\varphi_{\overline{B}}) = \deg(\varphi_A \varphi_{B\setminus A}) = \deg(B). \end{array}$$

It follows that the automorphism degree spectrum of $\mathcal{B}$ equals

$$\{\mathbf{0}, \deg(A), \deg(B \setminus A), \deg(\overline{B})\}.$$

We then apply Lemma 6.9 to turn $\mathcal{B}$ into a computable structure $\mathcal{B}_{\exists\forall}$ with automorphisms of precisely the same degrees.

Finally, the structure $\mathcal{A}$ is identical to $\mathcal{B}$, only in a language with an extra constant symbol c added. We fix $c^{\mathcal{A}}$ to be any element a_i^0, $i \in A$. This eliminates the automorphism φ_A from the generating set of the automorphism group, leaving the identity, $\varphi_{B\setminus A}$, $\varphi_{\overline{B}}$, and $\varphi_{B\setminus A}\varphi_{\overline{B}}$, as the automorphisms. However, $\varphi_{B\setminus A}\varphi_{\overline{B}}$ leaves fixed precisely those a_i with $i \in A$, hence is Turing-equivalent to A, as required. Again, Lemma 6.9 turns $\mathcal{A}$ into a computable structure with this same automorphism spectrum, completing the proof of Theorem 6.7. ■

7 Upper Cones of Degrees

Proposition 7.1 *Let $\mathcal{A}$ be any computable non-rigid structure. Then there exists a computable structure $\mathcal{C}$ of finite signature such that AutSp$^*(\mathcal{C})$ contains precisely those Turing degrees $\mathbf{d}$ which compute a nontrivial automorphism of $\mathcal{A}$. That is, AutSp$^*(\mathcal{C})$ is the upward closure of AutSp$^*(\mathcal{A})$ under Turing reducibility.*

Proof. We begin by working in an infinite signature, containing unary predicates P_i for each $i \in \omega$, along with all symbols from the signature of $\mathcal{A}$. By Corollary 5.2, the result will also hold for a finite signature.

Our structure $\mathcal{C}$ will consist of countably many *planes*, and the predicate P_i will hold of all elements of the i-th plane. P_i contains precisely the i-th row $\{\langle i, x\rangle : x \in \omega\}$ of ω^2, and we make P_i into a copy of $\mathcal{A}$ using the second coordinate:

$$R^{\mathcal{C}}(\langle i_1, x_1\rangle, \ldots, \langle i_n, x_n\rangle) \iff i_1 = \cdots = i_n \ \&\ R^{\mathcal{A}}(x_1, \ldots, x_n);$$

$$f^{\mathcal{C}}(\langle i_1, x_1\rangle, \ldots, \langle i_n, x_n\rangle) = \begin{cases} \langle i_1, f^{\mathcal{A}}(x_1, \ldots, x_n)\rangle, & \text{if } i_1 = i_2 = \cdots = i_n, \\ \langle i_1, x_1\rangle, & \text{otherwise.} \end{cases}$$

In a relational signature, this $\mathcal{C}$ is just the cardinal sum of countably many copies of $\mathcal{A}$.

Now suppose that φ is an automorphism of $\mathcal{C}$. The predicates P_i ensure that φ maps each plane onto itself. Within a single plane P_i, φ may or may not fix the entire plane pointwise. Of course, if all planes are fixed pointwise, then φ is the identity. Otherwise, there exists a plane P_i such that $\varphi\restriction P_i$ is a nontrivial automorphism of the copy of $\mathcal{A}$ in that plane, so φ computes some degree in $\mathrm{AutSp}^*(\mathcal{A})$.

Conversely, let D compute some nontrivial automorphism ψ of $\mathcal{A}$, and fix an $a \in \mathcal{A}$ such that $\psi(a) \neq a$. We define an automorphism φ of $\mathcal{C}$ by:

$$\varphi(\langle i, x\rangle) = \begin{cases} \langle i, x\rangle, & \text{if } i \in D, \\ \langle i, \psi(x)\rangle, & \text{if } i \notin D. \end{cases}$$

Thus $\varphi \leq_T D$, and conversely $D \leq_T \varphi$, because $i \in D$ iff $\varphi(\langle i, a\rangle) = \langle i, a\rangle$. So $\deg(D) = \deg(\varphi) \in \mathrm{AutSp}^*(\mathcal{C})$. This proves the proposition. ■

Corollary 7.2 *Let $\mathcal{A}$ be any computable rigid structure, and assume that $\mathcal{B}$ is a computable copy of $\mathcal{A}$ such that the unique isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is of degree $\boldsymbol{d}$. Then there exists a computable structure $\mathcal{C}$ of finite signature such that AutSp$^*(\mathcal{C})$ contains precisely those Turing degrees which compute $\boldsymbol{d}$.* ■

In concert with this result, Proposition 3.3 yields:

Corollary 7.3 *For any Π^0_1-function singleton f, there exists a computable structure whose automorphism spectrum is the upper cone of degrees $\geq_T$ deg(f). In particular, for any ordinal $\alpha < \omega_1^{CK}$ and any Turing degree $\boldsymbol{d}$ with $\mathbf{0}^{(\alpha)} \leq_T \boldsymbol{d} \leq_T \mathbf{0}^{(\alpha+1)}$, the upper cone of degrees $\geq_T \boldsymbol{d}$ forms an automorphism spectrum. For Σ^0_{n+1} degrees, the construction is uniform in any $\emptyset^{(n)}$-computable enumeration of a set of that degree.* ■

With these results, we may now show that a union of upper cones above incomparable degrees can be the automorphism spectrum of a computable structure.

Corollary 7.4 *For any finite set $\{\boldsymbol{d}_i : i < n\}$ of Turing degrees, each with structures $\mathcal{A}_i$ and $\mathcal{B}_i$ as described in Corollary 7.2, there exists a computable structure $\mathcal{D}$ of finite signature such that*

$$\mathit{AutSp}^*(\mathcal{D}) = \{\boldsymbol{c} : (\exists i)\ \boldsymbol{d}_i \leq_T \boldsymbol{c}\}.$$

Indeed, the same result holds for any countable collection $\{\boldsymbol{d}_i : i \in \omega\}$ of such degrees, provided that the structures $\mathcal{A}_i$ and $\mathcal{B}_i$ can be given uniformly in i.

Proof. Let the structure $\mathcal{A}$ be the cardinal sum of one copy of each $\mathcal{A}_i$ and one copy of each $\mathcal{B}_i$, with a predicate P_i (for each i) which holds precisely of the elements of $\mathcal{A}_i \cup \mathcal{B}_i$. Then for each i there is a nontrivial automorphism of $\mathcal{A}$ of degree $\boldsymbol{d}_i$, which interchanges $\mathcal{A}_i$ and $\mathcal{B}_i$ and fixes everything else. Conversely, every nontrivial automorphism of $\mathcal{A}$ computes some $\boldsymbol{d}_i$. Apply Proposition 7.1 to $\mathcal{A}$. ∎

Corollary 7.5 *There exists a computable structure $\mathcal{M}$ whose spectrum is the union of the upper cones above each degree of an infinite antichain of Σ^0_1 degrees. The same holds in general for Σ^0_n degrees, and also for arbitrary finite antichains of degrees of Π^0_1-function singletons.*

Proof. This follows from Proposition 7.1 and Corollary 4.10. For $n > 1$, it uses a uniform sequence of Σ_n formulas $\alpha_i(x)$ defining sets which form a countable antichain under $\leq_T$. ∎

On the other hand, we have restrictions on which upper cones, and which countable unions of cones, can be automorphism spectra of computable structures. Indeed, our results apply more generally to minimal degrees of such spectra. (To clarify: here *minimal degree* means "minimal in $\mathrm{AutSp}^*(\mathcal{C})$ under $\leq_T$," as opposed to the notion of a degree which is minimal among all nonzero Turing degrees.)

Theorem 7.6 (Minimal Degree Theorem) *Let $\mathcal{C}$ be a computable structure. If there are no more than countably many Turing degrees $\boldsymbol{d}$ which are minimal under $\leq_T$ within AutSp$^*(\mathcal{C})$, then each such minimal $\boldsymbol{d}$ is hyperarithmetical.*

Proof. Let $\boldsymbol{d}$ be such a degree. Since $\mathcal{C}$ is a computable structure in a computable language, being a nontrivial automorphism of $\mathcal{C}$ is arithmetically definable. (We denote the set of nontrivial automorphisms by $\mathrm{Aut}^*(\mathcal{C})$.) Thus the set

$$\{f \in \mathrm{Aut}^*(\mathcal{C}) : (\forall e)[\Phi_e^f \in \mathrm{Aut}^*(\mathcal{C}) \implies f \leq_T \Phi_e^f]\}$$

contains precisely the automorphisms of $\mathcal{C}$ of minimal degree in $\mathrm{AutSp}^*(\mathcal{C})$. By assumption, there are at most countably many such automorphisms, so the Perfect Set Theorem shows that the elements of this set are hyperarithmetical. ∎

Corollary 7.7 *For a computable structure $\mathcal{C}$, if AutSp$^*(\mathcal{C})$ is the upper cone of degrees $\geq_T$ $\boldsymbol{d}$, then $\boldsymbol{d}$ is hyperarithmetical.* ■

Finally, since Aut$(\mathcal{C})$ is definable, the Perfect Set Theorem yields one more condition equivalent to those of Theorem 3.2.

Proposition 7.8 *For a computable structure $\mathcal{C}$, AutSp$^*(\mathcal{C})$ is at most countable iff it contains only hyperarithmetical degrees.* ■

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